Lectures on

Supersymmetric Path Integrals

Pyeong-Chang Summer Institute

3—9 July, 2016

Kazuo Hosomichi

National Taiwan University

- In the field of supersymmetric field theories,
  there has been a big technical breakthrough
  which allows us to compute many SUSY-preserving observables exactly
  using explicit path integrals.

- Key idea – **SUSY localization**

Topics to be covered :

1. SUSY on curved space
2. Three-sphere partition function
3. Squashings
4. Yang-Mills Instantons
5. Four-sphere partition function

I. SUSY on Curved Spaces

We study the example of 3D $\mathcal{N}=2$ theories in detail.

contents:
- 3D N=2 SUSY theories
- SUSY on curved spaces
- Geometry of 3-sphere

Preliminaries

Spinors in 3D

- Gamma matrices for $SO(1,2)$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}$$

- Generators of Lorentz transformation in **spinor representation**

$$\frac{1}{2}\gamma^{ab} \equiv \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$$

- Lorentz transformation on **vectors** and **spinors**:

$$\delta V^a = \omega^a{}_b V^b, \qquad \delta\psi^\alpha = \frac{1}{4}\omega_{ab}(\gamma^{ab})^\alpha{}_\beta \psi^\beta.$$

- $\{\gamma^{01}, \gamma^{02}, \gamma^{12}\}$ span the linear space of 2x2 real traceless matrices, so

$$SO(1,2) \simeq SL(2,\mathbb{R})$$

Spinors in 3D

- Invariant inner products of spinors:

$$\xi\psi \equiv \xi^\alpha C_{\alpha\beta}\psi^\beta, \qquad C \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Proof of Lorentz invariance:

$$\xi'^\alpha = \Lambda^\alpha{}_\beta \xi^\beta, \quad \psi'^\alpha = \Lambda^\alpha{}_\beta \psi^\beta \quad (\Lambda \in SL(2,\mathbb{R}))$$

$$\longrightarrow \xi'\psi' \equiv C_{\alpha\beta}\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta\, \xi^\gamma\psi^\delta = (\det\Lambda) C_{\gamma\delta}\, \xi^\gamma\psi^\delta = \xi\psi.$$

The inner product is usually defined using **Dirac conjugate** $\bar{\psi} \equiv \psi^\dagger\gamma^0$.

Here we used **Majorana conjugate** $\psi_\alpha \equiv \psi^\beta C_{\beta\alpha}$.

The two are equivalent for Majorana spinors.

3D SUSY

- **Simple SUSY**

Fermionic conserved charges Q^α satisfying $(Q^\alpha)^\dagger = Q^\alpha$ and

the anti-commutation relation

$$\{Q^\alpha, Q^\beta\} = -2(\gamma^a C^{-1})^{\alpha\beta} P_a.$$

$(P_0, P_1, P_2) = (-E, P^1, P^2)$: energy-momentum

- **Extended SUSY** $Q^{\alpha I} \quad (I = 1, \cdots, \mathcal{N})$

$$\{Q^{\alpha I}, Q^{\beta J}\} = -2\delta^{IJ}(\gamma^a C^{-1})^{\alpha\beta} P_a.$$

R-symmetry : The anti-commutator is invariant under

$$Q^{\alpha I} \to R^I{}_J Q^{\alpha J}, \quad R \in SO(\mathcal{N})$$

3D N=2 SUSY

For $\mathcal{N}=2,$ the R-symmetry is $SO(2) \simeq U(1).$

One can write the algebra using

$$Q^\alpha \equiv \frac{1}{2}(Q^{\alpha 1} + iQ^{\alpha 2}), \quad \text{U(1) R-charge} \quad (-1)$$

$$\bar{Q}^\alpha \equiv \frac{1}{2}(Q^{\alpha 1} - iQ^{\alpha 2}). \qquad (+1)$$

$$(Q^\alpha)^\dagger = \bar{Q}^\alpha$$

- **N=2 SUSY algebra**

$$\{Q^\alpha, \bar{Q}^\beta\} = -(\gamma^a C^{-1})^{\alpha\beta} P_a, \quad \{Q^\alpha, Q^\beta\} = \{\bar{Q}^\alpha, \bar{Q}^\beta\} = 0.$$

$(\xi Q + \bar{\xi}\bar{Q})^2 = \xi\gamma^a\bar{\xi}\, P_a$ for Grassmann-even spinors $\xi, \bar{\xi}.$

3D N=2 SUSY Theories

we first discuss theories on **Euclidean space** $\mathbb{R}^3$.

Spinor conventions (again)

- Gamma matrices for SO(3)

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \{\gamma^a, \gamma^b\} = 2\delta^{ab}.$$

- Generators of SO(3) in **spinor representation**

$$\frac{1}{2}\gamma^{ab} \equiv \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$$

- $\{i\gamma^{12}, i\gamma^{23}, i\gamma^{13}\}$ span the linear space of 2x2 Hermite traceless matrices, so

$$SO(3) \simeq SU(2)$$

Spinor conventions (again)

- Invariant inner products of spinors:

$$\xi\psi \equiv \xi^\alpha C_{\alpha\beta}\psi^\beta, \qquad C \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- We will write

$$\xi\psi \equiv \xi^\alpha C_{\alpha\beta}\psi^\beta, \quad \xi\gamma^a\psi \equiv \xi^\alpha C_{\alpha\beta}(\gamma^a)^\beta{}_\gamma\psi^\gamma, \quad \cdots$$

- Note: $\quad C_{\alpha\beta} = -C_{\beta\alpha}, \quad (C\gamma^a)_{\alpha\beta} = (C\gamma^a)_{\beta\alpha}.$

  So, for Grassmann odd spinors one has

$$\xi\psi = \psi\xi, \quad \xi\gamma^a\psi = -\psi\gamma^a\xi.$$

Free Wess-Zumino model

Let us consider a simple model,

Fields:

$\phi, \bar{\phi}$: complex scalar

$\psi, \bar{\psi}$: complex spinor

Transformation rules:

$$\delta\phi = \xi\psi, \qquad \delta\psi = i\gamma^m \bar{\xi}\partial_m \phi,$$

$$\delta\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \delta\bar{\psi} = i\gamma^m \xi\partial_m \bar{\phi}.$$

Lagrangian: $\mathcal{L} = \partial_m \bar{\phi}\partial_m \phi - i\bar{\psi}\gamma^m \partial_m \psi.$

- Check the invariance of the Lagrangian.

$$\delta\mathcal{L} = \partial_m(\delta\bar{\phi})\partial_m\phi + \partial_m\bar{\phi}\partial_m(\delta\phi) - i(\delta\bar{\psi})\gamma^m\partial_m\psi - i\bar{\psi}\gamma^m\partial_m(\delta\psi)$$

$$= \partial_m(\bar{\xi}\psi)\partial_m\phi + \partial_m\bar{\phi}\partial_m(\xi\psi) - i(-i\partial_n\bar{\phi}\xi\gamma^n)\gamma^m\partial_m\psi - i\bar{\psi}\gamma^m\partial_m(i\gamma^n\bar{\xi}\partial_n\phi)$$

partial integration:

$$\simeq -\bar{\xi}\psi\cdot\partial^2\phi - \partial^2\bar{\phi}\cdot\xi\psi + \partial_m\partial_n\bar{\phi}\cdot\xi\gamma^n\gamma^m\psi + \bar{\psi}\gamma^m\gamma^n\bar{\xi}\cdot\partial_m\partial_n\phi = 0.$$

Free Wess-Zumino model

$$\delta\phi = \xi\psi, \qquad \delta\psi = i\gamma^m\bar{\xi}\partial_m\phi,$$

$$\delta\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi}.$$

Can we reproduce the **SUSY algebra**

$$\delta^2 \equiv (\xi Q + \bar{\xi}\bar{Q})^2 = \xi\gamma^a\bar{\xi}\, P_a\, ?$$

($\xi, \bar{\xi}$ are Grassmann even here)

- Check:

= zero from EOM

$$\delta^2\phi = \delta(\xi\psi) = i\xi\gamma^m\bar{\xi}\partial_m\phi.$$

$$\delta^2\psi = \delta(i\gamma^m\bar{\xi}\partial_m\phi) = i\gamma^m\bar{\xi}\,(\xi\partial_m\psi) \;= i\partial_m\psi\;(\xi\gamma^m\bar{\xi})\boxed{-\,i\xi\;(\bar{\xi}\gamma^m\partial_m\psi).}$$

Use Fierz identity: $\epsilon\;(\chi\eta) + \chi\;(\eta\epsilon) + \eta\;(\epsilon\chi) = 0.$

$$\gamma^m\bar{\xi}\;(\xi\partial_m\psi) + \xi\;(\partial_m\psi\gamma^m\bar{\xi}) + \partial_m\psi\;(-\bar{\xi}\gamma^m\xi) = 0.$$

The SUSY algebra $\delta^2 = i\xi\gamma^m\bar{\xi}\;\partial_m$ is realized **on-shell.**

Free Wess-Zumino model

The SUSY algebra can be realized **off-shell** by adding auxiliary fields.

$$\boxed{\begin{aligned} &\delta\phi = \xi\psi, \qquad &&\delta\psi = i\gamma^m\bar{\xi}\partial_m\phi + \xi F, \qquad &&\delta F = i\bar{\xi}\gamma^m\partial_m\psi, \\ &\delta\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad &&\delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F}, \qquad &&\delta\bar{F} = i\xi\gamma^m\partial_m\bar{\psi}. \end{aligned}}$$

$\delta^2 = i\xi\gamma^m\bar{\xi}\,\partial_m$ without assuming EOM.

- Free Lagrangian: $\mathcal{L} = \partial_m\bar{\phi}\partial_m\phi - i\bar{\psi}\gamma^m\partial_m\psi + \bar{F}F.$

- **Supermultiplets** = set of fields on which the SUSY is realized irreducibly.

chiral multiplet : (ϕ, ψ, F)

anti-chiral multiplet : $(\bar{\phi}, \bar{\psi}, \bar{F})$

Vector Multiplet

A_m : **gauge field** for the gauge group G

σ, D : **scalars** in the adjoint rep of G

$\lambda, \bar{\lambda}$: **spinors** in the adjoint rep of G

Lie algebra-valued.
(matrices)

- **SUSY transformation rule**

$$\delta A_m = -\frac{i}{2}(\xi\gamma_m\bar{\lambda} + \bar{\xi}\gamma_m\lambda), \qquad \delta\lambda = \frac{1}{2}\gamma^{mn}\xi F_{mn} - \xi D - i\gamma^m\xi D_m\sigma,$$

$$\delta\sigma = \frac{1}{2}(\xi\bar{\lambda} - \bar{\xi}\lambda), \qquad \delta\bar{\lambda} = \frac{1}{2}\gamma^{mn}\bar{\xi}F_{mn} + \bar{\xi}D + i\gamma^m\bar{\xi}D_m\sigma,$$

$$\delta D = \frac{i}{2}\xi(\gamma^m D_m\bar{\lambda} + [\sigma, \bar{\lambda}]) - \frac{i}{2}\bar{\xi}(\gamma^m D_m\lambda - [\sigma, \lambda]).$$

$$F_{mn} \equiv \partial_m A_n - \partial_n A_m - i[A_m, A_n],$$

$$D_m\lambda \equiv \partial_m\lambda - i[A_m, \lambda]$$

Matters Coupled To Gauge Fields

- The chiral multiplet fields can be coupled to gauge fields.

(ϕ, ψ, F) belong to the a rep $\mathbf{R}$ of G.

$$D_m\phi \equiv \partial_m\phi - iA_m\phi,$$

$$D_m\psi \equiv \partial_m\psi - iA_m\psi.$$

$$\boxed{\begin{aligned} \delta\phi &= \xi\psi, \\ \delta\psi &= i\gamma^m\bar{\xi}D_m\phi + i\bar{\xi}\sigma\phi + \xi F, \\ \delta F &= \bar{\xi}(i\gamma^m D_m\psi - i\sigma\psi - i\bar{\lambda}\phi). \end{aligned}}$$

$(\bar{\phi}, \bar{\psi}, \bar{F})$ belong to the a rep $\bar{\mathbf{R}}$ of G.

$$D_m\bar{\phi} \equiv \partial_m\phi + i\bar{\phi}A_m,$$

$$D_m\bar{\psi} \equiv \partial_m\bar{\psi} + i\bar{\psi}A_m.$$

$$\boxed{\begin{aligned} \delta\bar{\phi} &= \bar{\xi}\bar{\psi}, \\ \delta\bar{\psi} &= i\gamma^m\xi D_m\bar{\phi} + i\xi\bar{\phi}\sigma + \bar{\xi}\bar{F}, \\ \delta\bar{F} &= \xi(i\gamma^m D_m\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\lambda). \end{aligned}}$$

U(1) R-charges

We assign to $(\xi, \bar{\xi})$ the R-charges $(+1, -1)$.

Then it follows from $\delta A_m = -\frac{i}{2}(\xi\gamma_m\bar{\lambda} + \bar{\xi}\gamma_m\lambda)$

that $\boxed{\mathbf{R} = 0}$ $\boxed{1}$ $\boxed{-1}$ $\boxed{-1}$ $\boxed{1}$

Vectormultiplet

field	$\mathbf{R}$
A_m	0
σ	0
λ	$+1$
$\bar{\lambda}$	-1
D	0

chiral multiplet

field	$\mathbf{R}$
ϕ	r
ψ	$r-1$
F	$r-2$

anti-chiral multiplet

field	$\mathbf{R}$
$\bar{\phi}$	$-r$
$\bar{\psi}$	$-r+1$
$\bar{F}$	$-r+2$

r : arbitrary

Invariant Lagrangians

- Yang-Mills & Chern-Simons terms

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}F_{mn}F^{mn} + D_m\sigma D^m\sigma + D^2 + i\bar{\lambda}\gamma^m D_m\lambda - i\bar{\lambda}[\sigma,\lambda]\right),$$

$$\mathcal{L}_{\mathrm{CS}} = \frac{ik}{4\pi}\mathrm{Tr}\left[\varepsilon^{mnp}\left(A_m\partial_n A_p - \frac{2i}{3}A_m A_n A_p\right) - (\bar{\lambda}\lambda + 2\sigma D)\right].$$

- Fayet-Iliopoulos term * for U(1) vectormultiplet only

$$\mathcal{L}_{\mathrm{FI}} = -\frac{i\zeta}{\pi}D,$$

- Kinetic terms for chiral multiplets

$$\mathcal{L}_{\mathrm{mat}} = D_m\bar{\phi}D^m\phi + \bar{\phi}\sigma^2\phi - i\bar{\phi}D\phi + \bar{F}F - i\bar{\psi}\gamma^m D_m\psi + i\bar{\psi}\bar{\lambda}\phi - i\bar{\phi}\lambda\psi,$$

- F-term for chiral multiplets (ϕ_i, ψ_i, F_i)

$$\mathcal{L}_{\text{F-term}} = F_i\frac{\partial W}{\partial\phi_i} - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j},$$

W : **superpotential**
= gauge invariant function of $\{\phi_i\}$

Remarks

- **Standard trace** when writing gauge invariants

$$\begin{aligned} \mathrm{Tr}(\cdots) \;&\equiv\; \frac{1}{2h^\vee}\mathrm{Tr}_{(\mathrm{adj})}(\cdots) \\ &= \mathrm{Tr}_{(N\times N)}(\cdots) \qquad \text{for } SU(N), USp(N) \\ &= \frac{1}{2}\mathrm{Tr}_{(N\times N)}(\cdots) \quad \text{for } SO(N) \end{aligned}$$

- Chern-Simons action

$$S = \frac{ik}{4\pi}\int \mathrm{Tr}\left(AdA - \frac{2i}{3}A^3\right)$$

is gauge invariant up to shifts by $2\pi i\mathbb{Z}$ if $k \in \mathbb{Z}$.

3D N=2 SUSY Theories : Summary

- **N=2 SUSY algebra :**

There are 2-component supercharges $Q^\alpha, \bar{Q}^\alpha$ satisfying

$$\{Q^\alpha, \bar{Q}^\beta\} = -(\gamma^a C^{-1})^{\alpha\beta} P_a,$$

- **N=2 SUSY gauge theories :** can be constructed from

Vectormultiplet :	$(A_m, \sigma, \lambda, \bar{\lambda}, D)$	gauge group G	
Chiral multiplet :	(ϕ, ψ, F)	rep. $\mathbf{R}$,	U(1) R-charge r,
Anti-chiral multiplet :	$(\bar{\phi}, \bar{\psi}, \bar{F})$	rep. $\bar{\mathbf{R}}$,	U(1) R-charge $-r$,

Invariant Lagrangian : $\mathcal{L}_{\rm YM},\ \mathcal{L}_{\rm CS},\ \mathcal{L}_{\rm FI},\ \mathcal{L}_{\rm mat},\ \mathcal{L}_{\text{F-term}}.$

Field Theories on Curved Spaces

Field Theories on Curved Spaces

- Metric is curved : $ds^2 = g_{mn}(x)dx^m dx^n$

- There is no preferred choice of coordinates.

  The action should be written in a general covariant way.

$$S = \int d^3x \sqrt{g}\mathcal{L},$$

$$\partial_m \bar{\phi} \partial_m \phi \;\longrightarrow\; g^{mn} \partial_m \bar{\phi} \partial_n \phi,$$

$$\bar{\psi} \gamma^m \partial_m \psi \;\longrightarrow\; \bar{\psi} \gamma^m D_m \psi \equiv \bar{\psi} \gamma^a e_a^m \left(\partial_m + \frac{1}{4} \Omega_m^{ab} \gamma^{ab} \right) \psi.$$

- Need to define spinors on curved spaces.

  Spinors on curved spaces transform under **local Lorentz transformations.**

Vielbein and Local Lorentz Symmetry

- **Vielbein** $e^a_m(x)$ is a matrix-valued field satisfying

$$g_{mn}(x) = \delta_{ab}\, e^a_m(x) e^b_n(x) \qquad *$$

$$ds^2 = g_{mn}(x) dx^m dx^n = \delta_{ab}\, e^a e^b, \quad e^a \equiv e^a_m dx^m.$$

- Given $g_{mn}(x),$ the choice of $e^a_m(x)$ satisfying (*) is not unique.

