Modular Invariance

Torus partition function

$$Z(L,R) = \sum_{|\psi\rangle} \langle\psi|\, e^{H_{cyl} R} \cdot e^{-iP_{cyl}\theta} \cdot |\psi\rangle$$

$$H_{cyl} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right), \quad P_{cyl} = \frac{2\pi}{L}\left(L_0 - \bar{L}_0\right)$$

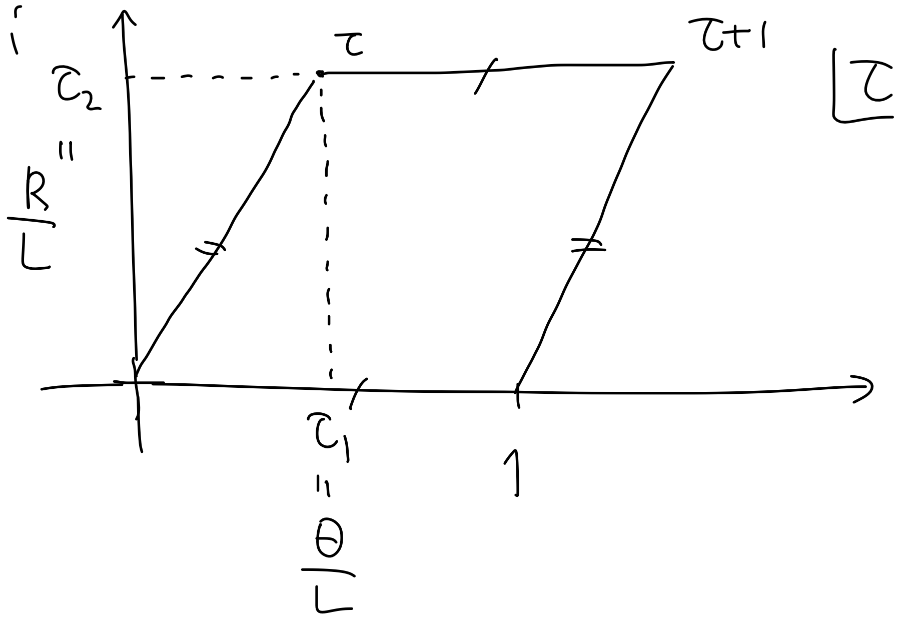
$$= \mathrm{Tr}\left[e^{\frac{2\pi}{L}(R - i\theta)\left(L_0 - \frac{c}{24}\right)} e^{\frac{2\pi}{L}(R+i\theta)\left(\bar{L}_0 - \frac{c}{24}\right)} \right]$$

define $\frac{R}{L} - i\frac{\theta}{L} \equiv i\tau \longrightarrow \frac{R}{L} + i\frac{\theta}{L} = -i\bar{\tau}$

$$= \mathrm{Tr}\left[e^{2\pi i\tau\left(L_0 - \frac{c}{24}\right)} e^{-2\pi i\bar{\tau}\left(\bar{L}_0 - \frac{c}{24}\right)} \right]$$

$$= \mathrm{Tr}\left[q^{\left(L_0 - \frac{c}{24}\right)} \bar{q}^{\left(\bar{L}_0 - \frac{c}{24}\right)} \right], \quad q \equiv e^{2\pi i\tau}$$

moduli of Torus



$Z(\tau)$ should be modular invariant $\because$

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d} \quad a,b,c,d \in \mathbb{Z} \quad (SL(2,\mathbb{Z})/\mathbb{Z}_2), \; a>0$$

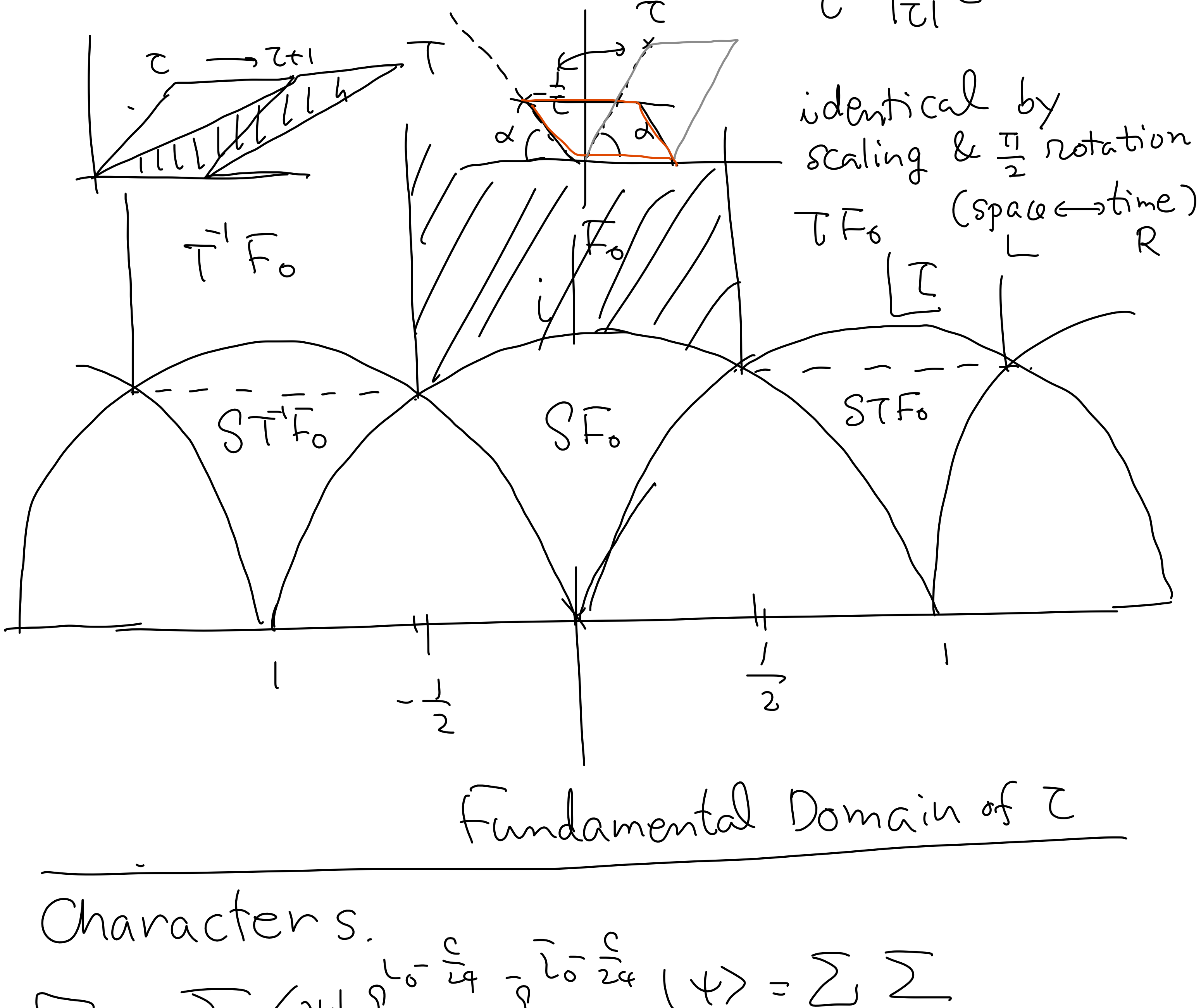
describe the same torus. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $b>0$ (periodicity)

$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rightarrow ST = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$: $(S^2 = 1, \; (ST)^3 = 1)$

$\downarrow$
generated by $T: \tau \rightarrow \tau + 1 \; ; \; \theta \rightarrow \theta + L$

$\underbrace{S: \tau \rightarrow -\frac{1}{\tau}}$ $\quad \tau = |\tau| e^{i\alpha}$

$-\frac{1}{\tau} = \frac{1}{|\tau|} e^{i(\pi-\alpha)}$



Fundamental Domain of τ

Characters.

$$Z = \sum_{|\psi\rangle} \langle \psi | q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} | \psi \rangle = \sum_{h,\bar{h}} \sum_{\underbrace{\{h\}\{\bar{h}\}}_{\text{descent}}}$$

$$= \sum_{h,\bar{h}} \underbrace{N_{h\bar{h}}}_{} \chi_h(q) \bar{\chi}_{\bar{h}}(\bar{q}) \; ; \; \boxed{\chi_h = \operatorname{Tr}_{\{h\}} q^{L_0 - \frac{c}{24}}}$$

multiplicity of $\Phi_{h\bar{h}}$ in the theory.

① Free boson on a torus (no zero-mode)

$c=1$: $V_\alpha = e^{i\sqrt{2}\alpha\varphi} \rightarrow h_\alpha = \alpha^2$

$$\chi_{h_\alpha}(q) = \mathrm{Tr}_{|h_\alpha\rangle}\, q^{L_0-\frac{c}{24}} = q^{h_\alpha-\frac{c}{24}} \underbrace{\sum_{N=0}^{\infty} q^N d(N)}_{} \leftarrow \text{\# of parti of } N$$

$$\longrightarrow \frac{1}{\prod_{n=0}^{\infty}(1-q^n)}$$

$$L_0 L_{-n_1}\cdots L_{-n_k}|h_\alpha\rangle = h_\alpha + \sum n_j$$

$$= \frac{q^{h_\alpha}}{q^{\frac{1}{24}}\prod_{n=0}^{\infty}(1-q^n)} = \frac{q^{h_\alpha}}{\eta}, \quad \left(\eta(\tau) \equiv q^{\frac{1}{24}}\prod_{n=0}^{\infty}(1-q^n)\right)$$

Dedekind

$$V_{\bar\alpha} = e^{i\sqrt{2}\bar\alpha\bar\varphi} \rightarrow h_{\bar\alpha} = \bar\alpha^2 \rightarrow \bar\chi_{h_{\bar\alpha}}(\bar q) = \frac{\bar q^{h_{\bar\alpha}}}{\bar\eta}$$

$$\therefore Z = \int d\alpha \int d\bar\alpha \begin{pmatrix}\delta_{\alpha\bar\alpha} \\ +\delta_{\alpha\bar\alpha}\end{pmatrix} \frac{e^{\alpha^2 2\pi i(\tau-\bar\tau)}}{|\eta|^2} = 2\int d\alpha \frac{e^{-4\pi\alpha^2\tau_2}}{(\eta|^2} = \frac{2}{|\eta|^2}\sqrt{\frac{1}{4\tau_2}}$$

$$= \frac{1}{\sqrt{\mathrm{Im}\tau}}\frac{1}{|\eta|^2},,$$

$\tau\to\tau+1$ is obvious

$\tau\to-\frac{1}{\tau}$

$\tau = \tau_1 + i\tau_2$

$-\frac{1}{\tau} = \frac{-\tau_1+i\tau_2}{\tau_1+\tau_2^2}$

$\sqrt{\tau_2} \rightarrow \sqrt{\frac{\tau_2}{\tau_1^2+\tau_2^2}}$

mod. of η-function

$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$

$|\eta(-\frac{1}{\tau})|^2 \Rightarrow \sqrt{\tau_1^2+\tau_2^2}\,|\eta|^2$

Path Integral : complicated

$$Z = \int[d\varphi]\; e^{-\frac{1}{2}\int d^2x \underbrace{(\nabla\varphi)^2}_{-\varphi\nabla^2\varphi}}$$

→ torus

let's expand $\varphi(x) = \sum_n c_n \psi_n(x)$

where $\nabla^2\psi_n = -\lambda_n\psi_n$

$$= \int \prod_i dc_i\; e^{-\frac{1}{2}\sum_n \lambda_n c_n^2 \langle\psi_n|\psi_n\rangle} \quad \prod_n \sqrt{\frac{2\pi}{\lambda_n\langle\psi_n|\psi_n\rangle}}$$

↑ diverge

| ζ-function regularization technique

define $G(s) \equiv \sum_n{}' \frac{1}{\lambda_n^s}$ (exclude $n \ni : \lambda_n = 0$)

$$G(s) = \sum_n{}' e^{s \ln \frac{1}{\lambda_n}} \to G'(s) = \sum_n e^{s \ln \frac{1}{\lambda_n}} \ln \frac{1}{\lambda_n} \to G'(0) = \sum{}' \ln \frac{1}{\lambda_n}$$
$$= \ln \prod_n \frac{1}{\lambda_n}$$

$$e^{\frac{1}{2}G'(0)} = \prod_n \frac{1}{\sqrt{\lambda_n}} \quad \therefore \quad Z = e^{\frac{1}{2}G'(0)}$$ upto overall c.