  Different choices are related to one another by local Lorentz transformations.

$$e'^a_m(x) = \Lambda^a{}_b(x) e^b_m(x), \quad \Lambda^a{}_b(x) \in SO(3)$$

- Infinitesimal local Lorentz transformation :

$$\delta e^a_m(x) = \omega^a{}_b(x) e^b_m(x), \quad \delta\psi(x) = \frac{1}{4}\omega_{ab}(x)\gamma^{ab}\psi(x).$$

Curved and Flat indices

We need to distinguish two kinds of indices

- **curved indices** $m, n, \cdots$

  transform under general coordinate transformations.

- **flat indices** $a, b, \cdots$

  transform under local Lorentz transformations.

Vielbein can convert one index to the other, for example

γ^a := constant matrices satisfying $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$.

γ^m := coordinate-dependent, satisfy $\{\gamma^m, \gamma^n\} = 2g^{mn}$.

$$\gamma^a = e^a_m \gamma^m, \quad \gamma^m = e^m_a \gamma^a.$$

Spin Connection & Covariant Derivative:

- **Covariant derivative** of spinor fields :

$$\boxed{D_m\psi \equiv \partial_m\psi + \frac{1}{4}\Omega_m^{ab}\gamma_{ab}\psi \qquad \Omega_m^{ab} = -\Omega_m^{ba} \ \text{: “spin connection”}}$$

is defined so that $D_m\psi$ transforms the same way as ψ

under local Lorentz transformations.

(Recall that the covariant derivatives of vectors are defined as

$$D_m V^n \equiv \partial_m V^n + \Gamma^n_{mk}V^k, \quad D_m V_n \equiv \partial_m V_n - \Gamma^k_{mn}V_k,$$

so that $D_m V^n,\ D_m V_n$ transform covariantly inder diffeomorphisms.)

Levi-Civita Connection & Spin Connection

- Levi-Civita connection Γ^k_{mn} is determined from $\Gamma^k_{mn} = \Gamma^k_{nm}$ and $D_k g_{mn} = 0$.

$$\Gamma^k_{mn} = \frac{1}{2} g^{kl} (\partial_m g_{nl} + \partial_n g_{ml} - \partial_l g_{mn})$$

[Recall]

$$\begin{aligned} 0 = D_k g_{mn} &= \partial_k g_{mn} - \Gamma^l_{km} g_{ln} - \Gamma^l_{kn} g_{ml} \\ &= \partial_k g_{mn} - \Gamma_{n,km} - \Gamma_{m,nk} \end{aligned}$$

$$\begin{aligned} \partial_k g_{mn} &= \Gamma_{n,km} + \Gamma_{m,nk} \\ \partial_m g_{nk} &= \Gamma_{k,mn} + \Gamma_{n,km} \\ \partial_n g_{km} &= \Gamma_{m,nk} + \Gamma_{k,mn} \end{aligned}$$

Levi-Civita Connection & Spin Connection

- Levi-Civita connection Γ^k_{mn} is determined from $\Gamma^k_{mn} = \Gamma^k_{nm}$ and $D_k g_{mn} = 0.$

$$\Gamma^k_{mn} = \frac{1}{2} g^{kl} (\partial_m g_{nl} + \partial_n g_{ml} - \partial_l g_{mn})$$

- Spin connection Ω^{ab}_m is determined from $D_m e^a_n = 0.$

$$0 = D_m e^a_n \equiv \partial_m e^a_n + \Omega^a{}_{bm} e^b_n - \Gamma^k_{mn} e^a_k$$

$$0 = D_{[m} e^a_{n]} = \partial_{[m} e^a_{n]} + \Omega^a{}_{b[m} e^b_{n]}$$

By introducing the 1-forms $e^a \equiv e^a_m dx^m, \; \Omega^a{}_b \equiv \Omega^a{}_{bm} dx^m$

one can write

$$\boxed{0 = De^a = de^a + \Omega^a{}_b \wedge e^b.}$$

SUSY on Curved Spaces

SUSY on Curved Spaces

- SUSY parameters $\xi, \bar{\xi}$

  are no longer constants, but solutions to **Killing spinor equations.**

- The simplest Killing spinor equation

$$D_m \xi \equiv \left(\partial_m + \frac{1}{4} \Omega_m^{ab} \gamma^{ab} \right) \xi = \gamma_a e_m^a \eta,$$

$$D_m \bar{\xi} \equiv \left(\partial_m + \frac{1}{4} \Omega_m^{ab} \gamma^{ab} \right) \bar{\xi} = \gamma_a e_m^a \bar{\eta} \qquad \text{for some } \eta, \bar{\eta}.$$

  has solutions on the **round sphere.**

An Important Exercise :

Show $\boxed{D_m \xi = \gamma_m \eta \longrightarrow \gamma^m D_m \eta = -\frac{1}{8} R \xi. \quad R \text{ : scalar curvature}}$

[proof]

- $\gamma^{mn} D_m D_n \xi = \gamma^{mn} D_m \gamma_n \eta = 2\gamma^m D_m \eta.$

- $\gamma^{mn} D_m D_n \xi = \frac{1}{2} \gamma^{mn} [D_m, D_n] \xi$

$$= \frac{1}{2} \gamma^{mn} \cdot \frac{1}{4} \gamma^{ab} R^{ab}_{mn} \xi \qquad \left(R^{ab}_{mn} \equiv \partial_m \Omega^{am}_n - \partial_n \Omega^{ab}_m + \Omega^a{}_{cm} \Omega^{cb}_n - \Omega^a_{cn} \Omega^{cb}_m \right)$$

use $\gamma^{ab} \gamma^{cd} = \gamma^{abcd} + (\delta^{bc} \gamma^{ad} - \delta^{bd} \gamma^{ac} - \delta^{ac} \gamma^{bd} + \delta^{ad} \gamma^{bc}) - (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc})$

and Bianchi identity

$$= -\frac{1}{4} R\, \xi. \qquad \left(R \equiv e^m_a e^n_b R^{ab}_{mn} \right)$$

Free WZ model revisited

Q. Is the simple general covariantization

$$\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\partial_n\phi - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F$$

invariant under the following?

$$\boxed{\begin{aligned} &\delta\phi = \xi\psi, \quad && \delta\psi = i\gamma^m\bar{\xi}\partial_m\phi + \xi F, \quad && \delta F = i\bar{\xi}\gamma^m D_m\psi, \\ &\delta\bar{\phi} = \bar{\xi}\bar{\psi}, \quad && \delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F}, \quad && \delta\bar{F} = i\xi\gamma^m D_m\bar{\psi}. \end{aligned}}$$

$$\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\partial_n\phi - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F$$

$$\boxed{\begin{aligned} &\delta\phi = \xi\psi, && \delta\psi = i\gamma^m\bar{\xi}\partial_m\phi + \xi F, && \delta F = i\bar{\xi}\gamma^m D_m\psi, \\ &\delta\bar{\phi} = \bar{\xi}\bar{\psi}, && \delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F}, && \delta\bar{F} = i\xi\gamma^m D_m\bar{\psi}. \end{aligned}}$$

$$\begin{aligned}
\delta\mathcal{L} &= g^{mn}\underline{\partial_m}(\delta\bar{\phi})\partial_n\phi - i(\delta\bar{\psi})\gamma^m D_m\psi + (\delta\bar{F})F \\
&\quad + g^{mn}\partial_m\bar{\phi}\underline{\partial_n}(\delta\phi) - i\bar{\psi}\gamma^m D_m(\delta\psi) + \bar{F}(\delta F) \\
&= -(\bar{\xi}\bar{\psi})g^{mn}D_m D_n\phi - i(-i\partial_n\bar{\phi}\xi\gamma^n + \underline{\bar{F}\bar{\xi}})\gamma^m D_m\psi + \underline{(i\xi\gamma^m D_m\bar{\psi})F} \\
&\quad - g^{mn}D_n D_m\bar{\phi}\,(\xi\psi) - i\bar{\psi}\gamma^m D_m(i\gamma^n\bar{\xi}\partial_n\phi + \underline{\xi F}) + \underline{\bar{F}(i\bar{\xi}\gamma^m D_m\psi)} \\
&= -(\bar{\xi}\bar{\psi})g^{mn}D_m D_n\phi - \partial_n\bar{\phi}\xi\gamma^n\gamma^m \underline{D_m}\psi \\
&\quad - g^{mn}D_n D_m\bar{\phi}\,(\xi\psi) + \bar{\psi}\gamma^m \underline{D_m}(\gamma^n\bar{\xi}\partial_n\phi) \\
&= \cancel{-(\bar{\xi}\bar{\psi})g^{mn}D_m D_n\phi} + \cancel{D_m D_n\bar{\phi}\xi\gamma^n\gamma^m\psi} + \partial_n\bar{\phi}D_m\xi\gamma^n\gamma^m\psi \\
&\quad \cancel{- g^{mn}D_n D_m\bar{\phi}\,(\xi\psi)} + \cancel{\bar{\psi}\gamma^m\gamma^n\bar{\xi}D_m D_n\phi} + \bar{\psi}\gamma^m\gamma^n D_m\bar{\xi}\partial_n\phi \\
&= \partial_n\bar{\phi}\psi\gamma^m\gamma^n \underline{D_m\xi} + \bar{\psi}\gamma^m\gamma^n \underline{D_m\bar{\xi}}\partial_n\phi \\
&= \partial_n\bar{\phi}\psi\gamma^m\gamma^n\gamma_m\eta + \bar{\psi}\gamma^m\gamma^n\gamma_m\bar{\eta}\partial_n\phi \;= -\partial_n\bar{\phi}\psi\gamma^n\eta - \partial_n\phi\bar{\psi}\gamma^n\bar{\eta}
\end{aligned}$$

: partial integration

cancel

$$\boxed{\begin{aligned} D_m\xi &= \gamma_m\eta, \\ D_m\bar{\xi} &= \gamma_m\bar{\eta} \end{aligned}}$$

$$\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\partial_n\phi - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F$$

$$\boxed{\begin{array}{lll} \delta\phi = \xi\psi, & \delta\psi = i\gamma^m\bar{\xi}\partial_m\phi + \xi F \underline{+ i\bar{\eta}\phi}, & \delta F = i\bar{\xi}\gamma^m D_m\psi, \\ \delta\bar{\phi} = \bar{\xi}\bar{\psi}, & \delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F} \underline{+ i\eta\bar{\phi}}, & \delta\bar{F} = i\xi\gamma^m D_m\bar{\psi}. \end{array}}$$

$$\delta\mathcal{L} = g^{mn}\partial_m(\delta\bar{\phi})\partial_n\phi - i(\delta\bar{\psi})\gamma^m D_m\psi + (\delta\bar{F})F$$

$$+g^{mn}\partial_m\bar{\phi}\partial_n(\delta\phi) - i\bar{\psi}\gamma^m D_m(\delta\psi) + \bar{F}(\delta F)$$

$$= -\partial_n\bar{\phi}\psi\gamma^n\eta - \partial_n\phi\bar{\psi}\gamma^n\bar{\eta} \; -i(i\eta\bar{\phi})\gamma^m D_m\psi - i\bar{\psi}\gamma^m D_m(i\bar{\eta}\phi)$$

$$= \bar{\phi}\psi\gamma^m D_m\eta + \phi\bar{\psi}\gamma^m D_m\bar{\eta}$$

$$= -\frac{1}{8}R(\bar{\phi}\psi\xi + \phi\bar{\psi}\bar{\xi}) = -\frac{1}{8}R\delta(\bar{\phi}\phi)$$

$$\boxed{\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\partial_n\phi \underline{+ \frac{1}{8}R\bar{\phi}\phi} - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F \quad \text{is invariant.}}$$

WZ model on Curved Space : Summary

- Lagrangian : $$\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\partial_n\phi + \frac{1}{8}R\bar{\phi}\phi - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F$$

- SUSY transformation :

$$\delta\phi = \xi\psi, \qquad \delta\psi = i\gamma^m\bar{\xi}\partial_m\phi + \xi F + \frac{i}{3}\gamma^m D_m\bar{\xi}\,\phi, \qquad \delta F = i\bar{\xi}\gamma^m D_m\psi,$$

$$\delta\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F} + \frac{i}{3}\gamma^m D_m\xi\,\bar{\phi}, \qquad \delta\bar{F} = i\xi\gamma^m D_m\bar{\psi}.$$

where $\xi, \bar{\xi}$ satisfy Killing spinor equation

$$D_m\xi = \gamma_m\eta, \quad D_m\bar{\xi} = \gamma_m\bar{\eta} \quad \text{for some } \eta, \bar{\eta}.$$

Generalization of Killing Spinors

- Let $g_{mn}(x)$: a metric on a 3D space M,

  $V_m(x), U_m(x)$: vector fields on M,

  $M(x)$: a scalar field on M.

- The background $\{g_{mn}, V_m, U_m, M\}$ is **supersymmetric** if

$$D_m\xi \equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - iV_m\right)\xi = iM\gamma_m\xi - iU_m\xi - \frac{1}{2}\varepsilon_{mnp}U^n\gamma^p\xi,$$

$$D_m\bar{\xi} \equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} + iV_m\right)\bar{\xi} = iM\gamma_m\bar{\xi} + iU_m\bar{\xi} + \frac{1}{2}\varepsilon_{mnp}U^n\gamma^p\bar{\xi},$$

  have solutions.

- Let us restrict to the case $\underline{U_m = 0}$ in the following.

- V_m is the gauge field for U(1) R-symmetry.

Generalization of Killing Spinors

The origin of the Killing spinor equation

$\{g_{mn}, V_m, U_m, M\}$ are the fields in 3D $\mathcal{N}=2$ supergravity multiplet.

Supergravity . . . a theory of graviton g_{mn}, gravitino $\psi_m, \bar{\psi}_m$ and other fields

which is invariant under local SUSY ($\xi, \bar{\xi}$ are arbitrary functions)

$$\delta e_m^a = \xi\gamma^a\bar{\psi}_m + \bar{\xi}\gamma^a\xi_m,$$

$$\delta\psi_m = \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - iV_m\right)\xi - iM\gamma_m\xi + iU_m\xi + \frac{1}{2}\varepsilon_{mnp}U^n\gamma^p\xi,$$

$$\delta\bar{\psi}_m = \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} + iV_m\right)\bar{\xi} - iM\gamma_m\bar{\xi} - iU_m\bar{\xi} - \frac{1}{2}\varepsilon_{mnp}U^n\gamma^p\bar{\xi},$$

$$\ldots$$

Killing spinor equation . . . $\delta\psi_m = 0,\ \delta\bar{\psi}_m = 0.$

Rigid SUSY on Curved Backgrounds

Proposal by Festuccia and Seiberg

- an off-shell local SUSY theory of gravity multiplet $\{g_{mn}, V_m, U_m, M; \psi_m, \bar{\psi}_m\}$ coupled to vector and chiral multiplets is known.
- There is a classical configuration $\{g_{mn}, V_m, U_m, M\}$ (and $\psi_m = \bar{\psi}_m = 0$) such that $\delta\psi_m = \delta\bar{\psi}_m = 0$ has solutions $(\xi, \bar{\xi})$.

Then the Lagrangian of the gravity and other multiplets

$$\mathcal{L}(\{g_{mn}, \cdots\}, \{A_m, \cdots\}, \{\phi, \cdots\})$$

- is invariant under the above δ,
- and the above δ does not change the value of gravity multiplet fields.

3D N=2 SUSY on General Curved Spaces

- **Vectormultiplet** :

$$\delta A_m = -\frac{i}{2}(\xi\gamma_m\bar{\lambda} + \bar{\xi}\gamma_m\lambda), \qquad \delta\lambda = \frac{1}{2}\gamma^{mn}\xi F_{mn} - \xi D - i\gamma^m\xi D_m\sigma,$$

$$\delta\sigma = \frac{1}{2}(\xi\bar{\lambda} - \bar{\xi}\lambda), \qquad \delta\bar{\lambda} = \frac{1}{2}\gamma^{mn}\bar{\xi}F_{mn} + \bar{\xi}D + i\gamma^m\bar{\xi}D_m\sigma,$$

$$\delta D = \frac{i}{2}\xi\left(\gamma^m D_m\bar{\lambda} + [\sigma,\bar{\lambda}] + \underline{iM\bar{\lambda}}\right) - \frac{i}{2}\bar{\xi}\left(\gamma^m D_m\lambda - [\sigma,\lambda] + \underline{iM\lambda}\right).$$

where $\quad D_m\lambda \equiv \partial_m\lambda + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\lambda - iV_m\lambda$

3D N=2 SUSY on General Curved Spaces

- **Chiral multiplet of R-charge r :**

$$\delta\phi = \xi\psi, \qquad \delta\psi = i\gamma^m\bar{\xi}D_m\phi + i\bar{\xi}\sigma\phi \underline{- 2rM\bar{\xi}}\phi + \xi F,$$

$$\delta\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \delta\bar{\psi} = i\gamma^m\xi D_m\bar{\phi} + i\xi\bar{\phi}\sigma \underline{- 2rM\xi\bar{\phi}} + \bar{\xi}\bar{F},$$

$$\delta F = \bar{\xi}\left\{i\gamma^m D_m\psi - i\sigma\psi - i\bar{\lambda}\phi + \underline{(2r-1)M\psi}\right\},$$

$$\delta\bar{F} = \xi\left\{i\gamma^m D_m\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\lambda + \underline{(2r-1)M\bar{\psi}}\right\},$$

where

$$D_m\phi \equiv \partial_m\phi - iA_m\phi - irV_m\phi,$$

$$D_m\bar{\phi} \equiv \partial_m\bar{\phi} + i\bar{\phi}A_m + irV_m\bar{\phi}.$$

3D N=2 SUSY on General Curved Spaces

- **Invariant Lagrangians**

$$\mathcal{L}_{\rm YM} = \frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}F_{mn}F^{mn} + D_m\sigma D^m\sigma + D^2 + i\bar{\lambda}\gamma^m D_m\lambda - i\bar{\lambda}[\sigma,\lambda] - M\bar{\lambda}\lambda\right]$$

$$\mathcal{L}_{\rm CS} = \frac{k}{4\pi}\mathrm{Tr}\left[\varepsilon^{mnp}(A_m\partial_n A_p - \frac{2i}{3}A_mA_nA_p) - \bar{\lambda}\lambda - 2\sigma D - 4M\sigma^2\right]$$

$$\mathcal{L}_{\rm FI} = -\frac{i\zeta}{\pi}(D + 4M\sigma)$$

$$\mathcal{L}_{\rm mat} = D_m\bar{\phi}D^m\phi + \bar{\phi}\sigma^2\phi + 4i(r-1)M\bar{\phi}\sigma\phi - 2r(2r-1)M^2\bar{\phi}\phi + \frac{rR}{4}\bar{\phi}\phi - i\bar{\phi}D\phi$$

$$+\bar{F}F - i\bar{\psi}\gamma^m D_m\psi + i\bar{\psi}\sigma\psi - (2r-1)M\bar{\psi}\psi + i\bar{\psi}\bar{\lambda}\phi - i\bar{\phi}\lambda\psi.$$

$$\mathcal{L}_{\text{F-term}} = F_i\frac{\partial W}{\partial\phi_i} - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} + \mathrm{h.c.}$$

W : gauge invariant function of ϕ_i of R-charge 2

SUSY on Curved Spaces : Summary

- SUSY parameters $\xi, \bar{\xi}$ on curved spaces are not constant
  but solutions to Killing spinor equations.