$A = \int d^2x \cdot 1 = \tau_2$

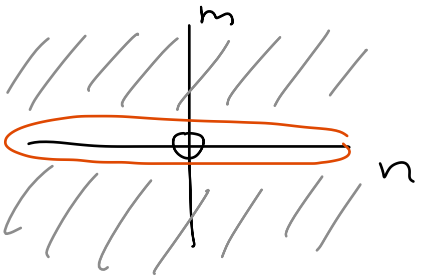
$$\psi_{(nm)}^{(z,\bar z)} = e^{2\pi i \frac{1}{2i\tau_2}\left(n(z-\bar z) - m(\tau \bar z - \bar\tau z)\right)}$$

which should satisfy periodicity

$$\psi(z+\tau, \bar z + \bar\tau) = \psi(z,\bar z), \psi(z+1, \bar z+1) = \psi(z,\bar z)$$

$$-\psi^\dagger_{nm}\nabla^2\psi_{nm} = -\left(\frac{2\pi i}{2i\tau_2}\right)^2 (n+\bar\tau m)(-n-m\tau)\overbrace{\int \underset{=1}{|\psi_{nm}|^2} d^2x}^{A}$$

$$= \underbrace{\left(\frac{\pi}{\tau_2}\right)^2 |n+m\tau|^2}_{\lambda_{nm}} A \equiv \lambda_{nm} A$$



$$G(s) = \left(\frac{\tau_2}{\pi}\right)^{2s} \frac{1}{A^s} \sum_{n,m}{}' \underbrace{\frac{1}{|n+m\tau|^{2s}}}_{(n,m\neq 0)}$$

$$= \left(\sum_{n=-\infty}^{\infty} \sum_{m\neq 0}{}' + \underset{\substack{(m=0)\\ \left(2\sum_{n=1}^{\infty}\right)}}{\sum_{n\neq 0}{}'}\right) \frac{1}{|n+m\tau|^{2s}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \equiv \zeta(2s)\right)$$

$$= 2\zeta(2s) + \sum_m{}' \left(\sum_n \underline{\underline{\frac{1}{|n+m\tau|^{2s}}}}\right)$$

let

$$f(x) \equiv \sum_n \frac{1}{|n+m\tau+x|^{2s}}$$

↑ periodic fct of x with $x \to x+1$ periodicity if

$$f(x+1) = f(x)$$

$$f(x) = \sum_p e^{2\pi i x p} \cdot \tilde{f}_p = \sum_p e^{2\pi i x p} \int_0^1 dy\, e^{-2\pi i p y} f(y)$$

$$\tilde{f}_q = \int_0^1 dy\, e^{-2\pi i q y} f(y) \leftarrow \left(\int_0^1 dy\, e^{2\pi i n y} = \delta_{n0} \right)$$

---

let $f(x) \equiv \sum_n \frac{1}{|n+m\tau+x|^{2s}} = \sum_p e^{2\pi i x p} \int_0^1 dy\, e^{-2\pi i p y} \sum_n \frac{1}{|n+m\tau+y|^{2s}}$

$$|n+m\tau+y|^2 = (n+m\tau_1+y)^2 + m^2\tau_2^2$$

$$\int_0^1 dy\, e^{-2\pi i p y} \sum_n \frac{1}{|n+m\tau+y|^{2s}} = \sum_n \int_0^1 dy\, e^{-2\pi i p y} \frac{1}{\left((n+m\tau_1+y)^2 + m^2\tau_2^2\right)^s}$$

$$= \int_{-\infty}^{\infty} dy \frac{e^{-2\pi i p y}}{\left((y+m\tau_1)^2 + m^2\tau_2^2\right)^s} \qquad n+y \equiv y$$

$$= \int_{-\infty}^{\infty} dy \frac{e^{2\pi i p (m\tau_1 - y)}}{(y^2+m^2\tau_2^2)^s}$$

$$\therefore f(0) = \sum_p \int_{-\infty}^{\infty} dy\, e^{2\pi i p (m\tau_1 - y)} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-t(y^2+m^2\tau_2^2)}$$

$$= \frac{1}{\Gamma(s)} \sum_p \int_0^\infty dt\, t^{s-1} e^{-m^2\tau_2^2 t} e^{2\pi i p m \tau_1} \qquad \left(\frac{\Gamma(s)}{x^s} \equiv \int_0^\infty dt\, t^{s-1} e^{-\boxed{xt}} \to t \right)$$

$$\times \underbrace{\int_{-\infty}^{\infty} dy\, e^{-(t y^2 + 2\pi i p y)}} \to \sqrt{\frac{\pi}{t}}\, e^{-\frac{\pi^2 p^2}{t}}$$

$$= \frac{\sqrt{\pi}}{\Gamma(s)} \sum_p \int_0^\infty dt\, t^{s-\frac{3}{2}} e^{-\left[\frac{\pi^2}{t} p^2 + m^2\tau_2^2 t - 2\pi i p m \tau_1\right]}$$

$$\sum_p = p{=}0 + \sum_{p(\neq 0)}{}'$$

$$p=0 \qquad \int_0^\infty dt\, t^{s-\frac{3}{2}} e^{-m^2\tau_2^2 t} = \Gamma\left(s-\tfrac{1}{2}\right)\left(m^2\tau_2^2\right)^{-(s-\frac{1}{2})}$$

$$\sum_m \; p=0: \quad \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_m{}' |m\tau_2|^{1-2s} = 2\sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} |\tau_2|^{1-2s} \zeta(2s-1) \;//$$

$$\& \; \frac{\sqrt{\pi}}{\Gamma(s)} \sum_m{}' \sum_p{}' e^{2\pi i p m \tau_1} \int_0^\infty \frac{dt}{t}\, t^{s-\frac{1}{2}} e^{-\left(\frac{\pi^2 p^2}{t} + m^2\tau_2^2 t\right)}$$

$$t \to t \left| \frac{p\pi}{m\tau_2} \right|$$

$$\int_0^\infty \frac{dt}{t}\, t^{s-\frac{1}{2}}\, e^{-\left(\frac{\pi^2 p^2}{t} + m^2\tau_2^2 t\right)} = \underbrace{\left|\frac{p\pi}{m\tau_2}\right|^{s-\frac{1}{2}} \int_0^\infty \frac{dt}{t}\, t^{s-\frac{1}{2}}\, e^{-\pi\tau_2|pm|\left(t+\frac{1}{t}\right)}}_{s\to 0\ \sqrt{\frac{\cancel{m\tau_2}}{|p|\cancel{\pi}}}\sqrt{\frac{\cancel{\pi}}{\sqrt{\pi}\,\cancel{\tau_2}|p\cancel{m}|}}\, e^{-2\pi\tau_2|pm|}}$$

$$t \to t\left|\frac{p\pi}{m\tau_2}\right| \qquad \to \frac{1}{|p|\sqrt{\pi}}$$

we need $G'(0)$ or $G(s) = G(0) + s\,G'(0)$

since $\Gamma(s) \sim \frac{1}{s}$ $\quad \frac{1}{\Gamma(s)} = s$ $\to$ we may evaluate the integral at $s=0$

$$\int_0^\infty \frac{dt}{t}\, t^{-\frac{1}{2}}\, e^{-\alpha\left(t+\frac{1}{t}\right)} = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x} e^{-2\alpha\cosh x} = \sqrt{\frac{\pi}{\alpha}}\, e^{-2\alpha}$$

$\alpha = \pi\tau_2|pm| \qquad t \equiv e^{x}$ //

$$s\sum_m{}'\sum_p{}' \frac{1}{|p|}\, e^{2\pi i p m \tau_1}\, e^{-2\pi|pm|\tau_2} \qquad p\neq 0$$

$$\therefore G(s) = \left(\frac{\sqrt{\tau_1}}{\pi}\right)^{2s}\left(\overbrace{2\,\zeta(2s)}^{-1} + \underbrace{2\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}|\tau_2|^{1-2s}\,\zeta(2s-1)} + s\sum_m{}'\sum_p{}' \frac{1}{|p|}\, e^{2\pi i p m\tau_1}\, e^{-2\pi|pm|\tau_2} + O(s^2)\right)$$

using Id

$$\left[\ \Gamma\left(s-\tfrac{1}{2}\right)\zeta(2s-1) = \pi^{2s-1}\,\Gamma(1-s)\,\zeta(2-2s)\right]$$

$$\underbrace{\frac{2}{\Gamma(s)}\left|\frac{\tau_2}{\pi}\right|^{1-2s}\Gamma(1-s)\,\zeta(2-2s)}_{\text{as } s\to 0}$$

$$2s\frac{\tau_2}{\pi}\cdot\underbrace{\zeta(2)}_{\frac{\pi^2}{6}} = \frac{\pi}{3}s\,\tau_2$$

$$\sum_{m>0,\,p>0}\frac{2}{p}\left(\underbrace{e^{2\pi i p m(\tau_1+i\tau_2)}}_{(q^m)^p} + \underbrace{e^{-2\pi i p m(\tau_1-i\tau_2)}}_{(\bar{q}^m)^p}\right)$$

$$= -2\sum_{m>0}\left[\ln(1-q^m) + \ln(1-\bar{q}^m)\right]$$

$$= -2\left(\ln\left[\prod_{m=1}^{\infty}(1-q^m)\right] + \ln\left[\prod_{m=1}^{\infty}(1-\bar{q}^m)\right]\right) \quad //$$

$$\eta(q) \equiv q^{\frac{1}{24}}\prod_n(1-q^n)$$

$$= G(0) + s\,G'(0) + \cdots$$

$$\therefore G'(0) = -2\ln|\eta|^2 + \cancel{\frac{1}{12}(\ln(q\bar{q}))} + \cancel{\frac{\pi}{3}\tau_2} - \ln\tau_2$$

$$\Rightarrow Z = \frac{1}{\sqrt{\operatorname{Im}\tau}\,|\eta|^2} \quad //$$

② Free boson compactified on a circle (zero-mode) (target space)

$$S = \frac{1}{2\pi}\int \partial\phi\,\bar\partial\phi$$ & ϕ is compactified on a circle

i.e. $\underline{\phi \equiv \phi + 2\pi R}$

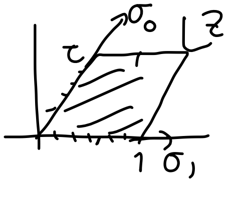
There are many different ϕ which belong to "instanton sectors"

$$\begin{pmatrix} \phi_{cl}(z+\tau, \bar z+\bar\tau) = \phi_{cl}(z,\bar z) + 2\pi R n' \\ \phi_{cl}(z+1, \bar z+1) = \phi_{cl}(z,\bar z) + 2\pi R n \end{pmatrix}$$

① periodicity in z

② $\begin{cases} z \to z+\tau \\ z \to z+1 \end{cases}$

solution ϕ_0 satisfy $\underline{\partial\bar\partial\phi_0 = 0}$ & these BC.



$$\phi_{cl}^{(n',n)}(z,\bar z) = \text{linear in } z, \bar z$$

$$= \frac{2\pi R}{2i\tau_2}\Big(n'\underbrace{(z-\bar z)}_{\text{inv. under ②}} + n\underbrace{(\tau\bar z - \bar\tau z)}_{\text{inv. under ①}}\Big)$$

$$Z = \int \overset{\overset{2\pi R}{\nwarrow} d\varphi_0 [D\tilde\phi]}{\mathcal{D}\phi}\, e^{-\frac{1}{2\pi}\int(\partial\phi)(\bar\partial\phi)} \qquad \phi = \phi_{cl}^{(n',n)} + \tilde\phi \leftarrow \text{no zero-mode}$$

$$= 2\pi R \sum_{n n'} e^{-S[\varphi_{cl}^{(n',n)}]} \cdot \underbrace{\int [D\tilde\varphi]\, e^{-\frac{1}{2\pi}\int(\partial\tilde\phi)(\bar\partial\tilde\phi)}}_{\hookrightarrow \frac{1}{\sqrt{\mathrm{Im}\tau}\,|\eta|^2}}$$

cross term $\left(\int \tilde\varphi \overbrace{\nabla^2 \varphi_{cl}}^{\overset{0}{\shortparallel}} = 0\right)$

$$= 2\pi R \sum_{n,n'} e^{-\frac{\overset{=\tau_2}{A}}{2\pi}\cdot\left[\left(\frac{2\pi R}{2i\tau_2}\right)^2 (n'-n\bar\tau)(-n'+n\tau)\right]} \frac{1}{\sqrt{\mathrm{Im}\tau}\,|\eta|^2}$$

$$= 2\pi R \sum_{n n'} e^{-\frac{R^2}{2}\frac{\pi}{\tau_2}|n'-n\tau|^2} \cdot \frac{1}{\sqrt{\mathrm{Im}\tau}\,|\eta|^2} = \frac{1}{|\eta|^2}\sum_{k,l\in\mathbb{Z}} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} \qquad p_{L,R} = \frac{k}{R} \pm \frac{lR}{2}$$

H.W.

$$|n'-n\tau|^2 = (n'-n\tau_1)^2 + n^2\tau_2^2 = n'^2 - 2n\tau_1 n' + n^2|\tau|^2$$

$$= 2\pi\cancel{R}\frac{\sqrt{2\tau_2}}{\sqrt{\cancel{R^2}}}\sum_n\sum_k e^{-\frac{2\tau_2\pi}{R^2}\underbrace{\left(k+i\frac{\tau_1 n}{\pi}\right)^2}_{k^2 + \frac{2i\tau_1 n}{\pi}k - \frac{\tau_1^2}{\pi^2}n^2} - \left(\frac{\pi R^2}{2\tau_2}|\tau|^2 - \frac{\tau_2\tau_1^2}{\pi R^2}\right)n^2}$$

use Poisson resum formula

$$\boxed{\sum_{n\in\mathbb{Z}} e^{(-\pi a n^2 + bn)} = \frac{1}{\sqrt a}\sum_k e^{-\frac{\pi}{a}\left(k+\frac{b}{2\pi i}\right)^2}}$$

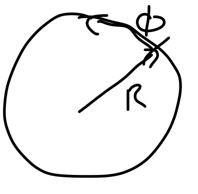
H.W. Prove using

$$\sum_{n\in\mathbb{Z}}\delta(x-n) = \sum_{k\in\mathbb{Z}} e^{2\pi i k x}$$

Modular Invariance : $T: \tau \to \tau+1$ $\frac{p_L^2}{2} - \frac{p_R^2}{2} = kl$ ✓

$S: \tau \to -\frac{1}{\tau}$; use Poisson ...

$\phi \equiv \phi + 2\pi R$



$V_{\alpha\bar\alpha} = e^{i\sqrt{2}(\alpha\varphi + \bar\alpha\bar\varphi)}$

$V_{kl} = e^{i\sqrt{2}(\alpha_{kl}\varphi + \bar\alpha_{kl}\bar\varphi)}$ $\qquad \alpha_{kl}^2 = \frac{p_L^{kl\,2}}{2}, \bar\alpha_{kl}^2 = \frac{p_R^{kl\,2}}{2}$

$\therefore \alpha_{kl} = \frac{1}{\sqrt2}\left(\frac{k}{R} + l\frac{R}{2}\right) \qquad \bar\alpha_{kl} = \frac{1}{\sqrt2}\left(\frac{k}{R} - l\frac{R}{2}\right)$

or $V_{kl} = e^{i\frac{k}{R}\phi + il\frac{R}{2}(\varphi - \bar\varphi)}$

k: momentum mode $\quad$ l: winding mode

$Z\left(\frac{R}{2}\right) = Z\left(\frac{1}{R}\right)$ "duality" $\quad \frac{R}{2} \longleftrightarrow \frac{1}{R}$

mom $\longleftrightarrow$ winding

$R = \sqrt2 \rightarrow$ self dual ; $h_{kl} = \frac14(k\pm l)^2$

String field [Mini-superspace approach]

$\Psi(\phi + 2\pi R, \phi') = \Psi(\phi, \phi') \rightarrow e^{i\frac{k}{R}\phi} f(\phi')$

duality $\quad \Psi\left(\phi, \phi' + \frac{4\pi}{R}\right) = \Psi(\phi, \phi') \rightarrow f(\phi') = e^{i\frac{lR}{2}\phi'}$

$$\varphi(z) = \varphi_0 - i a_0 \ln z + i\sum_{n\neq0}\frac{1}{n} a_n z^{-n}$$

$[\varphi_0, a_0] = i \quad a_0 = \frac{1}{i}\frac{\partial}{\partial\varphi_0}$

$[a_n, a_m] = n\,\delta_{n+m,0}$

$\partial\varphi = -\frac{1}{z}\frac{\partial}{\partial\varphi_0} + \cdots$

$T = -\frac12 :\partial\varphi\partial\varphi: \qquad L_0 = \oint T(z)\, z\, \frac{dz}{2\pi i} = -\frac12\frac{\partial^2}{\partial\varphi_0^2} + \cdots$

$\bar L_0 = \oint \bar T \bar z \frac{d\bar z}{2\pi i} = -\frac12\frac{\partial^2}{\partial\bar\varphi_0^2} + \cdots$ osc.)

$\phi_0 \equiv \varphi_0 + \bar\varphi_0, \quad \phi'_0 \equiv \varphi_0 - \bar\varphi_0$

$\frac{\partial}{\partial\varphi_0} = \frac{\partial}{\partial\phi_0} + \frac{\partial}{\partial\phi'_0}, \frac{\partial}{\partial\bar\varphi_0} = \frac{\partial}{\partial\phi_0} - \frac{\partial}{\partial\phi'_0}), \quad L_0 + \bar L_0 = \mathcal{H} = -\left(\frac{\partial^2}{\partial\phi_0^2} + \frac{\partial^2}{\partial\phi_0'^2}\right) + \cdots$

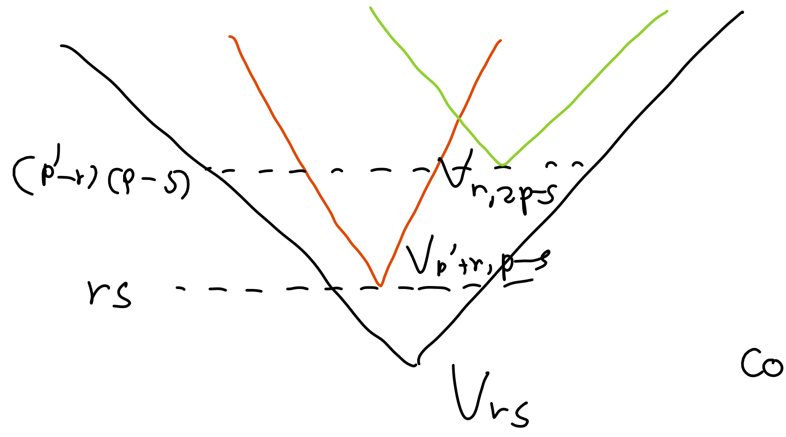
③ Minimal model ($c<1$)

$$c = 1 - 6\,\frac{(p-p')^2}{pp'} \qquad h_{rs} = \frac{(pr-p's)^2-(p-p')^2}{4pp'} \qquad \begin{matrix} 1 \leqq r \leqq p'-1 \\ 1 \leqq s \leqq p-1 \end{matrix}$$

$$h_{rs} + rs = h_{p'+r,p-s} = h_{p'-r,p+s} \overset{\uparrow}{=} h_{r,-s}$$

$$h_{rs} + (p'-r)(p-s) = h_{r\,2p-s} = h_{2p'-r,s}$$

$$h_{r,s} = h_{r+p',s+p}$$



$$\underset{\text{complicated}}{V_{r,s}} \Big/ \Big[\underset{\downarrow}{V_{p'+r,p-s}} \oplus \underset{\downarrow}{V_{r,2p-s}}\Big]$$

their own null states

$$\begin{tikzcd}
\Big[\underbrace{V_{p'+r,p-s}} \oplus \underbrace{V_{r,2p-s}}\Big] \equiv h_{r,-s} \arrow[dr] & \\
V_{2p'+r,s} \cup V_{r,2p+s} \arrow[r, bend right, "="'] & V_{\underset{\underset{2p'+r,\ s}{|||}}{p'-r},\,3p-s} \cup V_{3p'-r,p-s} \arrow[l, bend right, "\equiv"']
\end{tikzcd}$$

$$[r,s] = (r,s) - (p'+r,p-s) \cup (r,2p-s) + (2p'+r,s) \cup (r,2p+s) + \cdots$$

diagram ;

$$\begin{tikzcd}
 & (p'+r,p-s) \arrow[r] \arrow[ddr] & (2p'+r,s) \arrow[r, dashed] \arrow[ddr] & \left(kp'+r,\ \begin{matrix} \overset{k=\text{even}}{s} \\ \overset{k\,\text{odd}}{p-s} \end{matrix}\right)\cdots \\
(r,s) \arrow[ur] \arrow[dr] & \ominus & \oplus & (-1)^k \\
 & (r,2p-s) \arrow[r] \arrow[uur] & (r,2p+s) \arrow[r] \arrow[uur] & (r,(k+\delta_{k\,odd})p-(-1)^k s)\cdots
\end{tikzcd}$$

$$\chi_{(r,s)} = \frac{q^{-\frac{c}{24}}}{\underbrace{\prod_{n=1}^{\infty}(1-q^n)}_{\frac{q^{-(c-1)/24}}{\eta(q)}}}\left[q^{h_{rs}} + \sum_{k=1}^{\infty}(-1)^k\left(q^{h_{r+kp',\,p\delta_{k,odd}+(-1)^k s}} + q^{h_{r,(k+\delta_{k\,odd})p+(-1)^k s}}\right)\right]$$

$$h_{r+kp',\, p\delta_{k,odd}+(-1)^k s} \qquad h_{rs} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}$$

$$= -\frac{(p-p')^2}{4pp'} + \frac{1}{4pp'}\left(p(r+kp') - p'(p\delta_{kodd}+(-1)^k s)\right)^2$$

$$\overset{|||}{}\left[(k-\delta_{kodd})pp' + pr - p'(-1)^k s\right]^2$$

$$\downarrow$$

$$\left.\begin{array}{ll} k=2n-1 & 2(n-1)pp' + pr + p's \\ k=2n & 2npp' + pr - p's \end{array}\right\} n=1,2,\cdots$$

$$h_{r,(k+\delta_{kodd})p+(-1)^k s}$$

$$= -\frac{(p-p')^2}{4pp'} + \frac{1}{4pp'}\left(pr - p'\left((k+\delta_{kodd})p+(-1)^k s\right)\right)^2$$

$$\left[-(k+\delta_{kodd})pp' + pr - (-1)^k p's\right]^2$$

$$\left.\begin{array}{ll} k=2n-1 & -2n\,pp' + pr + p's \\ k=2n & -2npp' + pr - p's \end{array}\right\} \begin{array}{l} n=1,2,\cdots \\ \text{or } (-n)=-1,\cdots \end{array}$$

$n=0 \rightarrow (pr-p's)^2$ from h_{rs}

$$q^{-\overbrace{(c-1)}^{-6(p-p')^2/pp'}/24}\; q^{-\frac{(p-p')^2}{4pp}} = 1$$

$$\therefore\ \chi_{r,s} = \underbrace{K^{(pp')}_{r,s}}_{k=even} - \underbrace{K^{(p,p')}_{r,-s}}_{k=odd}$$

$$K^{(p,p')}_{rs}(q) = \frac{1}{\eta}\sum_{n=-\infty}^{\infty} q^{(2pp'n+pr-p's)^2/4pp'}$$

Modular transform

$$K^{(p,p')}_{rs}(q) = \frac{1}{\eta}\sum_{n=-\infty}^{\infty} q^{(\underset{\overset{\uparrow}{2pp'}}{N}n+\lambda_{rs})^2/2N} \qquad \lambda_{rs} \equiv pr - p's$$

$= \text{integer}$

$N = \text{even integer}$

① $T : \tau \to \tau+1$; $K_\lambda(\tau+1) = \frac{1}{\eta}\sum_n e^{2\pi i \underbrace{\frac{(Nn+\lambda)^2}{2N}}} \quad q^{(\ )^2/2N}$

$$\frac{N^2n^2}{2N} + n\lambda + \frac{\lambda^2}{2N} \equiv \frac{\lambda^2}{2N} \bmod 1 \qquad = e^{2\pi i\frac{\lambda^2}{2N}} K_\lambda(\tau)$$

$$\frac{\lambda^2_{r,-s}}{2N} - \frac{\lambda^2_{rs}}{2N} = \frac{1}{2\cancel{N}}\cancel{4p}p'\,rs = rs = \text{integer}$$

$= 0 \bmod 1$

$$\therefore \quad \chi^{(p,p')}_{rs}(\tau+1) = e^{2\pi i \underbrace{\left(\frac{\lambda^2_{r,s}}{2N} - \frac{1}{24}\right)}_{\boxed{h_{rs} - \frac{c}{24}}}} \chi^{(p,p')}_{rs}(\tau) = e^{2\pi i\left(h_{rs}-\frac{c}{24}\right)}\chi_{rs}(\tau) \quad //$$

$$\left(\begin{array}{l} \eta(q) \xrightarrow{\tau+1} e^{2\pi i\frac{1}{24}}\eta(q) \\[2ex] h_{r,s} = \dfrac{\lambda^2_{r,s} - (p-p')^2}{4pp'} = \dfrac{\lambda^2_{rs}}{2N} + \dfrac{c-1}{24} \end{array} \right)$$

② $S : \tau \to -\frac{1}{\tau}$: define $(r_0, s_0) \ni | \ pr_0 - p's_0 = 1$

$(p > p')$

let $\omega_0 \equiv pr_0 + p's_0 \bmod N = 2pp'$

$$\omega_0^2 = (pr_0 + p's_0)^2 = \underbrace{(pr_0 - p's_0)}_{1}{}^2 + \underbrace{4pp'}_{2N} r_0 s_0 = 1 \bmod 2N$$

$$\lambda_{r,-s} = pr + p's = (pr_0 - p's_0)(pr + p's) = \omega_0\lambda_{rs} \bmod N$$

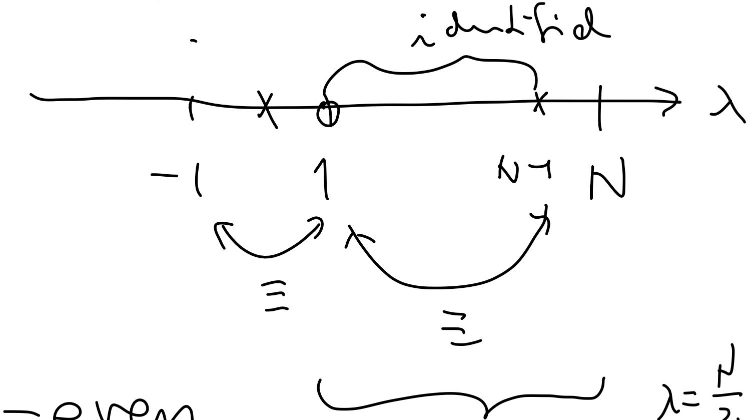
$$= (pr_0 + p's_0)(pr - p's) - \underbrace{2pp'}_{N}(s_0 r - r_0 s)$$

since $q^{(Nn+\overbrace{\lambda+lN})^2/2N} = q^{(Nn+\lambda)^2/2N}$

$$\boxed{\chi_\lambda = K_\lambda - K_{\omega_0\lambda}}$$

also $K_{\lambda+N} = K_\lambda = K_{-\lambda}$

$\therefore$



$\lambda = pr - p's$

$N = 2pp' \equiv$ even.