- The general form of Killing spinor equation has origin in supergravity.
  It can depend on the curved metric as well as other fields
  in gravity multiplet.

- The Lagrangians & transformation rules can be generalized
  from flat to curved spaces.

Geometry of 3-Sphere

Geometry of 3-Sphere

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

in $\mathbb{R}^4$ with metric

$$ds^2 = \ell^2(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2)$$

$$x_0 = \cos\theta\cos\varphi,$$
$$x_3 = \cos\theta\sin\varphi,$$
$$x_2 = \sin\theta\cos\chi,$$
$$x_1 = \sin\theta\sin\chi$$

- **symmetry :** $SO(4) \simeq SU(2)_L \times SU(2)_R$
- **metric :** $$ds^2 = \ell^2(\cos^2\theta d\varphi^2 + \sin^2\theta d^2\chi + d\theta^2)$$ $$= e^1e^1 + e^2e^2 + e^3e^3,$$
- **vielbein :** $e^1 = \ell\cos\theta d\varphi, \quad e^2 = \ell\sin\theta d\chi, \quad e^3 = \ell d\theta.$
- **spin connection :** solve $de^a + \Omega^{ab}\wedge e^b = 0.$

$$\Omega^{12} = 0, \quad \Omega^{13} = -\frac{1}{\ell}\sin\theta d\varphi, \quad \Omega^{23} = \frac{1}{\ell}\cos\theta d\chi.$$

3-Sphere and SU(2)

- 3-sphere is the group manifold SU(2)

$$g \equiv \begin{pmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{pmatrix} = \begin{pmatrix} \cos\theta e^{i\varphi} & \sin\theta e^{i\chi} \\ -\sin\theta e^{-i\chi} & \cos\theta e^{-i\varphi} \end{pmatrix} \in SU(2)$$

- **metric** : $ds^2 = \ell^2 dx_a dx_a = \frac{\ell^2}{2}\mathrm{tr}\left(dg^\dagger dg\right) = -\frac{\ell^2}{2}\mathrm{tr}(g^{-1}dg)^2.$

- **Left-invariant 1-forms :** $g^{-1}dg = i\gamma^a\mu^a$

$$\mu^1 = -\sin(\varphi - \chi)\, d\theta + \frac{1}{2}\cos(\varphi - \chi)\sin 2\theta\,(d\varphi + d\chi),$$
$$\mu^2 = +\cos(\varphi - \chi)\, d\theta + \frac{1}{2}\sin(\varphi - \chi)\sin 2\theta\,(d\varphi + d\chi),$$
$$\mu^3 = \frac{1}{2}(d\varphi - d\chi) + \frac{1}{2}\cos 2\theta\,(d\varphi + d\chi).$$

- Convenient choice of **vielbein** : $ds^2 = \ell^2\mu^a\mu^a, \quad e^a \equiv \ell\mu^a.$

3-Sphere and SU(2)

Let us determine the **spin connection** from

$$de^a + \Omega^{ab} \wedge e^b = 0. \quad \left(e^a = \ell\mu^a\right)$$

$$i\gamma^a \mu^a = g^{-1}dg$$

$$i\gamma^a d\mu^a = d(g^{-1}dg) = -g^{-1}dgg^{-1}dg = -(i\gamma^a\mu^a)^2$$

$$= \gamma^{ab}\mu^a \wedge \mu^b = i\varepsilon^{abc}\gamma^c\mu^a \wedge \mu^b$$

$$de^a = \frac{1}{\ell}\varepsilon^{abc}e^b \wedge e^c = -\Omega^{ab} \wedge e^b,$$

$$\Omega^{ab} = \frac{1}{\ell}\varepsilon^{abc}e^c.$$

Killing Spinors

Constant spinors satisfy $d\xi = 0.$

$$D\xi = d\xi + \frac{1}{4}\gamma^{ab}\Omega^{ab}\xi \;= \frac{1}{4}\gamma^{ab}\cdot\frac{1}{\ell}\varepsilon^{abc}e^{c}\xi \;= \frac{i}{2\ell}\gamma^{c}e^{c}\xi$$

$$dx^m D_m\xi = \frac{i}{2\ell}\gamma^{a}(dx^m e_m^{a})\xi$$

$$D_m\xi = \frac{i}{2\ell}\gamma_m\xi$$

The round S^3 of radius ℓ with the b.g. fields $M = \dfrac{1}{2\ell},\ U_m = V_m = 0$

has 2 Killing spinors for both ξ and $\bar{\xi}$.

Killing Vectors

- Define $\mathcal{R}^a \equiv \mathcal{R}^{am}\dfrac{\partial}{\partial x^m}$ by the property $\mathcal{R}^a g = ig\gamma^a$.

$$g \equiv \begin{pmatrix} \cos\theta e^{i\varphi} & \sin\theta e^{i\chi} \\ -\sin\theta e^{-i\chi} & \cos\theta e^{-i\varphi} \end{pmatrix}$$

$$\mathcal{R}^1 = -\sin(\varphi-\chi)\partial_\theta + \cos(\varphi-\chi)\left(\tan\theta\partial_\varphi + \cot\theta\partial_\chi\right),$$

$$\mathcal{R}^2 = +\cos(\varphi-\chi)\partial_\theta + \sin(\varphi-\chi)\left(\tan\theta\partial_\varphi + \cot\theta\partial_\chi\right),$$

$$\mathcal{R}^3 = \partial_\varphi - \partial_\chi.$$

<u>Important properties :</u> 1) $\dfrac{1}{2i}\mathcal{R}^a$ satisfies SU(2) commutation relation.

2) $\mathcal{R}^{am}\mu^b_m = \delta^{ab}$. $\left(\mathcal{R}^{am} \sim \text{inverse vielbein}\right)$

[proof] $\mathcal{R}^{am}\mu^b_m = \mathcal{R}^{am}\cdot(-i/2)\,\mathrm{Tr}[g^{-1}\partial_m g\gamma^b]$

$$= (-i/2)\,\mathrm{Tr}[g^{-1}\cdot ig\gamma^a\cdot\gamma^b] = \delta^{ab}$$

Geometry of 3-Sphere : Summary

- Round sphere of radius ℓ with the background fields

  $M = \dfrac{1}{2\ell},\ U_m = V_m = 0$ is supersymmetric.

- Round sphere is $SO(4) \simeq SU(2) \times SU(2)$-symmetric.

  Metric, vielbein, Laplace operator, Dirac operator, . . . can be expressed

  in terms of $\mu^a, \mathcal{R}^a,$ which transform nicely under the $SU(2)$.

- The path integral for sphere partition function can be explicitly performed

  using SUSY localization.

  We only need the representation theory of $SU(2)$.

II. 3-Sphere Partition Function

We derive an exact formula for general N=2 SUSY theories.

contents:

- SUSY localization
- sphere partition function
- applications

SUSY Localization

SUSY Localization

- We define **partition function** of gauge theories on 3-sphere by **path integrals** over bosonic & fermionic fields.

- We begin by reviewing basic facts and techniques

  - Grassmann integral

  - Gaussian integral

  - Saddle point approximation

Grassmann Variables

- **Fermions are Grassmann numbers**

$\{\eta_1, \cdots, \eta_n\}$ are Grassmann numbers $\longrightarrow$ $\eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \eta_i = 0.$

- **Integrals over a Grassmann number** η

$$\int d\eta = 0, \qquad \int d\eta \, \eta = 1.$$

$$\int d\eta f(\eta) = \int d\eta (f_0 + \eta f_1) = f_1 = \frac{\partial}{\partial \eta} f(\eta)$$

- **Note :** - For Grassmann numbers, integral = differentiation.

  - Integral of total derivative is zero, $\displaystyle \int d\eta \, \frac{\partial}{\partial \eta} F(\eta) = 0.$

Gaussian Integrals

complex bosons

$$\int d^2z \exp\left(-A|z|^2\right) = \frac{\pi}{A},$$

$$\int d^{2n}z \exp\left(-\bar{z}_i A_{ij} z_j\right) = \frac{\pi^n}{\det A}.$$

$A \quad : \quad (n \times n)$ Hermite, positive definite matrix

Fermions

$$\int d\eta d\bar{\eta} \exp\left(A\bar{\eta}\eta\right) = A,$$

$$\int d^n\eta d^n\bar{\eta} \exp\left(\bar{\eta}_i A_{ij} \eta_j\right) = \det A.$$

Gaussian Integrals in Field Theories

- **path integral over free fields**

Bosons

$$\int \mathcal{D}\bar{\phi}\mathcal{D}\phi \exp\left(-\int d^3x\sqrt{g}g^{mn}\partial_m\bar{\phi}\partial_n\phi\right) = \det(-\nabla^2)^{-1},$$

Fermions

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-\int d^3x\sqrt{g}i\bar{\psi}\gamma^m D_m\psi\right) = \det(-i\gamma^m D_m).$$

- The determinants are the product of (infinitely many) eigenvalues.

  To compute them, we need the spectrum (= eigenvalues & eigenfunctions)

  of the operators $-\nabla^2, \ -i\gamma^m D_m$.

Non-Gaussian Integrals

- As a model of interacting field theories, let us consider

$$Z \equiv \int dx \exp\left(-\frac{1}{g^2} S(x)\right) = \int dx \exp\left(-\frac{1}{g^2}\left\{x^2 + t_3 x^3 + t_4 x^4 + \cdots\right\}\right)$$

$x = 0$ is a local minimum of $S(x)$.

Perturbation theory computes Z as power series in $\{g, t_3, t_4, \cdots\}$.

$$Z = g \int dx \exp\left(-\left\{x^2 + g t_3 x^3 + g^2 t_4 x^4 + \cdots\right\}\right)$$

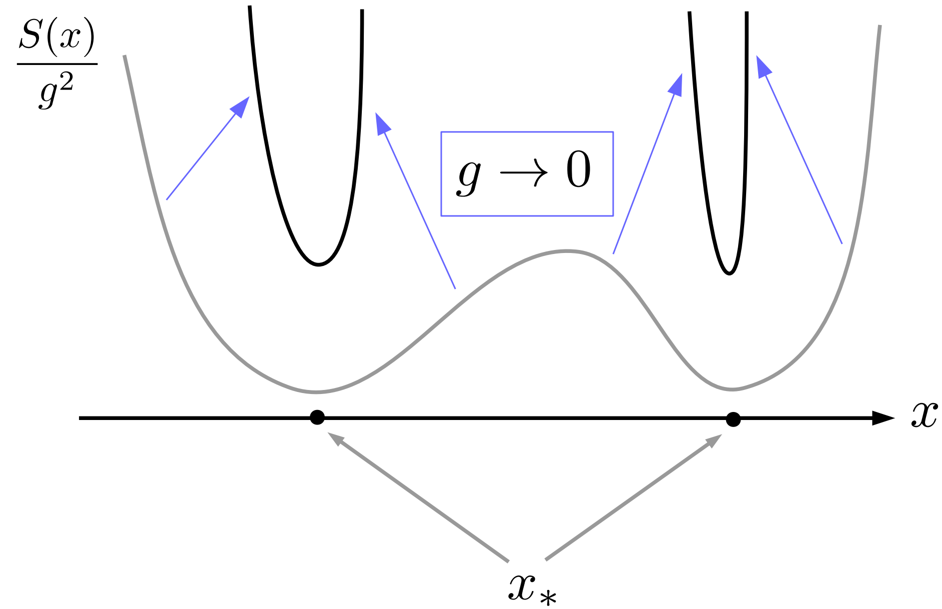
$$= g \int dx \exp(-x^2) \sum_{n_3, n_4, \cdots} \frac{(-g t_3 x^3)^{n_3}}{n_3!} \frac{(-g^2 t_4 x^4)^{n_4}}{n_4!} \cdots .$$

In the weak coupling (g small), the terms of higher order in $\{t_3, t_4, \cdots\}$ becomes negligible.

Saddle Point Approximation

$x = x_*$ is a <u>saddle point</u> if

$$S'(x_*) = 0,\ S''(x_*) > 0.$$



We can approximate Z as

$$Z \equiv \int dx \exp\left(-\frac{1}{g^2}S(x)\right)$$

$$= \sum_{x_*} (\text{perturbation expansion around } x = x_*)$$

$$= \sum_{x_*} e^{-S(x_*)/g^2} \sqrt{\frac{2\pi g^2}{S''(x_*)}} \left\{1 + \cdots (\text{series in } g)\right\}$$

This approximation becomes reliable in the limit $g \to 0$.

SUSY Localization

In SUSY path integrals, one can use $\mathbf{Q}$ (one of the SUSY) to show

The non-zero contribution to the path integral localizes onto

SUSY saddle points

= bosonic field configurations satisfying

$\mathbf{Q}(\mathrm{fermion}) = 0$ for all the fermions.

The idea of SUSY localization

Consider a SUSY integral over bosons $\{\phi_i\}$ and fermions $\{\psi_i\}$.

$$Z \equiv \int \prod_i d\phi_i \prod_j d\psi_j \exp\left(-S(\phi_i, \psi_j)\right),$$

$$\mathbf{Q}S = \mathbf{Q}\left(\prod_i d\phi_i \prod_j d\psi_j\right) = 0.$$

[Transformation rule]

(ϵ: Grassmann parameter)

$$\delta\phi_i = \epsilon\mathbf{Q}\phi_i = \epsilon\cdot(\text{fermionic})_i = \epsilon\cdot F_{ij}(\phi)\psi_j + \cdots,$$

$$\delta\psi_j = \epsilon\mathbf{Q}\psi_j = \epsilon\cdot(\text{bosonic})_j \quad = \epsilon\cdot P_j(\phi) + \cdots.$$

If we can move to a new coordinate system in which

ϵ is one of the fermionic coordinates, then the integral is **zero.**

$$\mathbf{Q}S = \frac{\partial S}{\partial\epsilon} = 0, \qquad Z = \int \left[d\epsilon\cdots\right] \exp\Big(-S(\epsilon\text{-independent})\Big) = 0.$$

One can do this change of variables as long as there are ψ_j such that $P_j(\phi) \neq 0$.

This change of variable is impossible where $P_j(\phi) = 0$ for all j.

The idea of SUSY localization

Consider a SUSY integral over bosons $\{\phi_i\}$ and fermions $\{\psi_i\}$.

$$Z \equiv \int \prod_i d\phi_i \prod_j d\psi_j \exp\left(-S(\phi_i, \psi_j)\right),$$

$$\mathbf{Q}S = \mathbf{Q}\left(\prod_i d\phi_i \prod_j d\psi_j\right) = 0.$$

[Transformation rule]

(ϵ: Grassmann parameter)

$$\delta\phi_i = \epsilon\mathbf{Q}\phi_i = \epsilon \cdot (\mathrm{fermionic})_i = \epsilon \cdot F_{ij}(\phi)\psi_j + \cdots,$$

$$\delta\psi_j = \epsilon\mathbf{Q}\psi_j = \epsilon \cdot (\mathrm{bosonic})_j \quad = \epsilon \cdot P_j(\phi) + \cdots.$$

The integral localizes to **SUSY saddle points,**

$$\mathbf{Q}\psi_j = 0 \quad \text{for all } \psi_j.$$

(which means $P_j(\phi) = 0$ for all j.)

Computing Saddle-Point Contributions

- SUSY localization implies that the saddle point approximation (Gaussian approximation) becomes exact.

- SUSY path integrals can be simplified as follows :

  1. choose a $\mathbf{Q}$-exact "localization term" $\Delta S = \mathbf{Q}\mathcal{V}$ $(\mathbf{Q}^2\mathcal{V} = 0)$ such that

     * its bosonic part has positive definite real part

     * its bosonic part vanishes only at SUSY saddle points.

     standard choice : $\mathcal{V} = \sum_{\Psi:\text{fermions}} (\mathbf{Q}\Psi)^\dagger \Psi.$

$$\mathbf{Q}\mathcal{V} = \sum_{\Psi:\text{fermions}} |(\mathbf{Q}\Psi)|^2 + \cdots .$$

Computing Saddle-Point Contributions

- SUSY localization implies that the saddle point approximation (Gaussian approximation) becomes exact.

- SUSY path integrals can be simplified as follows :

2. deform the path integral by $\Delta S = \mathbf{Q}\mathcal{V}$

$$Z = \int d(\text{measure}) \exp(-S - t\mathbf{Q}\mathcal{V})$$

The integral is independent of *t*, because

$$\frac{\partial Z}{\partial t} = \int d(\text{measure}) \exp(-S - t\mathbf{Q}\mathcal{V}) \cdot (-\mathbf{Q}\mathcal{V})$$

$$= -\int d(\text{measure}) \mathbf{Q}\Big(\exp(-S - t\mathbf{Q}\mathcal{V})\mathcal{V}\Big) = 0$$

* the measure is $\mathbf{Q}$-invariant.

3. Approximate ΔS by quadratic function (exact in the limit $t \to \text{large}$)

An Application:

Sphere Partition Function

Sphere Partition Function

Let us now apply the localization argument to $\mathcal{N}=2$ SUSY theories on S^3.

$$Z = \int \mathcal{D}(\text{fields}) \exp\left(-S\right)$$

$$S = \text{linear combination of } S_{\text{YM}}, S_{\text{CS}}, S_{\text{FI}}, S_{\text{mat}}, S_{\text{F-term}}$$

$(\text{fields}) =$ vector multiplet : $(A_m, \sigma, \lambda, \bar{\lambda}, D)$

chiral multiplet : $(\phi, \psi, F; \bar{\phi}, \bar{\psi}, \bar{F})$

$\mathbf{Q} =$ SUSY transformation

for a specific pair of Killing spinors $\xi, \bar{\xi}$.

Where are the SUSY saddle points?