$\lambda = \frac{N}{2}$

$\therefore \frac{N}{2}+1$ independent K_λ //

$\therefore \chi_\lambda = \chi_{\lambda+N} = \chi_{-\lambda} = -\chi_{\omega_0\lambda}$

$(K_{-\omega_0\lambda} \equiv K_{\omega_0\lambda})$ $(\uparrow\ \omega_0^2\lambda = \lambda)$

$\frac{(p-1)(p'-1)}{2} \in \mathbb{Z}$ ($\because p, p'$ not both even)

$(1 \le r \le p-1,\ 1 \le s \le p'-1 \Rightarrow$ double spectrum$)$

$$K_\lambda(\tau) = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} e^{2\pi i\tau\frac{(Nn+\lambda)^2}{2N}}$$

mod. of η-function

$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$

$$K_\lambda(-\tfrac{1}{\tau}) = \frac{1}{\eta(-\frac{1}{\tau})} \sum_{n\in\mathbb{Z}} e^{-2\pi i\frac{1}{\tau}\frac{(Nn+\lambda)^2}{2N}}$$

$$\boxed{\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{k\in\mathbb{Z}} e^{2\pi i kx}} \rightarrow \boxed{\begin{gathered} f(x) = \sum_{k\in\mathbb{Z}} e^{2\pi i kx}\tilde{f}_k \\ \tilde{f}_k = \int_0^1 f(x)e^{-2\pi i kx}dx \\ f(x) = \sum \delta(x-n) \\ \tilde{f}_k = 1 \end{gathered}}$$

$$= \frac{1}{\eta}\int dx \underbrace{\sum_n \delta(x-n)}_{\sum_{k\in\mathbb{Z}} e^{2\pi i kx}} e^{-\frac{2\pi i}{\tau}\frac{1}{2N}(Nx+\lambda)^2}$$

$$= \frac{1}{\eta(-\frac{1}{\tau})} \sum_k \underbrace{\int dx\, e^{-\frac{2\pi i}{\tau}\left(\frac{N}{2}x^2+\lambda x\right) - \frac{2\pi i}{\tau}\frac{\lambda^2}{2N} + 2\pi i kx}}_{\sqrt{\frac{\tau}{Ni}}\, e^{-\frac{(2\pi)^2(\frac{\lambda}{\tau}-k)^2\tau}{4\pi i N} - \cancel{\frac{2\pi i}{\tau}\frac{\lambda^2}{2N}} + \cancel{\frac{i2\pi\lambda^2}{\tau 2N}} - i2\pi\frac{\lambda k}{-N} + \frac{2i\pi k^2}{2N}\tau}} = \underbrace{\sqrt{\frac{\tau}{Ni}}\cdot\frac{1}{\sqrt{i\tau}\,\eta}}_{\to \frac{1}{\sqrt{N}}\frac{1}{\eta}\,//} \sum_{k\in\mathbb{Z}} e^{\frac{2\pi i k\lambda}{N}} q^{\frac{k^2}{2N}}$$

$$\therefore\ K_\lambda\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{N}} \sum_{k\in\mathbb{Z}} e^{-2\pi i \frac{k\lambda}{N}}\, q^{k^2/2N}$$

let $k = \mu + Nm$ w/ $m\in\mathbb{Z}$, $\mu = 0, \cdots, N-1$

$$= \frac{1}{\sqrt{N}} \sum_{m\in\mathbb{Z}} \sum_{\mu=0}^{N-1} e^{-2\pi i \frac{\mu\lambda}{N}}\, q^{\frac{(\mu+Nm)^2}{2N}}$$

$$K_\lambda\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} \underbrace{e^{-2\pi i \frac{\mu\lambda}{N}}}_{\lambda\to-\lambda \text{ using } K_\lambda = K_{-\lambda}} K_\mu(\tau)$$

$$K_{\omega_0\lambda}\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} \overset{e^{2\pi i \frac{\mu\lambda}{N}}}{e^{2\pi i \frac{\overset{\nu}{\overset{\shortparallel}{\boxed{\mu\omega_0}}}\lambda}{N}}}\ \underset{\underset{\omega_0\nu\ (\omega_0^2\equiv 1)}{\shortparallel}}{K_\mu}$$

$$= \frac{1}{\sqrt{N}} \sum_{\nu=0}^{N-1} e^{2\pi i \frac{\nu\lambda}{N}} K_{\omega_0\nu}$$

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{2\pi i \frac{\mu\lambda}{N}} \chi_\mu(\tau)$$

$\downarrow$

$\exists\, \mu \ni;\ \omega_0\mu = \pm\mu \mod N$

$\Rightarrow \chi_\mu = 0 \quad \Rightarrow \underset{\underset{2pp'-2}{\shortparallel}}{N-2}$ states

for μ which satisfy $\omega_0\mu \not\equiv \pm\mu \mod N$

---

① (r,s) $\boxed{\underbrace{(1\leq r\leq p-1,\ 1\leq s\leq p'-1)}_{2}\ \overset{\nearrow}{(r,s)\equiv(p-r,\ p'-s)}}$ $E_{pp'}$

(half of these) $= \frac{(p-1)(p'-1)}{2}$ states

② $[\omega_0①] \mod N$ $\quad \chi_{\omega_0\mu} = -\chi_\mu \overset{''}{\ } -e^{2\pi i \frac{\omega_0\mu\lambda}{N}} \overset{''}{\ }$

③ $\mu \to N - \mu$

" $e^{-2\pi i \lambda \mu / N}$ "

④ $\mu \to N - \omega_0 \mu \quad \to \quad - e^{-2\pi i \lambda \omega_0 \mu / N}$

let $\mu = p\rho - p'\sigma \to \omega_0 \mu = p\rho + p'\sigma \mod N$

$$\therefore \quad 2\left[\cos \frac{2\pi\lambda(p\rho - p'\sigma)}{N} - \cos \frac{2\pi\lambda(p\rho + p'\sigma)}{N}\right]$$

$$= 4 \sin \frac{2\pi\lambda p\rho}{\underbrace{N}} \sin \frac{2\pi\lambda p'\sigma}{\underbrace{N}}$$

$\lambda = pr - p's \qquad \sin \frac{\pi(pr - p's)\rho}{p'} = (-1)^{s\rho} \sin \frac{\pi p r \rho}{p'}$

$\sin \frac{2\pi\lambda p'\sigma}{2pp'} = \sin \frac{\pi(pr - p's)\sigma}{p} = (-)^{r\sigma + 1} \sin \frac{\pi p' s\sigma}{p}$

☆
$$\boxed{\chi_{r,s}\left(-\frac{1}{\tau}\right) = 2\sqrt{\frac{2}{pp'}} \sum_{\rho,\sigma} (-1)^{1 + r\sigma + s\rho} \sin\left(\pi \frac{p}{p'} r\rho\right) \sin\left(\pi \frac{p'}{p} s\sigma\right) \chi_{\rho\sigma}(\tau)}$$

$$S_{(rs)}^{(\rho\sigma)} = 2\sqrt{\frac{2}{pp'}} (-1)^{1 + r\sigma + s\rho} \sin\left(\pi \frac{p}{p'} r\rho\right) \sin\left(\pi \frac{p'}{p} s\sigma\right)$$

(ex) IM : $p = 4, \ p' = 3$

$$\begin{tikzcd}
(1,1) \arrow[rrd, leftrightarrow] & (2,1) \arrow[d, leftrightarrow] & (3,1) \arrow[lld, leftrightarrow] \\
(1,2) & (2,2) & (3,2)
\end{tikzcd}$$

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \to S^2 = 1 \quad \checkmark$$

Modular Invariant Partition function

so far : holo ⟷ antiholo. "separate"

sewing mechanism : MIPF

$$Z(\tau) = \sum_{\lambda,\bar{\lambda}} N^{\lambda\bar{\lambda}} \chi_{\lambda}(q)\, \bar{\chi}_{\bar{\lambda}}(\bar{q}) \qquad \text{(nonnegative integers)}$$

$\uparrow$ $\quad$ $\nwarrow$ multiplicity of "$\Phi_{\lambda\bar{\lambda}}(z,\bar{z})$"

$$Z(\tau+1) = Z(\tau), \quad Z(-\tfrac{1}{\tau}) = Z(\tau)$$

① T : $\chi_{\lambda} \to \chi_{\lambda} e^{2\pi i (h_{\lambda} - \frac{c}{24})}$, $\bar{\chi}_{\bar{\lambda}} \to \bar{\chi}_{\bar{\lambda}} e^{-2\pi i (h_{\bar{\lambda}} - \frac{c}{24})}$

$$\therefore \quad N_{\lambda\bar{\lambda}}\, e^{2\pi i (h_{\lambda} - h_{\bar{\lambda}})} = N_{\lambda\bar{\lambda}}$$

or $N_{\lambda\bar{\lambda}} = 0$ unless $h_{\lambda} = h_{\bar{\lambda}}$ mod N

(ex) boson. $h_{kl} = \frac{p_L^2}{2}$ $\quad \bar{h}_{kl} = \frac{p_R^2}{2}$ $\quad \to h_{kl} - \bar{h}_{kl} = \frac{1}{\cancel{2}} \cancel{2} kl = \mathbb{Z}$

$$\left(\frac{k}{R} + \frac{lR}{2}\right)^2$$

② $$\chi_{\lambda}(-\tfrac{1}{\tau}) = \sum_{\lambda'} S_{\lambda}^{\lambda'} \chi_{\lambda'}(\tau)$$

$$\bar{\chi}_{\bar{\lambda}}(-\tfrac{1}{\bar{\tau}}) = \sum_{\bar{\lambda}'} S^{*\,\bar{\lambda}'}_{\bar{\lambda}}\, \bar{\chi}_{\bar{\lambda}'}(\bar{\tau})$$

$$\boxed{N^{\lambda\bar{\lambda}} S_{\lambda}^{\lambda'} S^{*\,\bar{\lambda}'}_{\bar{\lambda}} = N^{\lambda'\bar{\lambda}'}}$$

for given S : can solve N for min. CFT & SU(2) WZW (later)

3 different types of N : A-D-E classification

[later]

Verlinde's Formula

$$[\phi_i][\phi_j] = \sum_k N_{ij}{}^k \, [\phi_k]$$ "fusion rule"

(min. model)

$$\phi_{(r_1 s_1)} \times \phi_{(r_2 s_2)} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=\text{odd}}}^{k_{max}} \; \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=\text{odd}}}^{l_{max}} \overset{N}{\downarrow}\!\! \underline{1} \; \phi_{(k,l)}$$

$(r_1+r_2-1) \equiv r$

$$k_{max} = \min(\overset{r}{r_1+r_2-1}, \overset{2p'-2-r}{2p'-1-r_1-r_2})$$

$$l_{max} = \min(s_1+s_2-1, 2p-1-s_1-s_2)$$

$$\checkmark \begin{cases} r < 2p'-2-r \\ \quad \to r < \underline{\underline{p'-1}} \\ \underbrace{2p'-2-r}_{} < r \to r > p'-1 \\ \therefore 2p'-2-r < \underline{\underline{p'-1}} \end{cases}$$

Verlinde

$$\boxed{N_{r_1 s_1, r_2 s_2}{}^{r_3 s_3} = \sum_{rs} \frac{S_{r_1 s_1}^{rs} \, S_{r_2 s_2}^{rs} \, S_{rs}^{r_3 s_3}}{S_{11}^{rs}}}$$

(pf) $N_{ij}{}^k \equiv (N_i)_j^k$ $\quad N_i =$ i-th matrix

above is just eigenvector equation

$$\underbrace{(N_i)\vec{v}^{(k)}}_{} = (\lambda_i^k)\vec{v}^{(k)} \quad \to \quad \sum_k \vec{v}^{(k)\dagger} \vec{v}^{(k)} = \mathbb{1}$$

$\uparrow$ no sum over k $\qquad$ completeness

$$\sum_m (N_i)_l^m \, v_m^{(k)} = (\lambda_i^k) \, v_l^{(k)}$$

multiply both $(\vec{v}^{(k)\dagger})^n$ & $\sum_k$

need to show

$$\left\{ \begin{matrix} \dfrac{S_i^k}{S_1^k} & S_l^k & (S^\dagger)_k^n \\ \uparrow & \uparrow & \uparrow \end{matrix} \right.$$

$$\sum_m (N_i)_l^m \, \delta_m^n = (N_i)_l^n = \sum_k \lambda_i^k \, v_l^{(k)} (\vec{v}^{(k)\dagger})^n$$

$$\overset{\shortparallel}{N_{il}{}^n} = \sum_k \frac{S_i^k \, S_l^k (S^\dagger)_k^n}{S_1^k}$$

$$\phi_i \times \phi_j = N_{ij}{}^{k}\,\phi_k$$

$$\underbrace{\phi_{(2,1)} \times \phi_{(r,s)}}_{} \times \phi_j = \phi_{(2,1)} \times N_{(r,s)j}{}^{k}\,\phi_k$$

$$\parallel \qquad \qquad = \underbrace{N_{(2,1)w}{}^{\ell}\, N_{(r,s)j}{}^{k}}_{}\,\phi_\ell$$

$$\underbrace{\phi_{(r+1,s)} + \phi_{(r-1,s)}}_{\ell} \qquad \qquad (N_{(r,s)} \underset{\underset{N_{(2,1)}N_{(r,s)}}{\parallel}}{N_{(2,1)}})_j{}^{\ell}\,\phi_\ell$$

$$N_{(r+1,s)j} + N_{(r-1,s)j}{}^{\ell}\,\phi_\ell$$

$$\boxed{\text{all } [N_i, N_j] = 0}$$

$$\underset{\underset{X}{\equiv}}{N_{(2,1)}}\, N_{(r,s)} = N_{(r+1,s)} + N_{(r-1,s)}$$

$$\underset{\underset{Y}{\equiv}}{N_{(1,2)}}\, N_{(r,s)} = N_{(r,s+1)} + N_{(r,s-1)}$$

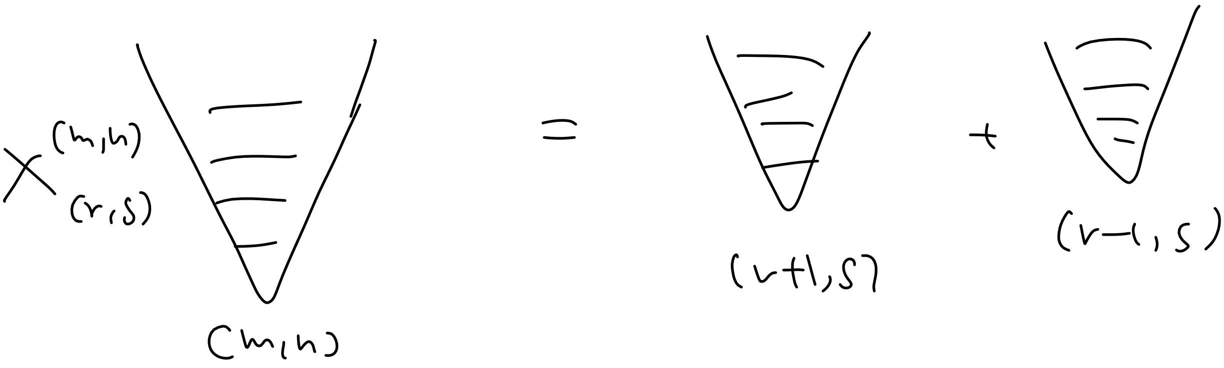
$$N(r,0) = N(0,s) \equiv 0$$

$$N_{(1,1)} = \mathbb{1}$$

$$N_{(2,1)}\,N_{(1,2)} = N_{(2,2)} = XY$$

$$N_{(1,3)} = Y^2 - \mathbb{1}, \quad N_{(3,1)} = X^2 - \mathbb{1} \quad \text{etc} \quad N_{(r,s)} = \overset{\text{polynom}}{P}(X,Y)$$

$$(\underbrace{\phi_{(2,1)} \times \phi_{(r,s)}}_{X^{(m,n)}_{(r,s)}\,\phi_{(m,n)}|0\rangle})\,|0\rangle = [\phi_{(r+1,s)} + \phi_{(r-1,s)}]\,|0\rangle$$



$$\therefore \sum_{(m,n)} X^{(m,n)}_{(r,s)}\,\chi_{(m,n)}(\tau) = \chi_{(r+1,s)}(\tau) + \chi_{(r-1,s)}(\tau)$$

$\tau \to -\frac{1}{\tau}$

$$\therefore \sum_{(m,n)} X^{(m,n)}_{(r,s)} \underbrace{\chi_{(m,n)}\left(-\tfrac{1}{\tau}\right)} = \chi_{(r+1,s)}\left(-\tfrac{1}{\tau}\right) + \chi_{(r-1,s)}\left(-\tfrac{1}{\tau}\right)$$

$$\sum_{(\rho,\sigma)} S^{(\rho,\sigma)}_{(m,n)} \chi_{(\rho,\sigma)}(\tau) = \sum_{\rho,\sigma}\left(S^{(\rho,\sigma)}_{(r+1,s)} + S^{(\rho,\sigma)}_{(r-1,s)}\right)\chi_{(\rho,\sigma)}(\tau)$$

for any τ $\chi_{(\rho,\sigma)}(\tau)$ are all linearly indep.

$$\sum_{(m,n)} X^{(m,n)}_{(r,s)} S^{(\rho,\sigma)}_{(m,n)} = S^{(\rho,\sigma)}_{(r+1,s)} + S^{(\rho,\sigma)}_{(r-1,s)}$$

$$= 2(-1)^{\sigma} \cos\left[\pi \frac{p}{p'} \rho\right] S^{(\rho,\sigma)}_{(r,s)}$$

Similarly

$$\sum_{(m,n)} Y^{(m,n)}_{(r,s)} S^{(\rho,\sigma)}_{(m,n)} = S^{(\rho,\sigma)}_{(r,s+1)} + S^{(\rho,\sigma)}_{(r,s-1)}$$

$$= 2(-1)^{\rho} \cos\left[\pi \frac{p'}{p} \sigma\right] S^{(\rho,\sigma)}_{(r,s)}$$

$$\therefore \left(\vec{S}^{(\rho,\sigma)}\right)_{(m,n)} = S^{(\rho,\sigma)}_{(m,n)}$$

$\left\{\vec{S}^{(\rho,\sigma)}\right\}$: simultaneous eigenvectors of X & Y $\to \therefore$ all $N_{(r,s)}$

$$\therefore N_{(r,s)} \cdot \vec{S}^{(\rho,\sigma)} = n^{(\rho,\sigma)}_{(r,s)} \vec{S}^{(\rho,\sigma)}$$

$$\Rightarrow N_{(r,s)} = \sum_{(\rho,\sigma)} \left(\vec{S}^{(\rho,\sigma)}\right)^{\dagger} n^{(\rho,\sigma)}_{(r,s)} \vec{S}^{(\rho,\sigma)}$$

$$N_{(r,s)}{}^{(mn)}_{(11)} = \sum_{(\rho,\sigma)} \left(\vec{S}^{(\rho,\sigma)}\right)^{\dagger\,(m,n)} n^{(\rho,\sigma)}_{(r,s)} \underbrace{\left(\vec{S}^{(\rho,\sigma)}\right)_{(11)}}$$

$$\shortparallel$$

$$N_{(1,1)(r,s)}^{(m,n)} = \delta^{m}_{r}\delta^{n}_{s} = \sum_{(\rho,\sigma)} S^{(m,n)}_{(\rho,\sigma)} n^{(\rho,\sigma)}_{(r,s)} S^{(\rho,\sigma)}_{(1,1)}$$

$$\delta^m_r \delta^n_s = \sum_{(\rho,\sigma)} S^{(m,n)}_{(\rho,\sigma)} n^{(\rho,\sigma)}_{(r,s)} S^{(\rho,\sigma)}_{(1,1)} \qquad S^2 = 1$$

both sides

$$\sum_{(m,n)} S^{(r',s')}_{(m,n)} \Rightarrow S^{(r',s')}_{(r,s)} = \sum_{(\rho,\sigma)} \delta^{r'}_{\rho} \delta^{s'}_{\sigma} n^{(\rho,\sigma)}_{(r,s)} S^{(\rho,\sigma)}_{(1,1)}$$

$$S^{(r',s')}_{(r,s)} = n^{(r',s')}_{(r,s)} S^{(r',s')}_{(1,1)}$$

$$\therefore \; n^{(r',s')}_{(r,s)} = \frac{S^{(r',s')}_{(r,s)}}{S^{(r',s')}_{(1,1)}}$$

Simpler $\quad [N_i, N_j] = 0 \to \{ |\psi_\ell\rangle \} \quad i, j, \ell = 1, \cdots, r$

\# of primary

$$\exists : \; N_i |\psi^{(\ell)}\rangle = \lambda^{(\ell)}_i |\psi^{(\ell)}\rangle \qquad \langle \psi^{(\ell)} | \psi^{(m)} \rangle = \delta_{m\ell}$$

e.value e.vector

$$\Downarrow$$

$$N_i = \sum_\ell \lambda^{(\ell)}_i |\psi^{(\ell)}\rangle \langle \psi^{(\ell)}|$$

$$\underbrace{\langle j | N_i | k \rangle}_{N_{ij}{}^{k}} = \sum_\ell \lambda^{(\ell)}_i \langle j | \psi^{(\ell)} \rangle \langle \psi^{(\ell)} | k \rangle$$

$$N_{ij}{}^{k} = \sum_\ell \lambda^{(\ell)}_i S^{\ell}_j \underbrace{S^{\ell *}_k}_{(S^\dagger)^k_\ell}$$

let $j = 1$

$$N_{i1}{}^{k} = \delta^k_i = \sum_\ell \lambda^{(\ell)}_i S^\ell_1 (S^\dagger)^k_\ell$$

$$\sum_k \delta^k_i S^m_k = S^m_i = \sum_\ell \lambda^{(\ell)}_i S^\ell_1 \underbrace{\sum_k (S^\dagger)^k_\ell S^m_k}_{\delta^m_\ell} = \lambda^{(m)}_i S^m_1$$

$$\therefore \; \boxed{\lambda^{(m)}_i = \frac{S^m_i}{S^m_1}}$$

Verlinde: $N \longrightarrow S$ as eigenvectors.

More on $c=1$

$R=\sqrt{2} \longrightarrow h^{L,R}_{kl} = \frac{1}{4}(k\pm l)^2 \longrightarrow h^L_{11} = h^L_{-1-1} = 1$

$\therefore J^{\pm}(z) = e^{\pm i\sqrt{2}\varphi}$; $\bar{J}^{\pm}(\bar{z}) = e^{\pm i\sqrt{2}\bar{\varphi}}$

$\uparrow$ $(1,0)$ $(0,1)$

single valued ; $(\varphi \equiv \varphi + \sqrt{2}\pi)$

let $J^3 = i\partial\varphi$

$$J^+(z)J^-(w) \sim \frac{1}{(z-w)^2} + \frac{\sqrt{2}}{z-w}\, i\partial_w\varphi$$

$$J^3(z)J^\pm(w) \sim \frac{\sqrt{2}}{z-w} J^\pm(w)$$

defin $J^{1,2} \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$

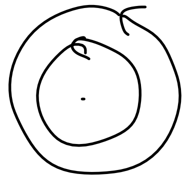
$$\Rightarrow J^i(z)J^j(w) = \frac{\delta^{ij}}{(z-w)^2} + \frac{i\sqrt{2}\,\varepsilon^{ijk}J^k(w)}{(z-w)}$$

---

this is $SU(2)_1$ Kac-Moody algebra [Later]

by defing $J^i(z) = \underline{\underline{\sum_{n\in\mathbb{Z}}}} J^i_n z^{-n-1}$, $J^i_n = \oint \frac{dz}{2\pi i} J^i(z)$

$$[J^i_n, J^j_m] = i\sqrt{2}\,\varepsilon^{ijk}J^k_{n+m} + n\,\delta^{ij}\delta_{n+m,0}$$



(Similarly for antiholo).

$$\alpha_{kl} = \frac{1}{\sqrt{2}}\left(\frac{k}{R} + l\frac{R}{2}\right)$$

$\underline{R=\sqrt{2}}$ $SU(2)\times SU(2)$ (enhanced)

$\underline{R=\sqrt{3}}$ $h=\alpha^2_{02} = \frac{3}{2} \rightarrow N=2$ SUSY $\psi^{\pm}\partial\varphi$