A Shortcut to Saddle Points

Both of the following Lagrangians are **<u>SUSY exact.</u>**

(for any choice of Killing spinors $\xi, \bar{\xi}$ such that $\bar{\xi}\xi \neq 0$)

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}F_{mn}F^{mn} + D_m\sigma D^m\sigma + D^2 + \cdots\right]$$

$$\mathcal{L}_{\mathrm{mat}} = D_m\bar{\phi}D^m\phi + \bar{\phi}\left(\sigma^2 + 4i(r-1)M\sigma - 2r(2r-1)M^2 + \frac{rR}{4} - iD\right)\phi + \bar{F}F + \cdots .$$

Use $R = 6/\ell^2,\ M = 1/2\ell$ for a sphere of radius ℓ.

$$= D_m\bar{\phi}D^m\phi + \bar{\phi}\left(\sigma^2 + \frac{r(2-r)}{\ell^2} + \frac{2i}{\ell}(r-1)\sigma - iD\right)\phi + \bar{F}F + \cdots .$$

Both of them can be written as $\mathbf{Q}(\mathrm{fermion})$.

Their bosonic parts have to be zero at saddle points.

SUSY saddle points

- **vector multiplet**

$$0 = \frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}F_{mn}F^{mn} + D_m\sigma D^m\sigma + D^2\right] \text{ at saddle points}$$

$$\longrightarrow \quad A_m = 0,\ \sigma = \text{const.},\ D = 0 \quad \text{up to gauge transformations.}$$

* One can assume σ is in the Cartan subalgebra.

For example, for $SU(N)$ gauge group, σ is a $N \times N$ diagonal traceless matrix.

- **chiral multiplet**

$$0 = D_m\bar{\phi}D^m\phi + \bar{\phi}\left(\sigma^2 + \frac{r(2-r)}{\ell^2} + \frac{2i}{\ell}(r-1)\sigma - iD\right)\phi + \bar{F}F \quad \text{at saddle points}$$

$$\longrightarrow \quad \phi = F = 0. \qquad \text{(at least if } 0 < r < 2\text{)}$$

The Path Integral Simplifies

Let $a \equiv (\text{constant value of } \sigma)$ . . . parametrizes the saddle points

$r \equiv \text{rank(gauge group)}$

$$Z_{S^3} = \int d^r a \int \mathcal{D}(\text{others}) \lim_{t \to \infty} \exp\left\{ -t(S_{\text{YM}} + S_{\text{mat}}) - S \right\}$$

$$= \int d^r a \exp\{-S(a)\} \boxed{\cdot Z_{\text{1-loop}}(a).}$$

Here $S(a) =$ (the value of the original action at the saddle point)

$$S_{\text{CS}}(a) = -i\pi k \ell^2 \text{Tr}(a^2), \quad S_{\text{FI}}(a) = -4\pi i \zeta \ell^2 a,$$

$$S_{\text{YM}}(a) = S_{\text{mat}}(a) = S_{\text{F-term}}(a) = 0.$$

Note: Z_{S^3} is independent of YM coupling and F-term couplings.

It remains to compute the **"1-loop determinant"** $Z_{\text{1-loop}}(a)$,

for which the Gaussian approximation is exact.

Exact Formula (Kapustin-Willett-Yaakov '09, Jafferis '10, Hama-KH-Lee '10)

For a theory with

gauge group G,

the j-th chiral multiplet: R-charge r_j, rep R_j,

action S,

the sphere partition function is given by the formula with $b := 1$.

$$Z_{S^3} = \frac{1}{|\mathcal{W}|}\int \prod_{i=1}^{r} da_i \prod_{\boldsymbol{\alpha}\in\Delta_+} 4\sinh(\pi \boldsymbol{a}\cdot\boldsymbol{\alpha} b)\sinh(\pi\boldsymbol{a}\cdot\boldsymbol{\alpha}/b)\cdot\exp\{-S(\boldsymbol{a})\}$$

$$\cdot\prod_j\left[\prod_{\boldsymbol{w}\in R_j} s_b\Big(\frac{i}{2}(b+b^{-1})(1-r_j)-\boldsymbol{a}\cdot\boldsymbol{w}\Big)\right]$$

Double-sine function: $s_b(x) \equiv \prod_{m,n\in\mathbb{Z}_{\geq 0}} \frac{mb+nb^{-1}+\frac{1}{2}(b+b^{-1})-ix}{mb+nb^{-1}+\frac{1}{2}(b+b^{-1})+ix}.$

(A Digression)

Simple Lie Algebras and Their Representations

Simple Lie algebras and their representations

Recall

- Lie algebras in general are characterized by the generators $\{T_a\}$ and commutators $[T_a, T_b] = i c_{ab}{}^{c} T_c$.
- For rank-*r* simple Lie algebras one can find *r* independent commuting generators $\{H_i \ _{(i=1,\cdots,r)}\}$ generating Cartan subalgebra.

Cartan-Weyl basis :

$$\{H_{i\ (i=1,\cdots,r)},\ E_{\boldsymbol{\alpha}\ (\boldsymbol{\alpha}\in\Delta)}\}$$

$[H_i, H_j] = 0,$ $\quad H_i$: Cartan generator

$[H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}},$ $\quad E_{\boldsymbol{\alpha}}$: ladder operator

$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_r) \in \Delta$: “root”

Standard normalization:

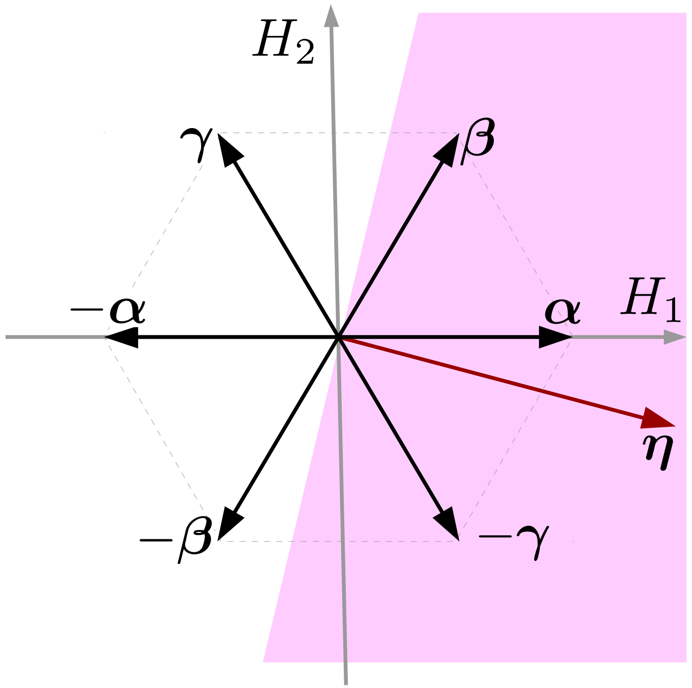
$$\mathrm{Tr}(H_i H_j) = \delta_{ij}, \quad \mathrm{Tr}(E_{\boldsymbol{\alpha}} E_{-\boldsymbol{\alpha}}) = 2/|\boldsymbol{\alpha}|^2,$$

It is easy to derive the following:

$$[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] = N_{\boldsymbol{\alpha},\boldsymbol{\beta}} E_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \quad (\text{if } \boldsymbol{\alpha}+\boldsymbol{\beta}\in\Delta),$$

$$[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \frac{2}{|\boldsymbol{\alpha}|^2}\alpha_i H_i.$$

Positive roots, simple roots & Weyl group



- SU(3) has rank 2 and 6 roots $\Delta = \{\pm\boldsymbol{\alpha}, \pm\boldsymbol{\beta}, \pm\boldsymbol{\gamma}\}$.

- Choose an arbitrary vector $\boldsymbol{\eta}$, and divide the roots according to their inner product with $\boldsymbol{\eta}$.

$$\Delta_+ \text{ (set of positive roots)} = \{\boldsymbol{\beta}, -\boldsymbol{\gamma}, \boldsymbol{\alpha}\}.$$

- All the positive roots can be written as linear sums of *r* simple roots with $\mathbb{Z}_{\geq 0}$ coefficients.

$$B \text{ (set of simple roots)} = \{\boldsymbol{\beta},\, -\boldsymbol{\gamma}\}\,.$$

- Δ_+, B depend on the choice of $\boldsymbol{\eta}$.

  Different choices are related by Weyl reflections.

  Weyl group : $\mathcal{W} = S_3$ for SU(3).

Representations & weights

For any representation of Lie algebra, the basis vectors can be chosen so as to diagonalize H_i.

$$H_i|\boldsymbol{w}\rangle = w_i|\boldsymbol{w}\rangle,$$

$$E_{\boldsymbol{\alpha}}|\boldsymbol{w}\rangle = N_{\boldsymbol{\alpha},\boldsymbol{w}}|\boldsymbol{w}+\boldsymbol{\alpha}\rangle. \qquad \boldsymbol{w} = (w_1,\cdots,w_r) \in R \text{ : “weight”}$$

A Lie algebra representation R is a collection of weights.

. . . Now we know all the symbols in the formula!

$$Z_{S^3} = \frac{1}{|\mathcal{W}|}\int \prod_{i=1}^{r} da_i \prod_{\boldsymbol{\alpha}\in\Delta_+} 4\sinh(\pi\boldsymbol{a}\cdot\boldsymbol{\alpha}b)\sinh(\pi\boldsymbol{a}\cdot\boldsymbol{\alpha}/b)\cdot\exp\{-S(\boldsymbol{a})\}$$

$$\cdot\prod_j\left[\prod_{\boldsymbol{w}\in R_j} s_b\left(\frac{i}{2}(b+b^{-1})(1-r_j)-\boldsymbol{a}\cdot\boldsymbol{w}\right)\right]$$

One-Loop Determinant

Determinant: Chiral Multiplet

We wish to reproduce

$$Z_{\text{1-loop}} = \prod_{\boldsymbol{w}\in R} s_b\left(\frac{i}{2}(b+b^{-1})(1-r) - \boldsymbol{a}\cdot\boldsymbol{w}\right)\Big|_{b=1}$$

from the path integral over chiral multiplet in the rep. R and R-charge r.

- **U(1) case**

Consider a $U(1)$ vectormultiplet fixed at the saddle-point value.

$$\sigma = a \ (\text{const}), \quad A_m = D = 0.$$

The path integral over a chiral multiplet with the $U(1)$ charge +1 should give

$$Z_{\text{1-loop}} = s_b\left(\frac{i}{2}(b+b^{-1})(1-r) - a\right)\Big|_{b=1}$$
$$= \prod_{n\in\mathbb{Z}_+}\left(\frac{n+1-r+ia}{n+r-1-ia}\right)^n$$

Determinant: Chiral Multiplet

We evaluate the Gaussian path integral

$$Z_{\text{1-loop}} = \lim_{t\to\infty} \int \mathcal{D}(\text{chiral}) \exp\left(-tS_{\text{mat}}\right),$$

with the localization Lagrangian:

$$\mathcal{L}_{\text{mat}} = g^{mn}\partial_m\bar{\phi}\partial_n\phi + \bar{\phi}\left(a^2 + \frac{2i}{\ell}(r-1)a + \frac{r(2-r)}{\ell^2}\right)\phi$$
$$-i\bar{\psi}\gamma^m\left(\partial_m + \frac{1}{4}\gamma^{ab}\Omega_m^{ab} - a - i\frac{(2r-1)}{2\ell}\right)\psi + \bar{F}F$$

Note: a plays the role of mass.

The “real mass” for the matters can be introduced

by turning on a background vectormultiplet for flavor symmetries.

Determinant: Chiral Multiplet

- Rewrite the localization Lagrangian using $e_a^m \partial_m \equiv \mathcal{R}^a$.

$$g^{mn}\partial_m\bar{\phi}\partial_n\phi = \frac{1}{\ell^2}\mathcal{R}^a\bar{\phi}\cdot\mathcal{R}^a\phi = \frac{4}{\ell^2}\bar{\phi}\cdot J_{\mathcal{R}}^a J_{\mathcal{R}}^a\phi$$

$$-i\bar{\psi}\gamma^m\left(\partial_m + \frac{1}{4}\gamma^{ab}\Omega_m^{ab}\right)\psi = \frac{1}{\ell}\bar{\psi}\left(-i\gamma^a\mathcal{R}^a + \frac{3}{2}\right)\psi = \frac{1}{\ell}\bar{\psi}\left(4S^a J_{\mathcal{R}}^a + \frac{3}{2}\right)\psi.$$

$J_{\mathcal{R}}^a \equiv \frac{1}{2i}\mathcal{R}^a, \quad S^a \equiv \frac{1}{2}\gamma^a$ ... satisfy SU(2) commutation relation.

- The localization Lagrangian becomes

$$\mathcal{L}_{\mathrm{mat}} = \frac{1}{\ell^2}\bar{\phi}\left\{4\boldsymbol{J}_{\mathcal{R}}^2 + 1 - (1 - r + i\ell a)^2\right\}\phi + \frac{1}{\ell}\bar{\psi}\left\{4\boldsymbol{J}_{\mathcal{R}}\cdot\boldsymbol{S} + 2 - r + i\ell a\right\}\psi + \bar{F}F$$

- We need the spectrum of the operators

$$\boxed{K_\phi \equiv \frac{1}{\ell^2}\left[4\boldsymbol{J}_{\mathcal{R}}^2 + 1 - (1 - r + i\ell a)^2\right],\ K_\psi \equiv \frac{1}{\ell}\left[4\boldsymbol{J}_{\mathcal{R}}\cdot\boldsymbol{S} + 2 - r + i\ell a\right],\ K_F \equiv 1.}$$

Spectrum of kinetic operators

Bosons: $$\boxed{K_\phi \equiv \frac{1}{\ell^2}\left[4\boldsymbol{J}_{\mathcal{R}}^2 + 1 - (1 - r + i\ell a)^2\right], \quad K_F \equiv 1.}$$

- ϕ, F can be expanded into spherical harmonics $Y_{m\tilde{m}}^{n/2} \quad (n \in \mathbb{Z}_{\geq 0})$

  ... eigenfunctions of $\boldsymbol{J}_{\mathcal{L}}^2 = \boldsymbol{J}_{\mathcal{R}}^2 = \frac{n}{2}(\frac{n}{2}+1),\ J_{\mathcal{L}}^3 = m,\ J_{\mathcal{R}}^3 = \tilde{m}.$

$$\begin{aligned} K_\phi &= \frac{1}{\ell^2}\left[4 \cdot \tfrac{n}{2}(\tfrac{n}{2}+1) + 1 - (1 - r + i\ell a)^2\right] \\ &= \frac{1}{\ell^2}(n + 2 - r + i\ell a)(n + r - i\ell a) \end{aligned}$$

$$\det(tK_\phi) \cdot \det(tK_F) = \prod_{n \in \mathbb{Z}_{\geq 0}} \left\{ \frac{t^2}{\ell^2}(n + 2 - r + i\ell a)(n + r - i\ell a) \right\}^{(n+1)^2}$$

Spectrum of kinetic operators

Fermions: $$\boxed{K_\psi \equiv \frac{1}{\ell}\big[4\boldsymbol{J}_{\mathcal{R}}\cdot\boldsymbol{S} + 2 - r + i\ell a\big]}$$

- ψ can be expanded into spinor spherical harmonics $Y^{j,\tilde{j}}_{m,\tilde{m}}$ $(\tilde{j} = j \pm \frac{1}{2})$

$$Y^{j,\tilde{j}}_{m,\tilde{m}} = \begin{pmatrix} (\mathrm{const})\cdot Y^{j}_{m,\tilde{m}-1/2} \\ (\mathrm{const})\cdot Y^{j}_{m,\tilde{m}+1/2} \end{pmatrix}$$

$$\boldsymbol{J}_{\mathcal{L}}^2 = j(j+1), \quad (\boldsymbol{J}_{\mathcal{R}} + \boldsymbol{S})^2 = \tilde{j}(\tilde{j}+1), \quad J^3_{\mathcal{L}} = m, \quad J^3_{\mathcal{R}} + S^3 = \tilde{m}.$$

$$K_\psi = \frac{1}{\ell}\big(n + 2 - r + i\ell a\big) \qquad \text{on} \quad Y^{\frac{n}{2},\frac{n+1}{2}}_{m,\tilde{m}}$$

$$K_\psi = \frac{1}{\ell}\big(-n - 1 - r + i\ell a\big) \quad \text{on} \quad Y^{\frac{n+1}{2},\frac{n}{2}}_{m,\tilde{m}}$$

$$\det(tK_\psi) = \prod_{n\in\mathbb{Z}_{\geq 0}} \left\{\frac{t^2}{\ell^2}(n+2-r+i\ell a)(-n-1-r+i\ell a)\right\}^{(n+1)(n+2)}$$

Determinant: Chiral Multiplet

We found

$$\det(tK_\psi) = \prod_{n\in\mathbb{Z}_{\geq 0}} \left\{ \frac{t^2}{\ell^2}(n+2-r+i\ell a)(-n-1-r+i\ell a) \right\}^{(n+1)(n+2)}$$

$$\det(tK_\phi)\cdot\det(tK_F) = \prod_{n\in\mathbb{Z}_{\geq 0}} \left\{ \frac{t^2}{\ell^2}(n+2-r+i\ell a)(n+r-i\ell a) \right\}^{(n+1)^2}$$

The total 1-loop determinant for a chiral multiplet

with $U(1)$ charge +1, R-charge *r* is

$$Z_{\text{1-loop}}(a) = \frac{\det(tK_\psi)}{\det(tK_\phi)\det(tK_F)}$$

$$= \prod_{n\in\mathbb{Z}_+} \frac{\left\{\frac{t}{\ell}(n+1-r+i\ell a)\right\}^n}{\left\{\frac{t}{\ell}(n-1+r-i\ell a)\right\}^n} = s_{b=1}\big(i(1-r)-\ell a\big).$$

Generalization

$$Z_{\text{1-loop}}(a) = s_{b=1}\big(i(1-r) - \ell a\big)$$

for a chiral multiplet of U(1) charge +1, R-charge *r*.

Consider a vectormultiplet for the gauge group G

fixed at a saddle point $\langle\sigma\rangle = a_i H_i$.

What is $Z_{\text{1-loop}}$ of the chiral multiplet in the rep. R?

The gauge group is broken from G to $U(1)^{\otimes r}$ at the saddle point.

The chiral multiplet in rep. R

= A collection of chiral multiplets with $U(1)^{\otimes r}$ charge $\boldsymbol{w} = (w_1, \cdots, w_r) \in R$.

$$Z_{\text{1-loop}}(\boldsymbol{a}) = \prod_{\boldsymbol{w}\in R} s_{b=1}\big(i(1-r) - \ell \boldsymbol{a}\cdot\boldsymbol{w}\big)$$

Determinant: Vectormultiplet

The goal is : $$\frac{1}{|\mathcal{W}|} d^r a \prod_{\boldsymbol{\alpha} \in \Delta_+} 4 \sinh^2(\pi \boldsymbol{a} \boldsymbol{\alpha})$$

Determinant: Vectormultiplet

We begin by studying the Lagrangian for 3D photon.

$$\int d^3x\sqrt{g}\frac{1}{4}F_{mn}F^{mn} = \int \frac{1}{2}F \wedge *F = \int \frac{1}{2}A \wedge (*d * dA).$$

The kinetic operator for the photon is $(*d)^2$,

where * is Hodge star operator which maps 2-forms to 1-forms.

It acts on the basis vielbein forms as

$$*(e^a \wedge e^b) = \varepsilon^{abc}e^c,$$

$$*(e^1 \wedge e^2) = e^3, \quad *(e^2 \wedge e^3) = e^1, \quad *(e^3 \wedge e^1) = e^2.$$

Let us study the spectrum of the operator $*d$.

Spectrum of the kinetic operator

- Let us study the operator $*d$ on the round 3-sphere.

$$A \equiv A_m dx^m \equiv A^a e^a$$

$$\longrightarrow \quad F = dA = dA^a \wedge e^a + A^a de^a$$

$$= \frac{1}{\ell} e^a \wedge e^b \left(\mathcal{R}^a A^b + \varepsilon^{abc} A^c \right)$$

$$\boxed{\begin{aligned} d &= dx^m \partial_m = \frac{1}{\ell} e^a \mathcal{R}^a, \\ de^a &= \frac{1}{\ell} \varepsilon^{abc} e^b \wedge e^c \end{aligned}}$$

$$\longrightarrow \quad *F = *dA = \frac{1}{\ell} e^a \left(\varepsilon^{abc} \mathcal{R}^b A^c + 2A^a \right)$$

$$*d \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} = \frac{1}{\ell} \begin{pmatrix} \mathcal{R}^2 A^3 - \mathcal{R}^3 A^2 + 2A^1 \\ \mathcal{R}^3 A^1 - \mathcal{R}^1 A^3 + 2A^2 \\ \mathcal{R}^1 A^2 - \mathcal{R}^2 A^1 + 2A^3 \end{pmatrix}$$

$$= \frac{1}{\ell} \begin{pmatrix} 2 & -\mathcal{R}^3 & \mathcal{R}^2 \\ \mathcal{R}^3 & 2 & -\mathcal{R}^1 \\ -\mathcal{R}^2 & \mathcal{R}^1 & 2 \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} = \frac{1}{\ell} \left(2 - i \mathcal{R}^a T^a \right) \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$$\boxed{*d = \frac{2}{\ell} (1 + \boldsymbol{J}_{\mathcal{R}} \cdot \boldsymbol{T}), \quad T^a = \text{generator of SU(2) in triplet rep.}}$$

Spectrum of the kinetic operator

$\boldsymbol{A} = (A^1, A^2, A^3)$ can be expanded into vector spherical harmonics $Y^{j,\tilde{j}}_{m,\tilde{m}}$

$$\boldsymbol{J}_{\mathcal{L}}^2 = j(j+1), \quad (\boldsymbol{J}_{\mathcal{R}} + \boldsymbol{T})^2 = \tilde{j}(\tilde{j}+1), \quad \tilde{j} = j \text{ or } j \pm 1.$$

$$J_{\mathcal{L}}^3 = m, \qquad\qquad J_{\mathcal{R}}^3 + T^3 = \tilde{m}$$

$$*d = \frac{2}{\ell}(1 + \boldsymbol{J}_{\mathcal{R}} \cdot \boldsymbol{T}) = \frac{n+2}{\ell} \quad \text{on } (j,\tilde{j}) = (\tfrac{n}{2}, \tfrac{n}{2}+1) \quad (n \in \mathbb{Z}_{\geq 0}) \qquad [1]$$

$$= 0 \quad \text{on } (j,\tilde{j}) = (\tfrac{n}{2}, \tfrac{n}{2}) \quad (n \in Z_{\geq 1}) \qquad [2]$$

$$= -\frac{n+2}{\ell} \quad \text{on } (j,\tilde{j}) = (\tfrac{n}{2}+1, \tfrac{n}{2}) \quad (n \in Z_{\geq 0}) \qquad [3]$$

The modes [2] are pure gauge, $dA = 0$.

The modes [1],[3] satisfy the Lorentz gauge condition $d * A = 0$.

Gauge-Fixing

The gauge field is now decomposed as $\boldsymbol{A} = \boldsymbol{A}_{[1]} + \boldsymbol{A}_{[2]} + \boldsymbol{A}_{[3]}$.

Gauge transformation acts on it as $\delta(\boldsymbol{A}_{[1]}, \boldsymbol{A}_{[2]}, \boldsymbol{A}_{[3]}) = (0, d\varphi, 0)$.

The pure gauge modes $\boldsymbol{A}_{[2]}$ need to be eliminated by gauge fixing.

$$\int \frac{\mathcal{D}\boldsymbol{A}}{(\text{gauge})} = \int \mathcal{D}\boldsymbol{A}_{[1]} \mathcal{D}\boldsymbol{A}_{[3]} \frac{\mathcal{D}\boldsymbol{A}_{[2]}}{\mathcal{D}'\varphi}$$

where $\mathcal{D}'\varphi$ indicates the constant mode is excluded.

- **Jacobian :**

$$\det \frac{\mathcal{D}\boldsymbol{A}_{[2]}}{\mathcal{D}'\varphi} = \frac{\int D\boldsymbol{A}_{[2]} \exp\left(- \int d^3x \sqrt{g} \boldsymbol{A}_{[2]}^m \boldsymbol{A}_{[2]m} \right)}{\int D'\varphi \exp\left(- \int d^3x \sqrt{g} \partial^m \varphi \partial_m \varphi \right)} = \det{}'(-\nabla^2)^{\frac{1}{2}} \, .$$

Determinant: Vectormultiplet

Consider now a vectormultiplet for a general gauge group.

- Localization Lagrangian:

$$\mathcal{L}_{\rm YM} = \frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}F_{mn}F^{mn} + D_m\sigma D^m\sigma + D^2 + i\bar{\lambda}\gamma^m D_m\lambda - i\bar{\lambda}[\sigma,\lambda] - M\bar{\lambda}\lambda\right]$$

  - Gaussian approximation
  - Lorentz gauge $\partial^m A_m = 0$ $(\boldsymbol{A}_{[2]} = 0)$
  - decompose $\sigma = a + \hat{\sigma}$

$$\simeq \frac{1}{g^2}\mathrm{Tr}\left[\boldsymbol{A}\cdot(*d)^2\boldsymbol{A} - [a,\boldsymbol{A}]^2 + \partial_m\hat{\sigma}\partial^m\hat{\sigma} + D^2 + i\bar{\lambda}\gamma^m D_m\lambda - i\bar{\lambda}[a,\lambda] - \tfrac{1}{2\ell}\bar{\lambda}\lambda\right]$$

- Integral over $\hat{\sigma}$ gives $\det'(-\nabla^2)^{-\frac{1}{2}\dim G}$

  which cancels with the Jacobian determinant $\det\dfrac{\mathcal{D}\boldsymbol{A}_{[2]}}{\mathcal{D}'\varphi}$.

Read off the kinetic operators

$$\mathcal{L}_{\rm YM} \simeq \frac{1}{g^2}\mathrm{Tr}\left[\boldsymbol{A}\cdot(*d)^2\boldsymbol{A} - [a,\boldsymbol{A}]^2 + \partial_m\hat\sigma\partial^m\hat\sigma + D^2 + i\bar\lambda\gamma^m D_m\lambda - i\bar\lambda[a,\lambda] - \tfrac{1}{2\ell}\bar\lambda\lambda\right]$$

Using the Cartan-Weyl basis one finds

$$= \frac{1}{g^2}\sum_{\alpha\in\Delta_+}\Big[\underline{\boldsymbol{A}_{-\boldsymbol{\alpha}}\{(*d)^2 + (\boldsymbol{a\alpha})^2\}\boldsymbol{A}_{\boldsymbol{\alpha}}} + \underline{\hat\sigma_{-\boldsymbol{\alpha}}(-\nabla^2)\hat\sigma_{\boldsymbol{\alpha}}} + \underline{D_{-\boldsymbol{\alpha}}D_{\boldsymbol{\alpha}}}$$
$$+\underline{\bar\lambda_{-\boldsymbol{\alpha}}\{i\gamma^m D_m - i\boldsymbol{a\alpha} - \tfrac{1}{2\ell}\}\lambda_{\boldsymbol{\alpha}}} + \underline{\bar\lambda_{\boldsymbol{\alpha}}\{i\gamma^m D_m + i\boldsymbol{a\alpha} - \tfrac{1}{2\ell}\}\lambda_{-\boldsymbol{\alpha}}}\Big]$$
$$+\frac{1}{g^2}\sum_{i=1}^{r}\Big[\underline{\boldsymbol{A}_i(*d)^2\boldsymbol{A}_i} + \underline{\hat\sigma_i(-\nabla^2)\hat\sigma_i} + \underline{D_i^2} + \underline{\bar\lambda_i\{i\gamma^m D_m - \tfrac{1}{2\ell}\}\lambda_i}\Big]$$

$$Z_{\text{1-loop}} = \prod_{\boldsymbol{\alpha}\in\Delta_+}\det\frac{\mathcal{D}\boldsymbol{A}_{\pm\boldsymbol{\alpha}[2]}}{\mathcal{D}\varphi_{\pm\boldsymbol{\alpha}}}\cdot\frac{\det\left[\frac{1}{g^2}(i\gamma^m D_m - i\boldsymbol{a\alpha} - \frac{1}{2\ell})\right]\det\left[\frac{1}{g^2}(i\gamma^m D_m + i\boldsymbol{a\alpha} - \frac{1}{2\ell})\right]}{\det\left[\frac{1}{g^2}((*d)^2 + (\boldsymbol{a\alpha})^2)\right]_{[1],[3]}\det\left[\frac{1}{g^2}(-\nabla^2)\right]\det\left[\frac{1}{g^2}\right]}$$

$$\times\prod_{i=1}^{r}\det\frac{\mathcal{D}\boldsymbol{A}_{i[2]}}{\mathcal{D}\varphi_i}\cdot\frac{\det\left[\frac{1}{g^2}(i\gamma^m D_m - \frac{1}{2\ell})\right]}{\det\left[\frac{1}{g^2}(*d)^2\right]^{1/2}_{[1],[3]}\det\left[\frac{1}{g^2}(-\nabla^2)\right]^{1/2}\det\left[\frac{1}{g^2}\right]^{1/2}}$$

Determinant: Vectormultiplet

After some cancellations one is left with

$$Z_{1\text{-loop}} = \prod_{\boldsymbol{\alpha}\in\Delta_+} \frac{\det\left[i\ell\gamma^m D_m - i\ell\boldsymbol{a}\boldsymbol{\alpha} - \frac{1}{2}\right]\det\left[i\ell\gamma^m D_m + i\ell\boldsymbol{a}\boldsymbol{\alpha} - \frac{1}{2}\right]}{\det\left[\ell^2(*d)^2 + \ell^2(\boldsymbol{a}\boldsymbol{\alpha})^2\right]_{[1],[3]}}$$

$$\times \frac{\det\left[i\ell\gamma^m D_m - \frac{1}{2}\right]^r}{\det\left[\ell^2(*d)^2\right]^{r/2}_{[1],[3]}}$$

$$= \prod_{n\in\mathbb{Z}_+}\left[n^{2r}\prod_{\boldsymbol{\alpha}\in\Delta_+}(n^2+\ell^2(\boldsymbol{a}\boldsymbol{\alpha})^2)\right] = (2\pi)^r \prod_{\boldsymbol{\alpha}\in\Delta_+}\left(\frac{2\sinh(\pi\ell\boldsymbol{a}\boldsymbol{\alpha})}{\ell\boldsymbol{a}\boldsymbol{\alpha}}\right)^2 .$$

Recall our goal is : $\frac{1}{|\mathcal{W}|}d^r a \prod_{\boldsymbol{\alpha}\in\Delta_+} 4\sinh^2(\pi\boldsymbol{a}\boldsymbol{\alpha})$

Close but not quite!

Residual gauge symmetry

Another factor $\dfrac{1}{|\mathcal{W}|}\prod_{\boldsymbol{\alpha}\in\Delta_+}(\boldsymbol{a}\boldsymbol{\alpha})^2$ arises from the **further gauge fixing.**

- we assumed $\langle\sigma\rangle \equiv a$ to be in Cartan subalgebra, not in the full Lie algebra.

- $\boldsymbol{a}$ (r -component vector in the root space)

  is subject to identification by Weyl reflections.

Vandermonde's determinant

Compute the Faddeev-Popov's determinant for the gauge fixing

$$a \in \mathrm{Cartan\ subalgebra}\ .$$

- original a : $\quad a = \sum_{r=1}^{r} a_i H_i + \sum_{\boldsymbol{\alpha}\in\Delta} a_{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}}$ (gauge condition : $a_{\boldsymbol{\alpha}} = 0$)

- ghosts and BRST symmetry :

$$\delta_{\mathrm{B}} a = [c, a], \quad c = \sum_{\boldsymbol{\alpha}\in\Delta} c_{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}}, \quad \delta_{\mathrm{B}} b_{\boldsymbol{\alpha}} = B_{\boldsymbol{\alpha}}, \quad \delta_{\mathrm{B}} B_{\boldsymbol{\alpha}} = 0.$$

- gauge-fixing term : $\delta_{\mathrm{B}}\Big(\sum_{\boldsymbol{\alpha}\in\Delta} b_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}\Big) = \sum_{\boldsymbol{\alpha}\in\Delta}\Big(B_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} + b_{\boldsymbol{\alpha}} (\boldsymbol{a\alpha}) c_{\boldsymbol{\alpha}}\Big)$

- Faddeev-Popov determinant :

$$\int [dBdbdc] \exp i \sum_{\boldsymbol{\alpha}\in\Delta} \big(B_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} + b_{\boldsymbol{\alpha}} (\boldsymbol{a\alpha}) c_{\boldsymbol{\alpha}}\big) = \prod_{\boldsymbol{\alpha}\in\Delta_+} (2\pi)^2 \delta^2(a_{\pm\boldsymbol{\alpha}}) \cdot \underline{(\boldsymbol{a\alpha})^2}\, .$$

Sphere Partition Function

Summary :

the sphere partition function is given by the formula with $b := 1.$

$$Z_{S^3} = \frac{1}{|\mathcal{W}|} \int \prod_{i=1}^{r} da_i \prod_{\boldsymbol{\alpha}\in\Delta_+} 4\sinh(\pi \boldsymbol{a}\cdot\boldsymbol{\alpha} b)\sinh(\pi \boldsymbol{a}\cdot\boldsymbol{\alpha}/b)\cdot\exp\{-S(\boldsymbol{a})\}$$

$$\cdot \prod_j \left[\prod_{\boldsymbol{w}\in R_j} s_b\left(\frac{i}{2}(b+b^{-1})(1-r_j) - \boldsymbol{a}\cdot\boldsymbol{w}\right)\right]$$

where $\quad e^{-S_{\mathrm{CS}}(a)} = e^{i\pi k \mathrm{Tr}(a^2)}, \qquad e^{-S_{\mathrm{FI}}(a)} = e^{4\pi i \zeta a},$

$$s_b(x) \equiv \prod_{m,n\in\mathbb{Z}_{\geq 0}} \frac{mb+nb^{-1}+\frac{1}{2}(b+b^{-1})-ix}{mb+nb^{-1}+\frac{1}{2}(b+b^{-1})+ix}.$$

* a, ζ are redefined to be dimensionless.

Application of Sphere Partition Function

- Multiple M2-branes
- F-theorem and F-maximization

Worldvolume theory of multiple M2-branes

M2-brane is a (2+1)-dim. fundamental dynamical object

in the 11-dim. quantum supergravity called M-theory.

The worldvolume theory for multiple M2-branes was not known

until 2007.

<u>AdS/CFT correspondence :</u>

The worldvolume theory on N coincident M2-branes is dual

to the 11-dim. supergravity on $AdS_4 \times S^7$.

Related to the above conjecture,

the free energy of the woldvolume theory of N M2-branes was believed

to behave, at strong coupling, as $F \sim \mathrm{const} \cdot N^{\frac{3}{2}}$.

Worldvolume theory of multiple M2-branes

ABJM model Aharony, Bergman, Jafferis, Maldacena, 2008

The theory of N M2-branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$

$$(z_1, z_2, z_3, z_4) \sim (\omega z_1, \omega z_2, \omega z_3, \omega z_4) \quad \left(\omega^k = 1\right)$$

was shown to be given by a Chern-Simons-matter theory

- Gauge group & Chern-Simons coupling : $U(N)_k \times U(N)_{-k}$
- 2 chiral multiplets of R-charge 1/2 in the rep. $(\Box, \overline{\Box})$
- 2 chiral multiplets of R-charge 1/2 in the rep. $(\overline{\Box}, \Box)$
- a quartic superpotential

All the fields are $N \times N$ matrices. Why $F \sim N^{3/2}$?