$\frac{1}{2}$ 1 $= \frac{3}{2}$

CFT on other geometries

- Cylinder
- $\frac{1}{2}$-plane $\Rightarrow$ strip

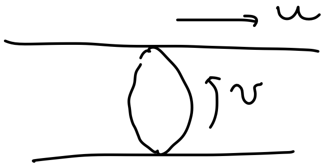
① Cylinder: $w = \frac{L}{2\pi}\ln z$ or $z = e^{\frac{2\pi}{L}w}$ $\left(\begin{matrix} iw \equiv x-t \\ z \equiv e^{-2\pi i(\frac{x-t}{L})} \end{matrix} \right)$

$$F_L = f_0 L - \frac{\pi c}{6L}$$

$$\langle \phi(w_1)\phi(w_2)\rangle = \left(\frac{dw}{dz}\right)^{-h}_{w_1}\left(\frac{dw}{dh}\right)^{-h}_{w_2}\langle\phi(z_1)\phi(z_2)\rangle$$

$$= \left(\frac{L}{2\pi}\right)^{-2h} e^{-\frac{2\pi}{L}(w_1+w_2)h} \frac{1}{\left(e^{\frac{2\pi}{L}w_1} - e^{\frac{2\pi}{2}w_2}\right)^{2h}} = \left(\frac{2\pi}{L}\right)^{2h}\frac{1}{\left(2\sinh\left[\frac{\pi(w_1-w_2)}{L}\right]\right)^{2h}}$$

let $w = w_1 - w_2 = u + iv$



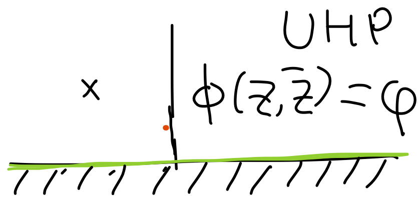
both holo + antiholo with $h = \bar{h} = \frac{\Delta}{2}$

$$\langle\phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle = \left(\frac{2\pi}{L}\right)^{2\Delta}\Big[\underbrace{4\,\mathrm{sh}\frac{\pi w}{L}\,\mathrm{sh}\frac{\pi\bar{w}}{L}}\Big]^{-\Delta}$$

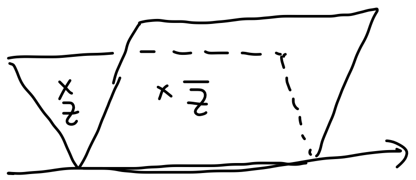
$$= \quad '' \quad \cdot \left[2\cosh\frac{2\pi u}{L} - 2\cos\frac{2\pi v}{L}\right]^{-\Delta}$$

if $u \gg L$ $\quad \langle\phi\phi\rangle \cong \left(\frac{2\pi}{L}\right)^{2\Delta} e^{-\frac{2\pi\Delta}{L}u}$ $\quad (u \gg L)$

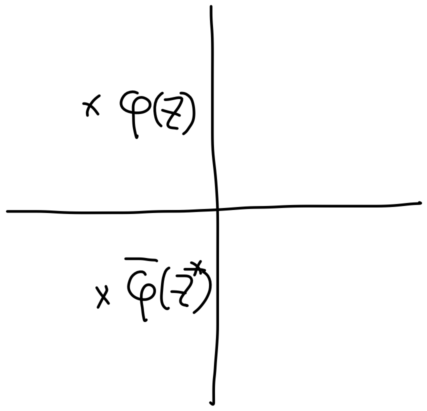
② Boundary CFT & Conformal BCs



$\phi(z,\bar{z}) = \varphi(z)\bar{\varphi}(\bar{z})$



$\Rightarrow$ unfolding



$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\,\epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\,\bar{\epsilon}(\bar{z})\langle\bar{T}X\rangle$$

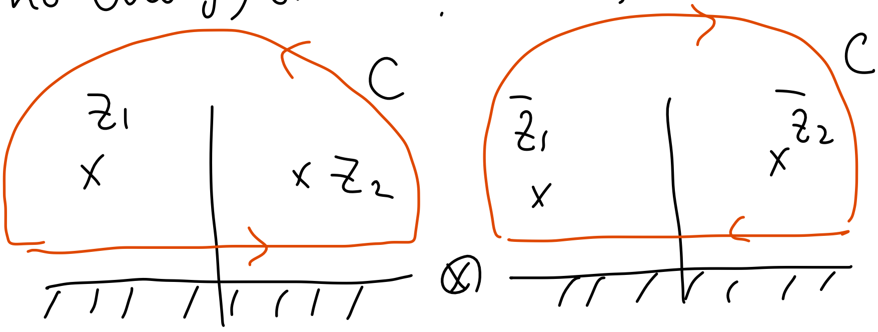
$\varphi_h(z)\bar{\varphi}_{\bar{h}}(\bar{z}) \longrightarrow \varphi_h(z)\varphi_{\bar{h}}(z^*)$

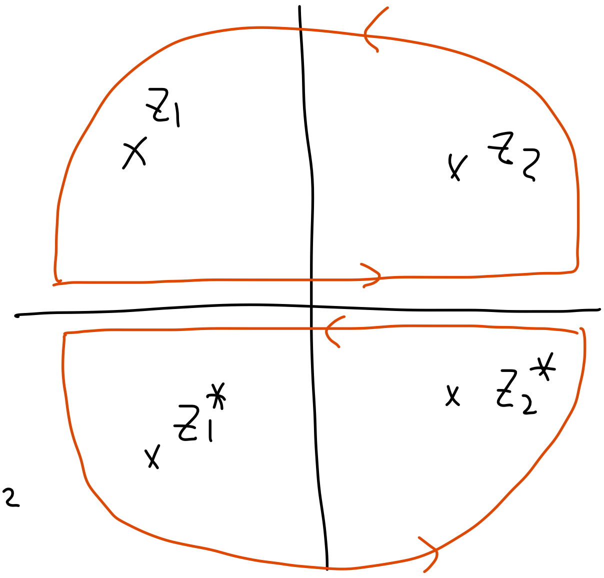
$z \in UHP, \bar{z} \in UHP'$ $\qquad z \in UHP, z^* \in LHP$

E-M tensor : $\bar{T}(\bar{z}) = T(z^*)$

$\Rightarrow$ on real axis ; $T = \bar{T} \rightarrow T_{xy} = 0$ on the axis

no energy or momentum flow across real axis





n-pt

$$\langle \phi_{h_1\bar{h}_1}(z_1,\bar{z}_1) \cdots \phi_{h_n\bar{h}_n}(z_n,\bar{z}_n)\rangle_{UHP}$$

$$\parallel$$

2n-pt

$$\langle \varphi_{h_1}(z_1)\varphi_{\bar{h}_1}(z_1^*) \cdots \varphi_{h_n}(z_n)\varphi_{\bar{h}_n}(z_n^*)\rangle_{\mathbb{C}^2}$$

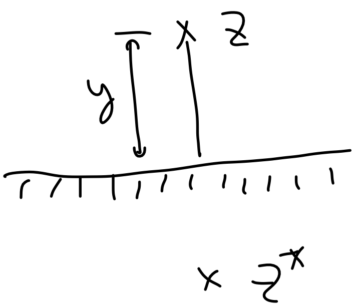
Ward Id.

$$\delta_\varepsilon \langle X \rangle = -\frac{1}{2\pi i}\oint_C dz\, \epsilon(z) \langle T(z) \underbrace{X'}_{\varphi_{h_1}(z_1)\bar{\varphi}_{\bar{h}_1}(z_1^*)\cdots\cdots\phi_{h_n}(z_n)\bar{\phi}_{h_n}(z_n^*)} \rangle$$

$\downarrow$

$\phi_1(z_1,\bar{z}_1)\cdots\cdots\phi_n(z_n,\bar{z}_n)$

(ex)① One-pt function



$$\langle \phi_{h\bar{h}}(z,\bar{z})\rangle_{\mathbb{C}^2} = 0 \qquad (h,\bar{h} \neq 0)$$

but $\langle \phi_{h\bar{h}}(z,\bar{z})\rangle_{UHP} \neq 0$

$$= \langle \varphi_h(z)\varphi_{\bar{h}}(z^*)\rangle = \frac{\delta_{h\bar{h}}}{(z-z^*)^{2h}} \sim \frac{1}{y^\Delta}$$

H.W. Coulomb gas on UHP

② two-pt. (IM)

$$\langle \sigma(z_1,\bar{z}_1)\,\sigma(z_2,\bar{z}_2)\rangle_{UHP} = \langle \sigma(z_1)\,\sigma(z_1^*)\,\sigma(z_2)\,\sigma(z_2^*)\rangle$$

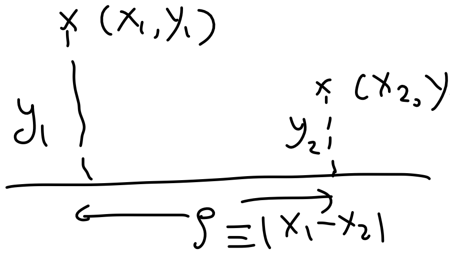
(cf)

$$G^{(4)} = \frac{1}{2}\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{\frac{1}{4}}\left[\,|1+\sqrt{1-x}\,| + |1-\sqrt{1-x}\,|\,\right]$$

$$= \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right)^{\frac{1}{8}}\left\{\alpha\sqrt{1+\sqrt{1-x}} + \beta\sqrt{1-\sqrt{1-x}}\right\}$$

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{(z_1-z_1^*)(z_2-z_2^*)}{(z_1-z_2)(z_1^*-z_2^*)}$$

$$= \left(\frac{|z_1-z_2|^2}{(z_1-z_1^*)(z_1^*-z_2)(z_2-z_2^*)(z_2^*-z_1)}\right)^{\frac{1}{8}}$$



$$x = \frac{2y_1\cdot 2y_2}{\rho^2+(y_1-y_2)^2}$$

$$= \left(\frac{\rho^2+(y_1-y_2)^2}{4y_1y_2\,(\rho^2+(y_1+y_2)^2)}\right)^{\frac{1}{8}}\left(\alpha\sqrt{1+\sqrt{1-x}} + \beta\sqrt{1-\sqrt{1-x}}\right)$$

α, β : determined by B.C.

$y_1, y_2 \to \infty$ $\overset{x\to\infty}{}$ $\langle\quad\rangle_{UHP} \to \langle\quad\rangle_{\mathbb{C}^2}$

with $y_1 - y_2$ = fixed

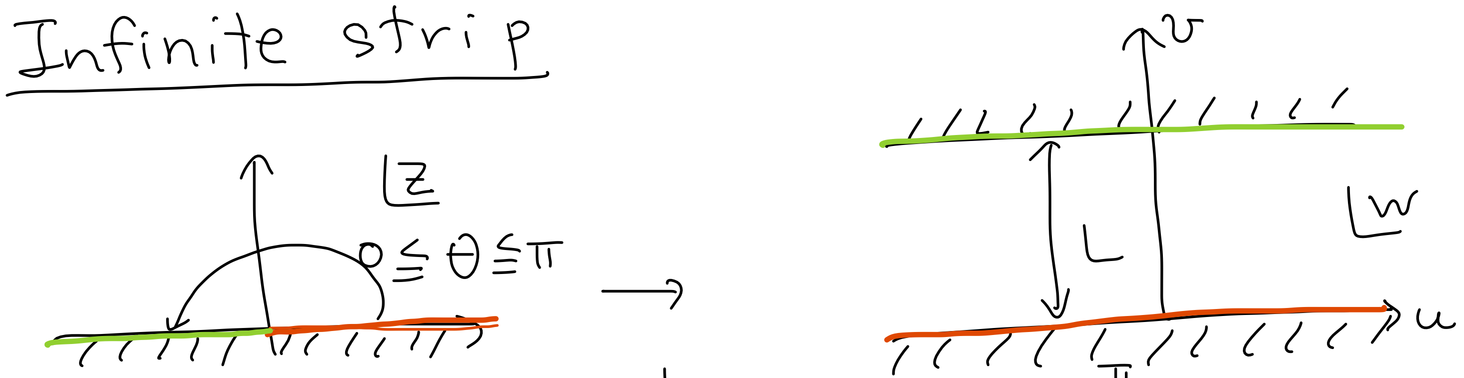
$$\underbrace{\left(\alpha e^{-\frac{i\pi}{4}} + \beta e^{-\frac{i3\pi}{4}}\right)}_{=\,1}\underbrace{\left(\frac{4\cancel{y_1y_2}}{L^2}\right)^{\frac{1}{4}}\left(\frac{L^2}{16\cancel{y_1^2y_2^2}}\right)^{\frac{1}{8}}}_{\frac{1}{L^{\frac{1}{4}}}}$$

$\rho\to\infty$, y_1, y_2 fixed ; $x \simeq \frac{4}{\rho^2}y_1y_2 \to 0$

$$\frac{\alpha\sqrt{2}}{(2y_1)^{\frac{1}{8}}(2y_2)^{\frac{1}{8}}} = \langle\sigma(z_1,\bar{z}_1)\rangle_{UHP}\,\langle\sigma(z_2,\bar{z}_2)\rangle_{UHP} \to \alpha = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \underset{\downarrow\,-\frac{1}{\sqrt{2}}}{\beta}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 1 \;\to\; \beta = -\frac{1}{\sqrt{2}}\,.$$

Infinite strip



$$w = \frac{L}{\pi}\ln z \rightarrow z = e^{\frac{\pi}{L}w}$$

$$\langle \phi(z,\bar z)\rangle_{UHP} = \langle \phi(z)\phi(\bar z)\rangle = \frac{1}{(z-\bar z)^{2h}}$$

$$\therefore \langle \phi(w,\bar w)\rangle_{strip} = \left(\frac{dz}{dw}\right)^h\left(\frac{d\bar z}{d\bar w}\right)^h \langle\phi(z)\phi(\bar z)\rangle$$

$$= \left(\frac{\pi}{L}\right)^{2h}\frac{e^{\frac{\pi h}{L}(w+\bar w)}}{\left(e^{\frac{\pi}{L}w}-e^{\frac{\pi}{L}\bar w}\right)^{2h}}$$

$w = u + iv$

$$= \left(\frac{\pi}{L}\right)^{2h}\frac{e^{\cancel{\frac{2\pi h}{L}u}}}{\cancel{e^{\frac{2\pi h u}{L}}}\left(2i\sin\frac{\pi v}{L}\right)^{\underbrace{2h}_{\Delta}}}$$

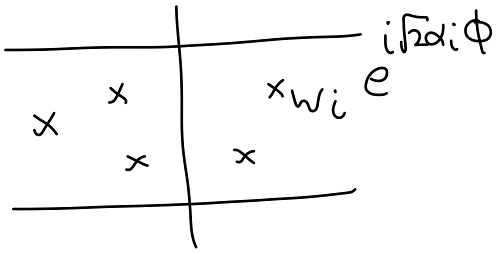
$$= \left(\frac{2iL}{\pi}\right)^{-\Delta}\frac{1}{\left(\sin\frac{\pi v}{L}\right)^{\Delta}}$$

(ex) IM : $\langle\sigma(w_1\bar w_1)\sigma(w_2\bar w_2)\rangle_{strip} = \left(\frac{\pi}{L}\right)^{\frac{1}{4}\leftarrow 4\times\frac{1}{16}}\left(e^{\frac{\pi u_1}{L}}e^{\frac{\pi u_2}{L}}\right)^{2\cdot\frac{1}{16}}$

$$\times \underbrace{\langle\sigma(z_1)\sigma(z_2)\sigma(z_1^*)\sigma(z_2^*)\rangle}_{\text{Computed before}}$$

if $u_2 - u_1 \gg L$ $\quad\sim e^{-\pi(u_2-u_1)/2L}$

H.W. Coulomb gas on strip

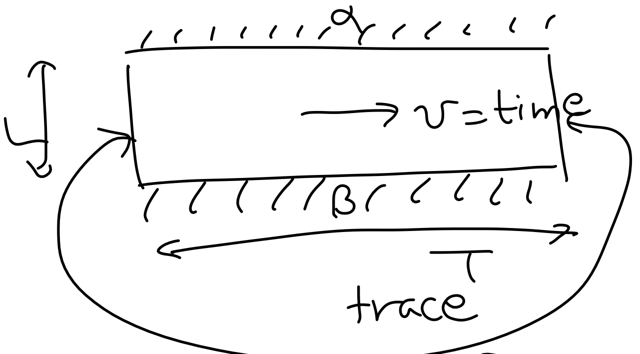


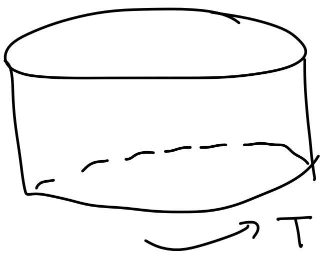
---

free energy on strip (cylinder $L \rightarrow 2L$)

$$F_L = f_0 L - \frac{\pi c}{24L}$$

Partition function on cylinder





define $-\frac{\pi T}{L} \equiv 2\pi i \tau$

$$Z_{\alpha\beta} = \mathrm{Tr}\, e^{-\frac{\pi}{L} H_{\alpha\beta} T}$$

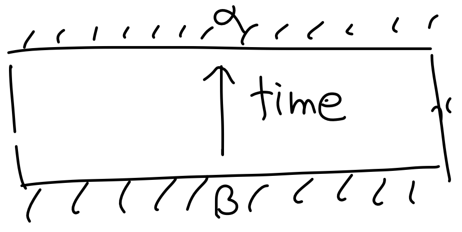
$H_{\alpha\beta}$: Hamiltonian w/ α, β conformal BCs.

$$= \mathrm{Tr}\left[q^{H_{\alpha\beta}} \right]$$

$$= \sum_i n^i_{\alpha\beta}\, \chi_i(q)$$

$n^i_{\alpha\beta} = \#$ of ϕ_i allowed in (α,β)

$$\chi_i(q) = \mathrm{Tr}_i\, q^{L_0 - \frac{c}{24}}$$



transition amplitude

$$\tilde Z_{\alpha\beta} = \langle \alpha | e^{-L \tilde H_{PBC}} | \beta \rangle$$

$\tilde H_{PBC}$ = Hamiltonian of cylinder $= \frac{2\pi}{T}\left(\tilde L_0 + \tilde{\bar L}_0 - \frac{c}{12}\right)$

$$\langle \alpha | e^{-\frac{2\pi L}{T}\left(\tilde L_0 + \tilde{\bar L}_0 - \frac{c}{12}\right)} | \beta \rangle \qquad -\frac{2\pi L}{T} = \frac{2\pi}{2i\tau} = \frac{-2i\pi}{2\tau}$$

$$= \langle \alpha | \left(\tilde q^{\frac{1}{2}}\right)^{\left(\tilde L_0 + \tilde{\bar L}_0 - \frac{c}{12}\right)} | \beta \rangle \qquad \left(\tilde q \equiv e^{-\frac{2\pi i}{\tau}}\right)$$

on boundary : $T_{cyl}(\underset{L}{0}, t) = \bar T_{cyl}(\underset{L}{0}, t)$

$$\downarrow$$

$$T_{pl}(\zeta)\zeta^2 = \bar T_{pl}(\bar\zeta)\bar\zeta^2 \qquad \left(e^{-\frac{2\pi i t}{T}} \equiv \zeta\right)$$

$$\sum L_n \zeta^{-n} \qquad \sum \bar L_n \bar\zeta^{-n} \overset{n\to -n}{=} \sum \bar L_{-n} \zeta^{-n}$$

$\bar\zeta = \zeta^{*n} = \left(\frac{1}{\zeta}\right)^n$

$$\boxed{\therefore \left(L_n - \bar L_{-n}\right)|\beta\rangle = 0}$$

for all n

let $|j\rangle$ satisfy $\underline{(L_n - \bar{L}_{-n})|j\rangle = 0}$

$$\Rightarrow |j\rangle = \sum_N |j;N\rangle \otimes \overline{|j;N\rangle}^*$$

$$|j,N\rangle = \{L_{-n_1}\cdots L_{-n_k}|j;0\rangle,\ \sum n_i = N\}$$

$$\overline{|j,N\rangle} = \{\bar{L}_{-n_1}\cdots \bar{L}_{-n_k}|\bar{j};0\rangle,\ \sum n_i = N\}$$

(Pf)

complete basis $\{|k;N_1\rangle \otimes \overline{|l;N_2\rangle}^*\}$

act $\langle k;N_1| \otimes {}^*\overline{\langle l;N_2|}(L_n - \bar{L}_{-n})\sum_N |j;N\rangle \otimes \overline{|j;N\rangle}^*$

$$= \sum_N \Big(\underbrace{\langle k;N_1|L_n|j;N\rangle}_{\propto \delta_{kj}} \underbrace{\big(\overline{\langle l;N_2|j;N\rangle}^*\big)}_{\delta_{NN_2}\delta_{lj}} - \underbrace{\langle k;N_1|j;N\rangle}_{\delta_{N_1N}\delta_{kj}}\, {}^*\underbrace{\overline{\langle l;N_2|\bar{L}_{-n}|j;N\rangle}}_{\propto \delta_{lj}}\Big)$$

$$= \delta_{kj}\delta_{lj}\Big(\langle j;N_1|L_n|j;N_2\rangle - \underbrace{{}^*\overline{\langle j;N_2|\bar{L}_{-n}|j;N_1\rangle}}_{\overline{\langle j;N_1|\bar{L}_n|j;N_2\rangle}}\Big) \qquad \bar{L}_n^\dagger = \bar{L}_{-n}$$

exactly equal.

$$= 0$$

$$(L_n - \bar{L}_{-n})\sum_N |j;N\rangle \otimes \overline{|j;N\rangle}^* = 0$$

$|\beta\rangle$ should be linear combination of $|j\rangle$'s

$$\tilde{Z}_{\alpha\beta} = \langle\alpha|\overset{\sum_i |i\rangle\langle i|}{\downarrow}\big(\tilde{q}^{\frac{1}{2}}\big)^{(L_0 + \tilde{L}_0 - \frac{c}{12})}\overset{\sum_j |j\rangle\langle j|}{\downarrow}|\beta\rangle$$

$$= \sum_{i,j}\langle\alpha|i\rangle\langle j|\beta\rangle\, \underline{\langle i|\tilde{q}^{\frac{1}{2}(L_0 + \tilde{L}_0 - \frac{c}{12})}|j\rangle}$$

$\uparrow$ $\{|j\rangle\}$ form complete basis for B.C. $|\alpha\rangle$

$$\langle i|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|j\rangle = \delta_{ij}\sum_N \langle j:N|\otimes^*\langle\overline{j:N}|\tilde{q}^{\underbrace{\frac{1}{2}(L_0+\tilde{L}_r-\frac{c}{12})}}|j:N\rangle\otimes\overline{|j,N\rangle}^*$$

$$|j\rangle \;\|\; \sum_N \underbrace{|j,N\rangle}_{\text{descendants}}\otimes\overline{|j,N\rangle}^*$$

$$\sum_N \underset{\uparrow}{d_N}\,\tilde{q}^{\frac{1}{2}(h_j+h_j+N+N-\frac{c}{12})} = \chi_j(\tilde{q})$$

\# of states at level N

$$\therefore\ \tilde{Z}_{\alpha\beta}(\tilde{q}) = \sum_j \langle\alpha|j\rangle\langle j|\beta\rangle\,\chi_j(\tilde{q})$$

$$\|$$

$$Z_{\alpha\beta}(q) = \sum_i n^i_{\alpha\beta}\,\chi_i(q)$$

$q\to\tilde{q}$; S transf : $\chi_i(q) = \sum_j S_{ij}\chi_j(\tilde{q})$

$$\therefore\ \boxed{\sum_i n^i_{\alpha\beta}S_{ij} = \langle\alpha|j\rangle\langle j|\beta\rangle}$$

Cardy Eq.

$\tilde{0}$

$i=0$

$\tilde{0}$

Solution

we define $|\tilde{0}\rangle$ $\Rightarrow$: $n^i_{00} = \delta^i_0 \leftarrow$ identity

$$\therefore\ S_{0j} = \langle\tilde{0}|j\rangle\langle j|\tilde{0}\rangle = |\langle j|\tilde{0}\rangle|^2$$

for unitary model : $S_{0j}>0 \to \langle j|\tilde{0}\rangle = \sqrt{S_{0j}}$

or $|\tilde{0}\rangle = \sum_j \sqrt{S_{0j}}\,|j\rangle$

also consider $\tilde{k} \Rightarrow n^i_{\tilde{0}\tilde{k}} = \delta^i_k \to S_{kj} = \langle\tilde{0}|j\rangle\langle j|\tilde{k}\rangle$

$$\therefore\ \langle j|\tilde{k}\rangle = \frac{S_{kj}}{\sqrt{S_{0j}}} \Rightarrow |\tilde{k}\rangle = \sum_j \frac{S_{kj}}{\sqrt{S_{0j}}}|j\rangle$$

$$\sum_i n^i_{\tilde k \tilde \ell}\, S_{ij} = \underset{\downarrow}{\langle \tilde k | j \rangle} \underbrace{\langle j | \tilde \ell \rangle}_{\frac{S_{\ell j}}{\sqrt{S_{0j}}}} = \frac{S_{kj} S_{\ell j}}{S_{0j}}$$

$$\frac{S_{k \overset{\text{real}}{\downarrow}{j}}}{\sqrt{S_{0j}}}$$

using $S^2 = \mathbb{1}$, act $\sum_j S_{jm}$ both sides

$$\therefore\ n^m_{\tilde k \tilde \ell} = \sum_j \frac{S_{jm} S_{kj} S_{\ell j}}{S_{0j}} = \underset{\uparrow}{N_{k\ell}{}^m}$$

by Verlinde fusion.