Worldvolume theory of multiple M2-branes

Exact free energy Drukker, Marino, Putrov, 2010

The free energy was evaluated using the exact formula for sphere partition function.

$$e^{-F} = \int \prod_{i=1}^{N} \frac{d\mu_i}{2\pi} \frac{d\tilde{\mu}_i}{2\pi} e^{\frac{ik}{4\pi}(\mu_i^2 - \tilde{\mu}_i^2)} \frac{\prod_{i<j}(2\sinh\frac{\mu_i-\mu_j}{2})^2 (2\sinh\frac{\tilde{\mu}_i-\tilde{\mu}_j}{2})^2}{\prod_{i,j}(2\cosh\frac{\mu_i-\tilde{\mu}_j}{2})^2}.$$

In the limit of large N, one can use the technique of large-N matrix integrals.

$$F \simeq \frac{\sqrt{2}\pi}{3} k^{\frac{1}{2}} N^{\frac{3}{2}}.$$

F-theorem and F-maximization

Free energy of a d-dim. CFT on the d-sphere is an important observable, though it is generally UV divergent.

$$F_d \equiv -\log Z_{S^d} = a_d(\Lambda r)^d + a_{d-2}(\Lambda r)^{d-2} + \cdots$$

Λ : UV cutoff

r : radius

Removing the power divergences one generally finds

$(d = \text{even})$ $\quad F_d = a \log(\Lambda r) + \text{finite.}$

$(d = \text{odd})$ $\quad F_d = \text{finite.}$

universal

a-theorem / F-theorem : For a pair of CFTs connected by a RG flow,

$(d = \text{even})$ $\quad a_{\text{UV}} > a_{\text{IR}}.$

$(d = \text{odd})$ $\quad F_{\text{UV}} > F_{\text{IR}}.$

F-theorem and F-maximization

- Consider a 3D $\mathcal{N}=2$ theory of vector and chiral multiplets, which is free in the UV and flows to a SCFT in the IR.
- Localization method allows us to compute the free energy as a function of the R-charges of the matter chiral multiplets.
- Consistent assignments of R-charges in the UV is not unique. Any two consistent choices are related by shifts by global symmetries.

$$R(t) = R_0 + \sum_a t_a Q_a \, .$$

- For which value of t_a is $F(t_a)$ the free energy of the IR SCFT?

$\longrightarrow$ The value which maximizes $\mathrm{Re}F(t_a)$.

Closset, Dumitrescu, Festuccia, Komargodski, Seiberg, 2012

III. Squashings

- Squashings
- Elliosoid Partition Function
- AGT Relation

Recap

The sphere partition function is given by the formula with $b := 1$.

$$Z_{S^3} = \frac{1}{|\mathcal{W}|} \int \prod_{i=1}^{r} da_i \prod_{\boldsymbol{\alpha}\in\Delta_+} 4\sinh(\pi \boldsymbol{a}\cdot\boldsymbol{\alpha} b)\sinh(\pi\boldsymbol{a}\cdot\boldsymbol{\alpha}/b)\cdot\exp\{-S(\boldsymbol{a})\}$$

$$\cdot \prod_j \left[\prod_{\boldsymbol{w}\in R_j} s_b\Big(\frac{i}{2}(b+b^{-1})(1-r_j) - \boldsymbol{a}\cdot\boldsymbol{w}\Big)\right]$$

where $\quad e^{-S_{\mathrm{CS}}(a)} = e^{i\pi k \mathrm{Tr}(a^2)}, \qquad e^{-S_{\mathrm{FI}}(a)} = e^{4\pi i \zeta a},$

$$s_b(x) \equiv \prod_{m,n\in\mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{1}{2}(b+b^{-1}) - ix}{mb + nb^{-1} + \frac{1}{2}(b+b^{-1}) + ix}.$$

The parameter b corresponds to a SUSY deformation away from the round sphere, called “squashing parameter”.

Squashing to Ellipsoid

The formula with general b is reproduced from the ellipsoid,

$$\frac{x_0^2+x_1^2}{\ell^2}+\frac{x_2^2+x_3^2}{\tilde{\ell}^2}=1 \quad \text{in flat } \mathbb{R}^4. \qquad \longrightarrow \qquad b \equiv \sqrt{\ell/\tilde{\ell}}.$$

- symmetry $\quad SO(4) \longrightarrow U(1) \times U(1)$
- coordinates $\quad (x_0, x_1, x_2, x_3) = (\ell \cos\theta \cos\varphi,\ \ell \cos\theta \sin\varphi,\ \tilde{\ell} \sin\theta \cos\chi,\ \tilde{\ell} \sin\theta \sin\chi)$
- metric $\quad ds^2 = \ell^2 \cos^2\theta d\varphi^2 + \tilde{\ell}^2 \sin^2\theta d\chi^2 + f(\theta)^2 d\theta^2,$
- vielbein $\quad e^1 = \ell \cos\theta d\varphi, \quad e^2 = \tilde{\ell} \sin\theta d\chi, \quad e^3 = f(\theta) d\theta.$

$$\left(f(\theta) = \sqrt{\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta} \right)$$

- spin connection $\quad \Omega^{12} = 0, \quad \Omega^{13} = -\frac{\ell}{f} \sin\theta d\varphi, \quad \Omega^{23} = \frac{\tilde{\ell}}{f} \cos\theta d\chi.$

Killing spinors on the ellipsoid

- We **choose** the Killing spinors as

$$\xi := \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{\frac{i}{2}(\chi-\varphi+\theta)} \\ e^{\frac{i}{2}(\chi-\varphi-\theta)} \end{pmatrix}, \quad \bar{\xi} := \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(-\chi+\varphi+\theta)} \\ e^{\frac{i}{2}(-\chi+\varphi-\theta)} \end{pmatrix}, \quad \bar{\xi}\xi = 1.$$

and choose the SUGRA background fields (M, V_m, U_m) so that they satisfy Killing spinor equation.

- On the round sphere they satisfy

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\xi = \frac{i}{2\ell}\gamma_m\xi, \qquad \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\bar{\xi} = \frac{i}{2\ell}\gamma_m\bar{\xi},$$

- On the ellipsoid they satisfy

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - iV_m\right)\xi = iM\gamma_m\xi, \quad \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} + iV_m\right)\bar{\xi} = iM\gamma_m\bar{\xi},$$

where $V_m dx^m = -\frac{1}{2}\left(1 - \frac{\ell}{f}\right)d\varphi + \frac{1}{2}\left(1 - \frac{\tilde{\ell}}{f}\right)d\chi, \quad M = \frac{1}{2f}.$

"Traditional" Squashing

- Another, more well-studied, deformation of the sphere is

$$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^2\mu^3\mu^3. \qquad \text{(Recall } g^{-1}dg = i\mu^a\gamma^a\text{)}$$

  This is traditionally called the *squashed sphere.*

- Isometry $\quad SO(4) \longrightarrow SU(2)_{\mathcal{L}} \times U(1)_{\mathcal{R}}$

- Killing spinors : On the round sphere, there is a pair $\xi, \bar{\xi}$ satisfying

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\xi = \frac{i}{2\ell}\gamma_m\xi, \quad \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\bar{\xi} = \frac{i}{2\ell}\gamma_m\bar{\xi}.$$

  On the squashed sphere, the **same pair** satisfies

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - iV_m\right)\xi = iM\gamma_m\xi,$$

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} + iV_m\right)\bar{\xi} = iM\gamma_m\bar{\xi}, \qquad \text{with } \; V_m dx^m = \left(1 - \frac{\tilde{\ell}^2}{\ell^2}\right)\mu^3, \quad M = \frac{\tilde{\ell}}{2\ell^2}.$$

- The partition function is given by : $\boxed{b \equiv 1.}$

"Traditional" Squashing

- Another, more well-studied, deformation of the sphere is

$$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^2\mu^3\mu^3. \qquad \text{(Recall } g^{-1}dg = i\mu^a\gamma^a\text{)}$$

This is traditionally called the *squashed sphere.*

- Isometry $\quad SO(4) \longrightarrow SU(2)_{\mathcal{L}} \times U(1)_{\mathcal{R}}$

- Killing spinors : On the round sphere, there is **another pair** $\eta, \bar{\eta}$ satisfying

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\eta = -\frac{i}{2\ell}\gamma_m\eta, \quad \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\bar{\eta} = -\frac{i}{2\ell}\gamma_m\bar{\eta}.$$

On the squashed sphere, the **same pair** satisfies

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\eta = iM\gamma_m\eta - B^n\gamma_{mn}\eta, \qquad \text{with} \quad B_m dx^m = \frac{\tilde{\ell}}{\ell^2}\sqrt{\ell^2 - \tilde{\ell}^2}\mu^3,$$

$$\left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\bar{\eta} = iM\gamma_m\bar{\eta} + B^n\gamma_{mn}\bar{\eta}, \qquad M = -\frac{\tilde{\ell}}{2\ell^2}.$$

- The partition function is given by : $\boxed{b \equiv u - i\sqrt{1-u^2}.}$

Condition for a SUSY **Closset, Dumitrescu, Festuccia, Komargodski 2012**

In order for a 3-manifold to have a Killing spinor satisfying

$$\boxed{D_m\xi \equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - iV_m\right)\xi = iM\gamma_m\xi - iU_m\xi - \frac{1}{2}\varepsilon_{mnp}U^n\gamma^p\xi,}$$

The 3-manifold has to have an **almost contact metric structure**

= a triplet $(\eta_\mu, \xi^\mu = g^{\mu\nu}\eta_\nu, \Phi^\mu{}_\nu)$ satisfying

$$\eta_\mu\xi^\mu = 1, \quad \Phi^\mu{}_\rho\Phi^\rho{}_\nu = -\delta^\mu{}_\nu + \xi^\mu\eta_\nu,$$

satisfying an integrability condition $\Phi^\mu{}_\rho\mathcal{L}_\xi\Phi^\rho{}_\nu = 0.$

Such a manifold has local coordinate charts

$$(\tau, z, \bar{z}) \quad \big(\tau' = \tau - t(z,\bar{z}), \quad z' = f(z)\big)$$

and metric

$$ds^2 = (d\tau + h(\tau,z,\bar{z})dz + \bar{h}(\tau,z,\bar{z})d\bar{z})^2 + c(\tau,z,\bar{z})^2 dz d\bar{z}.$$

Example of Contact Manifolds

- **Seifert manifolds**

  = circle bundle over Riemann surface (with orbifold singularities)

  have **a pair** of Killing spinors of opposite R-charge.

$$\boxed{\begin{aligned} D_m\xi &\equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - iV_m\right)\xi = iM\gamma_m\xi - iU_m\xi - \frac{1}{2}\varepsilon_{mnp}U^n\gamma^p\xi, \\ D_m\bar{\xi} &\equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} + iV_m\right)\bar{\xi} = iM\gamma_m\bar{\xi} + iU_m\bar{\xi} + \frac{1}{2}\varepsilon_{mnp}U^n\gamma^p\bar{\xi}, \end{aligned}}$$

- metric $ds^2 = \Omega(z,\bar{z})^2(d\tau + h(z,\bar{z})dz + \bar{h}(z,\bar{z})d\bar{z})^2 + c(z,\bar{z})^2 dz d\bar{z}.$

  has an isometry ∂_τ.

Ellipsoid Partition Function

Localization argument works, but..

The SUSY saddle points are labeled by $a \equiv (\text{constant value of } \sigma)$.

$$Z_{\text{ell}} = \lim_{t\to\infty} \int d^r a\, \mathcal{D}(\text{others}) \exp\Big\{ - S - t(S_{\text{YM}} + S_{\text{mat}})\Big\}$$

$$= \int d^r a \exp\big\{ - S(a)\big\} \cdot Z_{\text{1-loop}}(a).$$

$$e^{-S_{\text{CS}}(a)} = e^{i\pi k \ell\tilde{\ell}\text{Tr}(a^2)}, \qquad e^{-S_{\text{FI}}(a)} = e^{4\pi i \zeta \ell\tilde{\ell} a}.$$

The computation of 1-loop determinant is harder

because the spherical harmonics do not diagonalize the kinetic operators.

* For squashed sphere with $SU(2)_{\mathcal{L}} \times U(1)_{\mathcal{R}}$ isometry,

  the spherical harmonics still diagonalize the kinetic term

  although the eigenvalue degeneracy is partially resolved.

Determinants from Index theorem

Actually, $Z_{\text{1-loop}}(a)$ can be computed *without* knowing the full spectrum by using the index theorem.

Let us compute it in the two examples,

- The theory of a chiral multiplet charged +1 under a background U(1) vectormultiplet fixed at the saddle point value.
- pure SYM theory

Before this, we need to study the **square of the SUSY** $\mathbf{Q}^2$.

Square of SUSY

To use the idea of index theorem, we need to know $\mathbf{Q}^2$.

$$\boxed{\begin{aligned}\mathbf{Q}^2 = i\boldsymbol{\mathcal{L}}_v &+ \mathbf{Lorentz}(iD_{[m}v_{n]} + iv^p\Omega_{p,mn}) \\ &+\mathbf{Gauge}(-i\sigma\bar{\xi}\xi + v^m A_m) + (v^m V_m + 2M\bar{\xi}\xi)\cdot\mathbf{R}_{U(1)} \\ &(v^m \equiv \bar{\xi}\gamma^m\xi) \qquad \text{on all the fields.}\end{aligned}}$$

Let us check this for a charged chiral multiplet using

$$\mathbf{Q}\phi = \xi\psi, \qquad \mathbf{Q}\psi = i\gamma^m\bar{\xi}D_m\phi + i\bar{\xi}\sigma\phi - 2rM\bar{\xi}\phi + \xi F,$$

$$\mathbf{Q}\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \mathbf{Q}\bar{\psi} = i\gamma^m\xi D_m\bar{\phi} + i\xi\bar{\phi}\sigma - 2rM\xi\bar{\phi} + \bar{\xi}\bar{F},$$

$$\mathbf{Q}F = \bar{\xi}\left\{i\gamma^m D_m\psi - i\sigma\psi - i\bar{\lambda}\phi + (2r-1)M\psi\right\},$$

$$\mathbf{Q}\bar{F} = \xi\left\{i\gamma^m D_m\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\lambda + (2r-1)M\bar{\psi}\right\}$$

and assuming $\xi, \bar{\xi}$ are Grassmann even.

Square of SUSY

$$\mathbf{Q}\phi = \xi\psi, \quad \mathbf{Q}\psi = i\gamma^m\bar{\xi}D_m\phi + i\bar{\xi}\sigma\phi - 2rM\bar{\xi}\phi + \xi F$$

$$\longrightarrow \quad \mathbf{Q}^2\phi = \xi\mathbf{Q}\psi$$

$$= i\xi\gamma^m\bar{\xi}D_m\phi + i\xi\bar{\xi}\sigma\phi - 2rM\xi\bar{\xi}\phi$$

$$= iv^m(\partial_m - iA_m - irV_m)\phi - i\bar{\xi}\xi\sigma\phi + 2rM\bar{\xi}\xi\phi$$

$$= iv^m\partial_m\phi + (-i\bar{\xi}\xi\sigma + v^m A_m)\phi + r(2M\bar{\xi}\xi + v^m V_m)\phi$$

$$= i\boldsymbol{\mathcal{L}}_v\phi + \mathbf{Gauge}(-i\bar{\xi}\xi\sigma + v^m A_m)\cdot\phi + (2M\bar{\xi}\xi + v^m V_m)\mathbf{R}_{U(1)}\cdot\phi.$$

Square of SUSY

To use the idea of index theorem, we need to know $\mathbf{Q}^2$.

$$\mathbf{Q}^2 = i\boldsymbol{\mathcal{L}}_v + \mathbf{Lorentz}(iD_{[m}v_{n]} + iv^p\Omega_{p,mn})$$
$$+\mathbf{Gauge}(-i\sigma\bar{\xi}\xi + v^m A_m) + (v^m V_m + 2M\bar{\xi}\xi)\cdot \mathbf{R}_{U(1)}$$
$$(v^m \equiv \bar{\xi}\gamma^m\xi)$$

on all the fields.

For our choice of ellipsoid background and Killing spinors,

$$v^m\partial_m = -\frac{1}{\ell}\partial_\varphi + \frac{1}{\tilde{\ell}}\partial_\chi,$$

$$\mathbf{Q}^2 = \boldsymbol{\mathcal{L}}_{iv} + \mathbf{Gauge}\Big(-i\sigma - \frac{1}{\tilde{\ell}}A_\chi + \frac{1}{\ell}A_\varphi\Big) + \Big(\frac{1}{2\ell} + \frac{1}{2\tilde{\ell}}\Big)\mathbf{R}_{U(1)}$$

Determinant: Chiral Multiplet

Consider a path integral over

$(\phi, \psi, F; \bar{\phi}, \bar{\psi}, \bar{F}) \equiv$ chiral multiplet of R-charge *r*,

charged **+1** under a background U(1) vectormultiplet

$$(A_m, \sigma, \lambda, \bar{\lambda}, D) = (0, a, 0, 0, 0).$$

Transformation rule:

$$\mathbf{Q}\phi = \xi\psi, \qquad \mathbf{Q}\psi = i\gamma^m\bar{\xi}D_m\phi + i\bar{\xi}\sigma\phi - 2rM\bar{\xi}\phi + \xi F,$$

$$\mathbf{Q}\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \mathbf{Q}\bar{\psi} = i\gamma^m\xi D_m\bar{\phi} + i\xi\bar{\phi}\sigma - 2rM\xi\bar{\phi} + \bar{\xi}\bar{F},$$

$$\mathbf{Q}F = \bar{\xi}\left\{i\gamma^m D_m\psi - i\sigma\psi - i\bar{\lambda}\phi + (2r-1)M\psi\right\},$$

$$\mathbf{Q}\bar{F} = \xi\left\{i\gamma^m D_m\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\lambda + (2r-1)M\bar{\psi}\right\}$$

We move to a new set of path-integration variables.