$\{ |\tilde k\rangle \}$ conformal Boundary states

$\tilde k$: CBC

$\downarrow$

$\#$ = $\#$ of primaries.

(Ex) IM :

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \end{pmatrix}$$

$\sum_N |0;N\rangle \otimes |0;N\rangle^*$

$\downarrow$

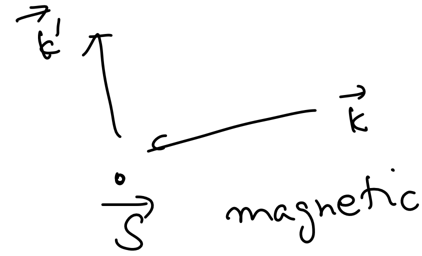
$$|\tilde 0\rangle = \frac{1}{\sqrt2}|0\rangle + \frac{1}{\sqrt2}|\varepsilon\rangle + \frac{1}{\sqrt{\sqrt2}}|\sigma\rangle$$

$$|\tilde{\tfrac{1}{2}}\rangle = \frac{1}{\sqrt2}|0\rangle + \frac{1}{\sqrt2}|\varepsilon\rangle - \frac{1}{\sqrt{\sqrt2}}|\sigma\rangle$$

$$|\tilde{\tfrac{1}{16}}\rangle = |0\rangle - |\varepsilon\rangle$$

Kondo Effect (Affleck & Ludwig)

$$H = \sum_{\vec{k},\alpha} \psi^{\dagger\alpha}_{\vec{k}} \psi_{\vec{k}\alpha} \epsilon(\vec{k}) + \lambda \vec{S} \cdot \sum_{\vec{k},\vec{k}'} \psi^{\dagger}_{\vec{k}} \frac{\vec{\sigma}}{2} \psi_{\vec{k}'}$$



Spherical symmetry; $\epsilon(\vec{k}) = \epsilon(k) \approx v_F (k - k_F)$

$$\psi(\vec{k}) = \frac{1}{\sqrt{4\pi}\, k} \underset{\substack{\uparrow \\ l=0 \text{ "s-wave"}}}{\psi_0(k)} + \cdots\cdot$$

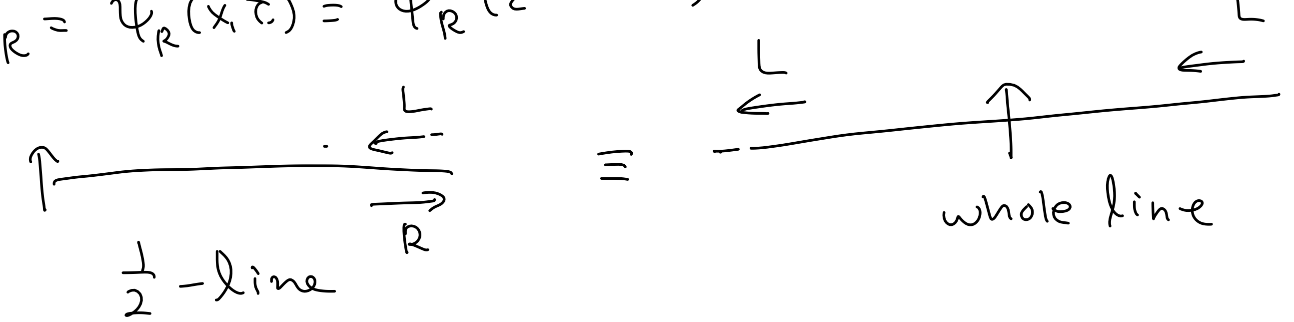
$$H = \underbrace{\int dk\, \psi^{\dagger}_{0k} \psi_{0k} \epsilon(k)}_{\text{excitation near Fermi surface}} + \lambda v_F \int dk dk'\, \psi^{\dagger}_{0k} \frac{\vec{\sigma}}{2} \psi_{0k'} \cdot \vec{S}$$

$$\underset{\text{F.T. back to real space}}{=} \frac{v_F}{2\pi} \underbrace{\int_0^\infty dr \left(\psi^{\dagger}_L i \frac{d}{dr} \psi_L - \psi^{\dagger}_R i \frac{d}{dr} \psi_R \right)} + \lambda v_F \vec{S} \cdot \psi^{\dagger}_L(0) \frac{\vec{\sigma}}{2} \psi_L(0)$$

$$(\psi_L(0) = \psi_R(0))$$

$$\psi_L = \psi_L(x,\tau) = \psi_L(z = \tau + ix) \qquad (v_F \equiv 1)$$

$$\psi_R = \psi_R(x,\tau) = \psi_R(\bar{z} = \tau - ix)$$



$$= \frac{v_F}{2\pi} \int_{-\infty}^{\infty} dx\, \psi^{\alpha\dagger}_L i \frac{d}{dx} \psi_{L\alpha} + \cdots$$

$$\mathcal{H} = \psi^{\alpha\dagger}_L i \frac{d}{dx} \psi_{L\alpha} + \lambda \psi^{\dagger\alpha}_L \frac{\vec{\sigma}^{\beta}_{\alpha}}{2} \psi_{L\beta} \cdot \vec{S}\, \delta(x)$$

spin- change separation

$J = :\psi^{\alpha\dagger} \psi_\alpha:$ U(1) change $\qquad \vec{J} = \psi^{\dagger\alpha} \frac{\vec{\sigma}^{\beta}_{\alpha}}{2} \psi_\beta$ SU(2)

using $\vec{\sigma}^{\beta}_{\alpha} \cdot \vec{\sigma}^{\delta}_{\gamma} = 2 \delta^{\beta}_{\gamma} \delta^{\delta}_{\alpha} - \delta^{\alpha}_{\beta} \delta^{\gamma}_{\delta}$

$$\vec{J}^2 = -\frac{3}{4} :\psi^{\dagger\alpha} \psi_\alpha \psi^{\dagger\beta} \psi_\beta: + \frac{3}{2} i \psi^{\alpha\dagger} \frac{d}{dx} \psi_\alpha$$

$$J^2 = :\psi^{\dagger\alpha} \psi_\alpha \psi^{\dagger\beta} \psi_\beta: + 2i\, \psi^{\alpha\dagger} \frac{d}{dx} \psi_\alpha$$

$$\mathcal{H} = \underbrace{\frac{1}{8\pi} J^2}_{\text{decouples}} + \frac{1}{6\pi}\vec{J}^2 + \lambda \vec{J}\cdot\vec{S}\,\delta(x)$$

↳ $\vec{J}(z)$ satisfy

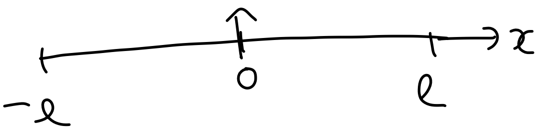
$SU(2)_1$ Kac-Moudy algebra

(later) ↓

$$\vec{J}(x) = \frac{1}{2\pi}\int_{-l}^{l} dx\, e^{\frac{in\pi}{l}x}\vec{J}_n \longrightarrow [J_n^a, J_m^b] = i\epsilon^{abc} J_{n+m}^c + \frac{nk}{2}\delta^{ab}\delta_{n,-m}$$

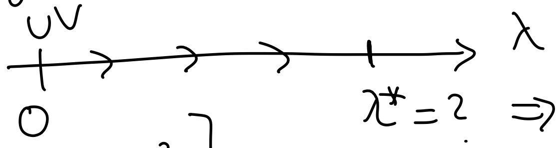
$$\mathcal{H}_S = \frac{1}{6\pi}\vec{J}^2 + \lambda\vec{J}\cdot\vec{S}\;\delta(x)$$

$$H = \int_{-\infty}^{\infty} H_S\,dx = \frac{\pi}{l}\left(\frac{1}{3}\sum_n \vec{J}_{-n}\cdot\vec{J}_n + \lambda\vec{J}_n\cdot\vec{S}\right)$$



RG flow of λ : $\frac{d\lambda}{d\lg\mu} = -\lambda^2 + \cdots < 0$

① nontrivial fixed point



if $\lambda = \frac{2}{3}$ one can square

$\lambda^* = 2 \Rightarrow \frac{2}{3}$ "

⇓

CFT !

$$= \frac{\pi}{3l}\cdot\sum_n\Big[(\vec{J}_{-n}+\vec{S})(\vec{J}_n+\vec{S}) - \underset{\substack{\downarrow\\ \frac{3}{4}\text{ if } S=\frac{1}{2}}}{\vec{S}^2}\Big]$$

still conformal. since $\vec{\mathcal{J}}_n \equiv \vec{J}_n + \vec{S}$ also satisfy Kac-Moudy algebra

② near IR. $\vec{\mathcal{J}}'(x) = \vec{J}(x) + 2\pi\delta(x)\vec{S}$

physical quantities near IR fixed point.

$$\mathcal{H}_S = \frac{1}{6\pi}\left(\vec{\mathcal{J}}'(x) - 2\pi\delta(x)\vec{S}\right)^2 + \lambda\left(\vec{\mathcal{J}} - 2\pi\delta(x)\vec{S}\right)\cdot\vec{S}\delta(x)$$

$$= \frac{1}{6\pi}\left(\vec{\mathcal{J}}(x)\right)^2 + (\vec{S}\text{ dependence disappears})$$

$$+ \underbrace{\lambda_1\vec{\mathcal{J}}(0)^2\delta(x)}_{\substack{\text{leading dim=2 operator}\\ \text{near IR FP.}}}$$

Susceptibility

$$\chi = \frac{1}{3T}\left\langle \int dx\left(\vec{\mathcal{J}}(x)\right)^2\right\rangle_{\lambda_1} = \chi_0 - \frac{\lambda_1}{3T^2}\left\langle\int dx\left(\vec{\mathcal{J}}(x)\right)^2\vec{\mathcal{J}}(0)^2\right\rangle$$

...

Multichannel Kondo effect.

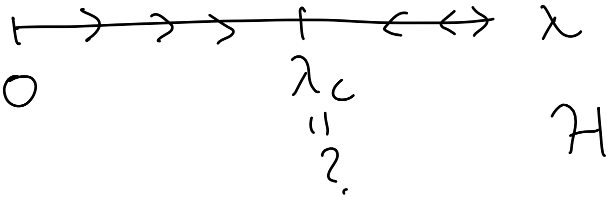
$$\underset{\substack{\uparrow \\ \text{spin}}}{\psi_{\alpha}}\,{}_{i}(\vec{k})$$

$\uparrow$ channel : different d-shell orbitals

$i = 1, \cdots, k$

① overscreened : $\frac{1}{2}\cdot k \left\{ \begin{matrix} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{matrix} \right. \Uparrow S \; = \; \downarrow$ if $\frac{k}{2} > S$

RG flow



$$\mathcal{H}_S = \frac{1}{2\pi(k+2)} \vec{J}^{\,2} + \lambda \vec{J}\cdot\vec{S}\,\delta(x)$$

by same trick ; $\lambda_c = \frac{2}{k+2}$

② Impurity entropy

$$\tilde{q} = e^{-\frac{4\pi l}{\beta}}$$

$$\tilde{Z}_{\alpha\beta}(\tilde{q}) = \sum_j \langle\alpha|j\rangle\langle j|\beta\rangle \underbrace{\chi_j(\tilde{q})}_{\tilde{q}^{-\frac{c}{24}+h_j}(1+q+\cdots)}$$

as $\tilde{q} \to 0$ $\quad \simeq \langle\alpha|0\rangle\langle 0|\beta\rangle\, e^{\frac{\pi l c}{6\beta}} + \cdots$

$$\rightarrow \underset{\substack{\shortparallel \\ \frac{1}{\beta}\log\tilde{Z}}}{F} = -\pi c\, \frac{T^2 l}{6} - T \underbrace{\ln\langle\alpha|0\rangle\langle 0|\beta\rangle}_{\text{impurity entropy}}$$

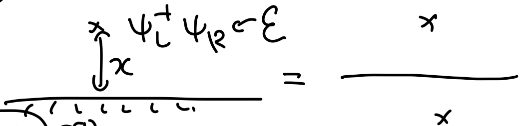
$$\downarrow$$

$$C = \frac{\pi c l}{3} T$$

$$\underset{\substack{\uparrow \\ \text{spin-}S\text{ state}}}{\langle\alpha|0\rangle = \langle S|0\rangle} = \frac{S_S^0}{S_0^0} \leftarrow \text{modular } S\text{-matrix}$$

$$S_j^{j'} = \sqrt{\frac{2}{k+2}}\,\sin\left(\frac{\pi(2j+1)(2j'+1)}{k+2}\right)$$

③ resistivity from boundary 1-pt



$= $ 2-pt on $\mathbb{C}$ $\quad \langle \psi_L^\dagger \psi_R \rangle = \boxed{\frac{S_S^{1/2}/S_0^{\frac{1}{2}}}{S_S^0/S_0^0}}^{S^{(1)}} \cdot \frac{1}{2x}$ & Kubo formula

$$\rho(0) = \left[\cdots\right]\left[\frac{1 - S^{(1)}}{2}\right]$$