Cohomological Variables

- The original integration variables: $(\phi, \psi, F; \bar{\phi}, \bar{\psi}, \bar{F})$

$$\mathbf{Q}\phi = \xi\psi \equiv \chi, \qquad \mathbf{Q}\chi = \mathbf{H}\phi,$$

$$\mathbf{Q}\bar{\phi} = \bar{\xi}\bar{\psi} \equiv \bar{\chi}, \qquad \mathbf{Q}\bar{\chi} = \mathbf{H}\bar{\phi},$$

SUSY algebra is just $\mathbf{Q}^2 = \mathbf{H}$, where

$$\mathbf{H} \equiv -\frac{i}{\ell}\partial_\varphi + \frac{i}{\tilde{\ell}}\partial_\chi - i\mathbf{Gauge}(a) + \Big(\frac{1}{2\ell} + \frac{1}{2\tilde{\ell}}\Big)\mathbf{R}_{U(1)}$$

$$\bar{\xi}\psi \equiv \chi', \qquad \mathbf{Q}\chi' = F + i\bar{\xi}\gamma^m\bar{\xi}D_m\phi \equiv \phi', \qquad \mathbf{Q}\phi' = \mathbf{H}\chi',$$

$$-\xi\bar{\psi} \equiv \bar{\chi}', \qquad \mathbf{Q}\bar{\chi}' = \bar{F} - i\xi\gamma^m\xi D_m\bar{\phi} \equiv \bar{\phi}', \qquad \mathbf{Q}\bar{\phi}' = \mathbf{H}\bar{\chi}'.$$

- The new integration variables : $\phi, \chi \ \ (\mathbf{R}_{U(1)} = +r), \quad \chi', \phi' \ \ (\mathbf{R}_{U(1)} = +r-2),$

$$\bar{\phi}, \bar{\chi} \ \ (\mathbf{R}_{U(1)} = -r), \quad \bar{\chi}', \bar{\phi}' \ \ (\mathbf{R}_{U(1)} = -r+2).$$

Note:
- the new variables are all Lorentz scalars.
- the change of variable is local and invertible.

$$\psi = -\bar{\xi}\,\chi + \xi\,\chi', \quad \bar{\psi} = \xi\,\bar{\chi} + \bar{\xi}\,\bar{\chi}'.$$

Determinant from Gaussian integral

The determinant is an integral over the fields $\{\phi, \bar{\phi}, \phi', \bar{\phi}'; \chi, \bar{\chi}, \chi', \bar{\chi}'\}$

with **any** convenient **Q**-exact weight function,

$$Z_{\text{1-loop}} = \int d(\text{fields}) \exp\left(-\mathbf{Q}\mathcal{V}\right)$$

Let us choose $\quad \mathcal{V} = \int d^3x \sqrt{g} (\bar{\phi} \mathbf{H}^\dagger \chi + \bar{\chi}' \phi')$

$$\mathbf{Q}\mathcal{V} = \int d^3x \sqrt{g} \Big(\bar{\phi} \mathbf{H}^\dagger (\mathbf{Q}\chi) + (\mathbf{Q}\bar{\phi}) \mathbf{H}^\dagger \chi + (\mathbf{Q}\bar{\chi}') \phi' - (\bar{\chi}' \mathbf{Q} \phi') \Big)$$

$$= \int d^3x \sqrt{g} \big(\bar{\phi} \mathbf{H}^\dagger \mathbf{H} \phi + \bar{\chi} \mathbf{H}^\dagger \chi + \bar{\phi}' \phi' - \bar{\chi}' \mathbf{H} \chi' \big)$$

$$Z_{\text{1-loop}} = \frac{\det(\mathbf{H}^\dagger)_r}{\det(\mathbf{H}^\dagger \mathbf{H})_r} \frac{\det(-\mathbf{H})_{r-2}}{\det(\mathbf{1})_{r-2}} = \frac{\det(-\mathbf{H})_{r-2}}{\det(\mathbf{H})_r}$$

The Ratio of Determinants

We obtained : $$Z_{\text{1-loop}} = \frac{\det(\mathbf{H})_{r-2}}{\det(\mathbf{H})_r}\ .$$

There is a pair of differential operators

$$\mathcal{J}^- \equiv \xi\gamma^m\xi D_m, \quad \mathcal{J}^+ \equiv \bar{\xi}\gamma^m\bar{\xi} D_m,$$

which commutes with $\mathbf{Q}^2$ and shifts the R-charge by ± 2.

Generic eigenmodes of $\mathbf{Q}^2$ are paired.

$$\longrightarrow \quad Z_{\text{1-loop}} = \frac{\det(\mathbf{H})_{r-2}}{\det(\mathbf{H})_r} = \frac{\det(\mathbf{H})_{\mathrm{Ker}\mathcal{J}^-}}{\det(\mathbf{H})_{\mathrm{Ker}\mathcal{J}^+}}\ .$$

$$\begin{tikzcd} \boxed{\phi \quad [\mathbf{R}_{U(1)} = r]} \arrow[d, shift right, "\mathcal{J}^+"'] \\ \boxed{\phi' \quad [\mathbf{R}_{U(1)} = r-2]} \arrow[u, shift right, "\mathcal{J}^-"'] \end{tikzcd}$$

Determinant and Index

- The ratio of determinants

$$Z_{\text{1-loop}} = \frac{\det(\mathbf{H})_{r-2}}{\det(\mathbf{H})_r} = \frac{\det(\mathbf{H})_{\mathrm{Ker}\mathcal{J}^-}}{\det(\mathbf{H})_{\mathrm{Ker}\mathcal{J}^+}} = \frac{\prod_i \tilde{\lambda}_i}{\prod_i \lambda_i},$$

is related to the **"equivariant index"**,

$$\mathrm{Str}(e^{\mathbf{H}}) \equiv \mathrm{Tr}(e^{\mathbf{H}})_{\mathrm{Ker}\mathcal{J}^+} - \mathrm{Tr}(e^{\mathbf{H}})_{\mathrm{Ker}\mathcal{J}^-} = \sum_i e^{\lambda_i} - \sum_i e^{\tilde{\lambda}_i},$$

which is a generalization of the ordinary **index.**

$$\mathrm{Ind}(\mathcal{J}^+) \equiv \mathrm{Dim}(\mathrm{Ker}\mathcal{J}^+) - \mathrm{Dim}(\mathrm{Ker}\mathcal{J}^-).$$

- Sometimes $\mathrm{Tr}(e^{\mathbf{H}})_r,\ \mathrm{Tr}(e^{\mathbf{H}})_{r-2}$ are not well-defined

  because of infinite degeneracy for each eigenvalue of **H**.

  The traces $\mathrm{Tr}(e^{\mathbf{H}})_{\mathrm{Ker}\mathcal{J}^{\pm}}$ may be well-defined even in such cases.

Evaluation of Determinant

Let us compute $\det(\mathbf{H})_{\mathrm{Ker}\mathcal{J}^+}$.

$$\mathcal{J}^+\phi \equiv \bar{\xi}\gamma^m\bar{\xi}D_m\phi = 0$$

$$e^{i(\varphi-\chi)}\left[\frac{i}{\ell}\frac{\sin\theta}{\cos\theta}(\partial_\varphi - irV_\varphi) + \frac{i}{\tilde{\ell}}\frac{\cos\theta}{\sin\theta}(\partial_\chi - irV_\chi) - \frac{1}{f}\partial_\theta\right]\phi = 0.$$

where $\quad V_\varphi = -\frac{1}{2}\Big(1-\frac{\ell}{f}\Big),\;\; V_\chi = \frac{1}{2}\Big(1-\frac{\tilde{\ell}}{f}\Big),\;\; f(\theta) \equiv \sqrt{\ell^2\sin^2\theta + \tilde{\ell}^2\cos^2\theta}.$

ansatz:

$$\phi := \Phi(\theta)e^{im\varphi-in\chi} \implies 0 = \left[\partial_\theta - \frac{f}{\ell}\frac{\sin\theta}{\cos\theta}(m-rV_\varphi) + \frac{f}{\tilde{\ell}}\frac{\cos\theta}{\sin\theta}(n+rV_\chi)\right]\Phi$$

Need to solve this over $\theta \in [0, \frac{\pi}{2}]$.

The behavior of Φ near the boundaries:

$$\Phi(\theta) \sim \cos^m\theta\sin^n\theta \quad \boxed{m, n \geq 0.}$$

Evaluation of Determinant

$\mathrm{Ker}\mathcal{J}^{+}$ is spanned by $\boxed{\phi = \Phi(\theta)e^{im\varphi - in\chi}, \quad \Phi(\theta) \sim \cos^m\theta\sin^n\theta \quad (m,n\geq 0)}$

$$\begin{aligned}\mathbf{H} &\equiv -\frac{i}{\ell}\partial_\varphi + \frac{i}{\tilde{\ell}}\partial_\chi - i\mathbf{Gauge}(a) + \Big(\frac{1}{2\ell} + \frac{1}{2\tilde{\ell}}\Big)\mathbf{R}_{U(1)} \\ &= \frac{m}{\ell} + \frac{n}{\tilde{\ell}} - ia + \frac{r}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\right) \ \text{ on } \phi.\end{aligned}$$

$$\det(\mathbf{H})_{\mathrm{Ker}\mathcal{J}^{+}} = \prod_{m,n\geq 0}\left\{\frac{m}{\ell} + \frac{m}{\tilde{\ell}} - ia + \frac{r}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\right)\right\}$$

Similarly,

$$\det(\mathbf{H})_{\mathrm{Ker}\mathcal{J}^{-}} = \prod_{m,n\geq 0}\left\{-\frac{m}{\ell} - \frac{n}{\tilde{\ell}} - ia + \frac{r-2}{2}\Big(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\Big)\right\}$$

$$\begin{aligned}Z_{\text{1-loop}} &\equiv \frac{\det(\mathbf{H})_{\phi'\in\mathrm{Ker}(\mathcal{J}^{-})}}{\det(\mathbf{H})_{\phi\in\mathrm{Ker}(\mathcal{J}^{+})}} = \prod_{m,n\in\mathbb{Z}_{\geq 0}}\frac{mb + nb^{-1} + \frac{2-r}{2}Q + i\hat{a}}{mb + nb^{-1} + \frac{r}{2}Q - i\hat{a}} \\ &= s_b\big(-\hat{a} + \tfrac{iQ}{2}(1-r)\big). \qquad \left(b \equiv \sqrt{\ell/\tilde{\ell}},\ Q \equiv b + b^{-1},\ \hat{a} \equiv \sqrt{\ell\tilde{\ell}}a\right)\end{aligned}$$

Determinant: Vector Multiplet

We next study the integral over the vectormultiplet fields $(A_m, \sigma, \lambda, \bar{\lambda}, D)$

around the saddle point $\langle\sigma\rangle = a.$ Recall

SUSY : $$\mathbf{Q}A_m = -\frac{i}{2}(\xi\gamma_m\bar{\lambda} + \bar{\xi}\gamma_m\lambda), \qquad \mathbf{Q}\lambda = \frac{1}{2}\gamma^{mn}\xi F_{mn} - \xi D - i\gamma^m\xi D_m\sigma,$$

$$\mathbf{Q}\sigma = \frac{1}{2}(\xi\bar{\lambda} - \bar{\xi}\lambda), \qquad \mathbf{Q}\bar{\lambda} = \frac{1}{2}\gamma^{mn}\bar{\xi}F_{mn} + \bar{\xi}D + i\gamma^m\bar{\xi}D_m\sigma,$$

$$\mathbf{Q}D = \frac{i}{2}\xi\left(\gamma^m D_m\bar{\lambda} + [\sigma, \bar{\lambda}] + iM\bar{\lambda}\right) - \frac{i}{2}\bar{\xi}\left(\gamma^m D_m\lambda - [\sigma, \lambda] + iM\lambda\right).$$

The square of SUSY : $$\mathbf{Q}^2 = i\boldsymbol{\mathcal{L}}_v - i\mathbf{Gauge}(\Phi) + \left(\frac{1}{2\ell} + \frac{1}{2\tilde{\ell}}\right)\mathbf{R}_{U(1)}$$

$$v^m\partial_m = -\frac{1}{\ell}\partial_\varphi + \frac{1}{\tilde{\ell}}\partial_\chi, \qquad \Phi \equiv \sigma + \frac{i}{\ell}A_\varphi - \frac{i}{\tilde{\ell}}A_\chi.$$

We need to gauge-fix.

Pestun's Gauge Fixing

We introduce the ghost multiplet $(c, \bar{c}, B)$ and BRST symmetry $\mathbf{Q}_{\mathrm{B}}$

- $\mathbf{Q}_{\mathrm{B}} = i\mathbf{Gauge}(c)$ on physical fields

$$\mathbf{Q}_{\mathrm{B}} A_m = D_m c, \quad \mathbf{Q}_{\mathrm{B}} \lambda = i\{c, \lambda\}, \cdots$$

- $\mathbf{Q}_{\mathrm{B}} c = icc, \quad \mathbf{Q}_{\mathrm{B}} \bar{c} = B, \quad \mathbf{Q}_{\mathrm{B}} B = 0.$ $\longrightarrow$ $\boxed{\mathbf{Q}_{\mathrm{B}}^2 = 0.}$

We determine the SUSY transformation rule of ghost so that

$$\widehat{\mathbf{Q}}^2 \equiv (\mathbf{Q} + \mathbf{Q}_{\mathrm{B}})^2 = i\mathcal{L}_v - i\mathbf{Gauge}(a) + \left(\frac{1}{2\ell} + \frac{1}{2\tilde{\ell}}\right)\mathbf{R}_{U(1)}$$

on the saddle point labeled by $\langle \sigma \rangle = a.$

$$\longrightarrow \boxed{\begin{array}{l} \mathbf{Q} c = \Phi - a, \qquad \left(\Phi \equiv \sigma + \frac{i}{\ell} A_\varphi - \frac{i}{\tilde{\ell}} A_\chi\right) \\ \mathbf{Q} \bar{c} = 0, \quad \mathbf{Q} B = i\mathcal{L}_v \bar{c} - i[a, \bar{c}]. \end{array}}$$

Pestun's Gauge Fixing

Let us derive $\mathbf{Q}c = \Phi - a.$ $\quad \left(\Phi \equiv \sigma + \frac{i}{\ell}A_\varphi - \frac{i}{\tilde{\ell}}A_\chi\right)$

We require

$$\widehat{\mathbf{Q}}^2 \equiv (\mathbf{Q}+\mathbf{Q}_{\rm B})^2 = i\mathcal{L}_v - i\mathbf{Gauge}(a) + \left(\frac{1}{2\ell}+\frac{1}{2\tilde{\ell}}\right)\mathbf{R}_{U(1)}$$

$$= i\mathcal{L}_v - i\mathbf{Gauge}(\Phi) + \left(\frac{1}{2\ell}+\frac{1}{2\tilde{\ell}}\right)\mathbf{R}_{U(1)} + \{\mathbf{Q},\mathbf{Q}_{\rm B}\}.$$

$$\{\mathbf{Q},\mathbf{Q}_{\rm B}\} = i\mathbf{Gauge}(\Phi - a).$$

Take any physical field *X*. Then

$$\{\mathbf{Q},\mathbf{Q}_{\rm B}\}X = \mathbf{Q}\big[i\mathbf{Gauge}(c)X\big] + i\mathbf{Gauge}(c)\big[\mathbf{Q}X\big] = i\mathbf{Gauge}(\mathbf{Q}c)X$$

$$= i\mathbf{Gauge}(\Phi - a)X. \longrightarrow \mathbf{Q}c = \Phi - a.$$

We use the **total supercharge** $\widehat{\mathbf{Q}} \equiv \mathbf{Q} + \mathbf{Q}_{\rm B}$ for localization.

Cohomological Variables

- The vector + ghost multiplet $\{A_m, \sigma, \lambda, \bar{\lambda}, D; c, \bar{c}, B\}$ consists of 6 bosons and 6 fermions.

- Move to a new set of fields (all Lorentz scalars).

		$\mathbf{R}_{U(1)}$
$\phi_2 \equiv \xi\gamma^m\xi A_m$	$\chi_2 \equiv \widehat{\mathbf{Q}}\phi_2 = \xi\gamma^m\xi D_m c - i\xi\lambda$	$+2$
$\phi_0 \equiv \bar{\xi}\gamma^m\xi A_m$	$\chi_0 \equiv \widehat{\mathbf{Q}}\phi_0 = \bar{\xi}\gamma^m\xi D_m c + \frac{i}{2}(\xi\bar{\lambda} - \bar{\xi}\lambda)$	0
$\phi_{-2} \equiv \bar{\xi}\gamma^m\bar{\xi} A_m$	$\chi_{-2} \equiv \widehat{\mathbf{Q}}\phi_{-2} = \bar{\xi}\gamma^m\bar{\xi} D_m c + i\bar{\xi}\bar{\lambda}$	-2
$\chi_0' \equiv \xi\bar{\lambda} + \bar{\xi}\lambda$	$\phi_0' \equiv \widehat{\mathbf{Q}}\chi_0' = -2D + \bar{\xi}\gamma^{mn}\xi F_{mn} + i\{c, \chi_0'\}$	0
$\bar{c}$	$\widehat{\mathbf{Q}}\bar{c} = B$	0
c	$\widehat{\mathbf{Q}}c = \sigma - a + i\bar{\xi}\gamma^m\xi A_m + icc$	0

Similar to a chiral multiplet in the adjoint rep, R-charge +2.

Determinant: Vectormultiplet

= the determinant for an adjoint chiral multiplet with $r = 2.$

$$Z_{\text{1-loop}} = \prod_{\boldsymbol{\alpha}\in\Delta} s_b(-\hat{\boldsymbol{a}}\boldsymbol{\alpha} - \tfrac{iQ}{2})$$

$$= \prod_{\boldsymbol{\alpha}\in\Delta_+} s_b(\hat{\boldsymbol{a}}\boldsymbol{\alpha} - \tfrac{ib}{2} - \tfrac{i}{2b}) s_b(-\hat{\boldsymbol{a}}\boldsymbol{\alpha} - \tfrac{ib}{2} - \tfrac{i}{2b})$$

use $$s_b(-\tfrac{ib}{2} + x) s_b(-\tfrac{ib}{2} - x) = 2\cosh(\pi b x),$$

$$s_b(-\tfrac{i}{2b} + x) s_b(-\tfrac{i}{2b} - x) = 2\cosh(\pi x/b),$$

$$= \prod_{\boldsymbol{\alpha}\in\Delta_+} 4\sinh(\pi b \hat{\boldsymbol{a}}\boldsymbol{\alpha}) \sinh(\pi b^{-1} \hat{\boldsymbol{a}}\boldsymbol{\alpha}).$$

Ellipsoid Partition Function: Summary

Ellipsoid: $$\frac{x_0^2+x_1^2}{\ell^2}+\frac{x_2^2+x_3^2}{\tilde{\ell}^2}=1$$

metric: $$ds^2=\ell^2\cos^2\theta d\varphi^2+\tilde{\ell}^2\sin^2\theta d\chi^2+f(\theta)^2d\theta^2$$

background fields: $$V_m dx^m=-\frac{1}{2}\Big(1-\frac{\ell}{f}\Big)d\varphi+\frac{1}{2}\Big(1-\frac{\tilde{\ell}}{f}\Big)d\chi,\quad M=\frac{1}{2f}.$$

Exact partition function:

$$Z_{\mathrm{ell}}=\frac{1}{|\mathcal{W}|}\int\prod_{i=1}^{r}d\hat{a}_i\prod_{\boldsymbol{\alpha}\in\Delta_+}4\sinh(\pi\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha}b)\sinh(\pi\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha}/b)\cdot\exp\{-S(\hat{\boldsymbol{a}})\}$$
$$\cdot\prod_j\Big[\prod_{\boldsymbol{w}\in R_j}s_b\Big(\frac{iQ}{2}(1-r_j)-\hat{\boldsymbol{a}}\cdot\boldsymbol{w}\Big)\Big]$$

$$e^{-S_{\mathrm{CS}}(\hat{a})}=e^{i\pi k\mathrm{Tr}(\hat{a}^2)},\qquad e^{-S_{\mathrm{FI}}(\hat{a})}=e^{4\pi i\hat{\zeta}a}.$$

$$s_b(x)\equiv\prod_{m,n\in\mathbb{Z}_{\geq 0}}\frac{mb+nb^{-1}+\frac{Q}{2}-ix}{mb+nb^{-1}+\frac{Q}{2}+ix}.$$

$$\boxed{b\equiv\sqrt{\ell/\tilde{\ell}}.}$$

IV. Yang-Mills Instantons

- Basics of Instantons
- Omega Deformation
- ADHM Construction
- Volume of Instanton Moduli Space

Motivations

- We would like to study 4D SUSY gauge theories through exact computations of observables (e.g. sphere partition functions)

- Localization principle reduces the path integral to a finite-dimensional integral over the space of saddle points, but there is an interesting **instanton** correction.

Preliminary: 4D Spinors

Gamma matrices $\{\gamma^a\}_{a=1}^4$ are 4×4 matrices satisfying $\{\gamma^a,\gamma^b\}=2\delta^{ab}$.

We choose them so that

$$\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} \mathbf{1}_{2\times 2} & 0 \\ 0 & -\mathbf{1}_{2\times 2} \end{pmatrix}.$$

A 4-component spinor decomposes into a chiral spinor $(\gamma^5=+1)$ and an anti-chiral spinor $(\gamma^5=-1)$.

The gamma matrices take the form

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}, \qquad \begin{aligned} \sigma^a\bar{\sigma}^b + \sigma^b\bar{\sigma}^a &= 2\delta^{ab}\mathbf{1}_{2\times 2}, \\ \bar{\sigma}^a\sigma^b + \bar{\sigma}^b\sigma^a &= 2\delta^{ab}\mathbf{1}_{2\times 2}. \end{aligned}$$

Our choice :

$$\sigma^i = -i\boldsymbol{\tau}^i,\ \sigma^4 = \mathbf{1},$$
$$\bar{\sigma}^i = +i\boldsymbol{\tau}^i,\ \bar{\sigma}^4 = \mathbf{1},$$

($\boldsymbol{\tau}^i$: Pauli's matrices)

We also use

$$\gamma^{ab} = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a) = \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \bar{\sigma}^{ab} \end{pmatrix}.$$

Note: $\sigma^{ab} = -\frac{1}{2}\varepsilon^{abcd}\sigma^{cd}, \quad \bar{\sigma}^{ab} = +\frac{1}{2}\varepsilon^{abcd}\bar{\sigma}^{cd}.$

Basics of Instantons

4D Euclidean Yang-Mills Theory

- **Action**

$$S = \int d^4x \mathcal{L}, \quad \mathcal{L} = \mathrm{Tr}\left[\frac{1}{2g^2} F_{mn}F_{mn} + \frac{i\theta}{16\pi^2} F_{mn}\tilde{F}_{mn}\right]$$

$$F_{mn} \equiv \partial_m A_n - \partial_n A_m - i[A_m, A_n], \quad \tilde{F}_{mn} \equiv \frac{1}{2}\varepsilon_{mnkl}F_{kl}$$

g : Yang-Mills coupling

θ : theta angle

- Tr is the standard trace.
- Theta angle is periodic, $\theta \sim \theta + 2\pi$, because . . .

Instanton number : $k \equiv -\dfrac{1}{16\pi^2}\displaystyle\int d^4x \mathrm{Tr}[F_{mn}\tilde{F}_{mn}] \in \mathbb{Z}.$

$$k \equiv -\frac{1}{2(2\pi)^2}\int \mathrm{Tr}\left[F \wedge F\right], \quad F \equiv \frac{1}{2}F_{mn}dx^m dx^n$$

Q. What is the action-minimizing configuration for a given k ?

Instantons

$$k \equiv -\frac{1}{16\pi^2}\int d^4x \mathrm{Tr}[F_{mn}\tilde{F}_{mn}].$$

$$\rightarrow S = \int d^4x \mathrm{Tr}\left[\frac{1}{2g^2}F_{mn}F_{mn} + \frac{i\theta}{16\pi^2}F_{mn}\tilde{F}_{mn}\right]$$

$$= \int d^4x \mathrm{Tr}\left[\frac{1}{g^2}F^{\pm}_{mn}F^{\pm}_{mn} + \left(\frac{i\theta}{16\pi^2} \mp \frac{1}{2g^2}\right)F_{mn}\tilde{F}_{mn}\right]$$

$$= \begin{cases} \int d^4x \mathrm{Tr}\left[\frac{1}{g^2}F^{+}_{mn}F^{+}_{mn}\right] - 2\pi i\tau k \\ \int d^4x \mathrm{Tr}\left[\frac{1}{g^2}F^{-}_{mn}F^{-}_{mn}\right] - 2\pi i\bar{\tau} k \end{cases}$$

(anti-)self-dual part

$$F^{\pm} \equiv \frac{1}{2}(F \pm \tilde{F})_{mn}$$

complex gauge coupling

$$\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$$

For $k > 0$, the action minimizing configuration satisfies

the anti-self-duality $F^{+}_{mn} = 0.$

The solutions are called "k-instanton solutions".

$|k|$ anti-instanton solutions are defined (for $k < 0$) in the same way.

The action : $\exp(-S) = \begin{cases} \exp(2\pi i\tau k) & (k > 0) \\ \exp(2\pi i\bar{\tau} k) & (k < 0) \end{cases}$

1-Instanton Solution for SU(2)

(Belavin,Polyakov, Schwarz, Tyupkin)

<u>Ansatz:</u>

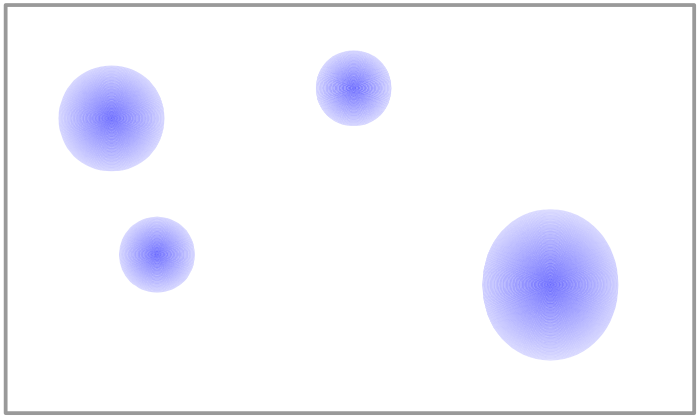
$$\boxed{A_m = if(x^2)\cdot \mathbf{x}\partial_m\bar{\mathbf{x}} \quad (\mathbf{x}\equiv x^m\sigma_m,\ \bar{\mathbf{x}}\equiv x^m\bar{\sigma}_m)}$$

$$\begin{aligned}
F_{mn} &\equiv \partial_m A_n - \partial_n A_m - i[A_m, A_n] \\
&= \partial_m\big(if\,\mathbf{x}\partial_n\bar{\mathbf{x}}\big) - \partial_n\big(if\,\mathbf{x}\partial_m\bar{\mathbf{x}}\big) + if^2\left\{\mathbf{x}\partial_m\bar{\mathbf{x}}\cdot\mathbf{x}\partial_n\bar{\mathbf{x}} - \mathbf{x}\partial_n\bar{\mathbf{x}}\cdot\mathbf{x}\partial_m\bar{\mathbf{x}}\right\} \\
&= i\left\{\partial_m f\cdot\mathbf{x}\partial_n\bar{\mathbf{x}} - \partial_n f\cdot\mathbf{x}\partial_m\bar{\mathbf{x}}\right\} + if\left\{\partial_m\mathbf{x}\partial_n\bar{\mathbf{x}} - \partial_n\mathbf{x}\partial_m\bar{\mathbf{x}}\right\} \\
&\quad + if^2\left\{\mathbf{x}\,\partial_m(\bar{\mathbf{x}}\mathbf{x})\partial_n\bar{\mathbf{x}} - \mathbf{x}\,\partial_n(\bar{\mathbf{x}}\mathbf{x})\partial_m\bar{\mathbf{x}}\right\} - if^2\cdot(\mathbf{x}\bar{\mathbf{x}})\left\{\partial_m\mathbf{x}\partial_n\bar{\mathbf{x}} - \partial_n\mathbf{x}\partial_m\bar{\mathbf{x}}\right\} \\
&= i\Big\{(\partial_m f + f^2\partial_m(x^2))\mathbf{x}\partial_n\bar{\mathbf{x}} - (m\leftrightarrow n)\Big\} + \underbrace{i(f - x^2f^2)\cdot 2\sigma_{mn}}_{\text{ASD}}
\end{aligned}$$

The first term vanishes when $f' + f^2 = 0$.

$$\boxed{f(x^2) = \frac{1}{x^2+\rho^2}, \quad \rho = \text{(size of instanton).}}$$

More general k-instanton solutions



. . . look like k blobs in $\mathbb{R}^4$.

. . . form a moduli space $\mathcal{M}_k$.

$\dim\mathcal{M}_k = 4Nk$ for k instantons of U(N).

- If saddle point approximation is accurate, the YM partition would be

$$Z = \sum_k q^k \int_{\mathcal{M}_k} d(\text{moduli}) \cdot Z_{\text{1-loop}} \cdot \Big\{1 + \cdots (\text{perturbative series})\Big\}$$
$$\sim \sum_k q^k \,\mathrm{vol}(\mathcal{M}_k). \qquad \left(q \equiv e^{2\pi i\tau}\right)$$

This is exact in "topologically twisted theories".

- However, $\mathrm{vol}(\mathcal{M}_k)$ has various source of infinities.

UV : instantons can become zero-size.

IR : instantons can move anywhere in $\mathbb{R}^4$.

Omega Deformation

Omega deformation (Nekrasov)

- The volume of a symplectic manifold (such as $\mathcal{M}_k$) can be deformed by using its symmetry.
- The **"Omega-deformed" volume** can be evaluated as a sum over fixed point contributions.

[def] symplectic manifold

= a (2*n*)-dimensional manifold with a non-degenerate closed 2-form

$$\omega \equiv \frac{1}{2}\omega_{ij}(x)dx^i \wedge dx^j \quad \big(\omega_{ij} : \text{ regular matrix}\big)$$

symplectic volume form: $\dfrac{\omega^n}{n!}$.

(example) phase space, $\omega = \sum_i dp_i \wedge dq_i$

Omega deformation

Let *M* be a symplectic manifold and v a vector field generating its symmetry,

$$\delta x^i = v^i(x), \quad \delta\omega = \mathcal{L}_v\omega = 0.$$

$$\mathcal{L}_v\omega = (di_v + i_v d)\omega = d(i_v\omega) = 0.$$

A function *H* is called the moment map function for *v* if it satisfies

$$dH = i_v\omega, \text{ or in components, } \partial_j H = v^i\omega_{ij}.$$

The **Omega-deformed volume** of *M* is then defined by

$$Z(\beta) \equiv \int_M \frac{\omega^n}{n!} e^{-\beta H}.$$

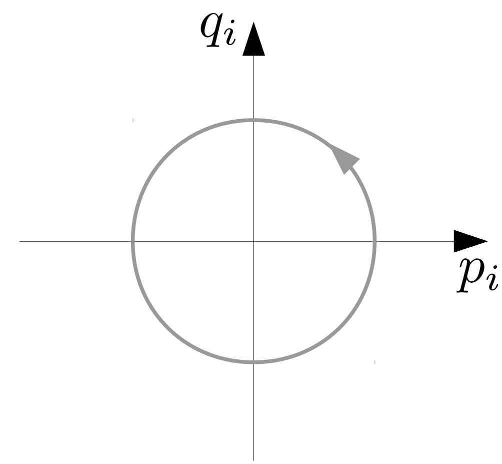
β is the parameter of Omega-deformation.

$Z(0)$ is the ordinary volume.

Omega deformation

[example 1] harmonic oscillators

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i, \quad H = \sum_i \frac{\epsilon_i}{2}(p_i^2 + q_i^2)$$



$$\partial_{p_i} H = \epsilon_i p_i = \omega_{p_i q_i} v^{q_i} = v^{q_i},$$

$$\partial_{q_i} H = \epsilon_i q_i = \omega_{q_i p_i} v^{p_i} = -v^{p_i},$$

$$\begin{pmatrix} v^{p_i} \\ v^{q_i} \end{pmatrix} = \begin{pmatrix} -\epsilon_i q_i \\ +\epsilon_i p_i \end{pmatrix}.$$

Naive volume of the phase space is infinite,

but Omega-deformed volume is finite.

$$\int \frac{\omega^n}{n!} = \infty \longrightarrow \int \frac{\omega^n}{n!} e^{-\beta H} = \left(\frac{2\pi}{\beta}\right)^n \frac{1}{\epsilon_1 \cdots \epsilon_n}.$$

(Gaussian integral)

Omega deformation can be used as an IR regulator.

Duistermaat-Heckman's Theorem

The omega-deformed volume is a sum of fixed-point contributions.

$$Z(\beta) = \int_M \frac{\omega^n}{n!} e^{-\beta H} = \left(\frac{2\pi}{\beta}\right)^n \sum_p \frac{e^{-\beta H(p)}}{e(p)}.$$

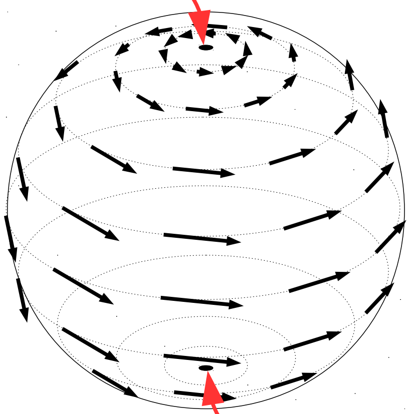
Here
- p is a fixed point at which $v = 0$.
- $e(p)$ is a product of weights calculated as follows,

If $\omega \simeq \prod_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}, \quad H \simeq \sum_{i=1}^{n} \frac{\epsilon_i}{2}(x_{2i-1}^2 + x_{2i}^2)$

near the fixed point $p: \; x_i = 0, \quad$ then $e(p) \equiv \prod_i \epsilon_i.$

This means that the saddle point approximation is exact.

Omega deformation



[example 2] unit sphere

$$\omega = \sin\theta d\theta d\varphi = d(-\cos\theta)d\varphi,$$

$$v = \partial_\varphi \longrightarrow H = -\cos\theta.$$

By an elementary integral we find

$$Z(\beta) = \int \omega e^{-\beta H} = \int d(-\cos\theta)d\varphi e^{\beta\cos\theta} = \frac{2\pi}{\beta}\left(e^{\beta} - e^{-\beta}\right)$$

The two terms in the last line can be interpreted as contributions from fixed points,

$$\text{North pole:} \quad \theta = 0, \ \text{weight} \ +1,$$

$$\text{South pole:} \quad \theta = \pi, \ \text{weight} \ -1.$$

DH Theorem and SUSY

The Omega-deformed volume can be written in the form of **SUSY integral.**

$$Z(\beta) = \int \frac{\omega^n}{n!} e^{-\beta H} = \int e^{\omega - \beta H}$$

$$= \int \prod_{i=1}^{2n} dx^i d\psi^i \exp\left[\tfrac{1}{2}\omega_{ij}(x)\psi^i\psi^j - \beta H(x)\right] = \int DxD\psi \exp(-S)$$

SUSY: $\mathbf{Q}x^i = \psi^i, \quad \mathbf{Q}\psi^i = v^i(x), \quad \mathbf{Q}S = 0.$

The integral localizes onto fixed points (zeroes of v).

Note: this **Q** is just an operator $d + \beta i_v$ acting on differential forms.

$$\mathbf{Q}S = 0 \iff (d + \beta i_v)(\omega - \beta H) = 0.$$

Since **Q** squares to $\beta\mathcal{L}_v,$ it defines a cohomology on the space of

$\mathcal{L}_v$ -invariant differential forms, called **equivariant cohomology**.

ADHM Construction

ADHM Construction

- we would like to compute the Omega-deformed volume of instanton moduli spaces.
- There is a nice parametrization of the moduli space (and construction of instanton solutions) due to Atiyah-Drinfeld-Hitchin-Manin.

There is an 1-1 correspondence between U(N) k-instanton solutions and a set of matrices called “**ADHM data**”.

ADHM data

is the following set of complex matrices

$$B_1,\ B_2\ (k \times k), \quad I\ (k \times N), \quad J\ (N \times K)$$

which satisfies the ADHM equation

$$\mu_{\mathbb{C}} := [B_1, B_2] + IJ = 0,$$
$$\mu_{\mathbb{R}} := [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0,$$

and is identified by the U(k) equivalence relation

$$(B_1, B_2, I, J) \sim (UB_qU^{-1}, UB_2U^{-1}, UI, JU^{-1})$$

- The number of independent parameters is

$$2k^2 + 2k^2 + 2kN + 2kN - 2k^2 - k^2 - k^2 = 4kN.$$

- Slightly different definition :

  matrices satisfying $\mu_{\mathbb{C}} = 0$, subject to the GL(k) equivalence

Construction of U(N) k-instantons

- Let X^1, X^2, X^3, X^4 : $k \times k$ Hermite matrices

  H_+, H_- : $k \times N$ complex matrices

  satisfying some conditions to be determined.

Identified later with (B_1, B_2, I, J)

- For $(x^1, x^2, x^3, x^4) \in \mathbb{R}^4$, define

$$\bar{\mathbf{x}} \equiv (x^a \bar{\sigma}^a) \otimes \mathbf{1}_k = \begin{pmatrix} (x^4 + ix^3)\mathbf{1}_k & (x^2 + ix^1)\mathbf{1}_k \\ -(x^2 - ix^1)\mathbf{1}_k & (x^4 - ix^3)\mathbf{1}_k \end{pmatrix},$$

$$\bar{\mathbf{X}} \equiv \bar{\sigma}^a \otimes X^a = \begin{pmatrix} X^4 + iX^3 & X^2 + iX^1 \\ -(X^2 - iX^1) & X^4 - iX^3 \end{pmatrix}, \quad \mathbf{H} \equiv \begin{pmatrix} H_+ \\ H_- \end{pmatrix}.$$

$$D(x) \equiv \left(\bar{\mathbf{x}} - \bar{\mathbf{X}}, \mathbf{H}\right) \quad : \quad 2k \times (2k + N) \text{ matrix}$$

$$= \begin{pmatrix} (x^4 + ix^3)\mathbf{1}_k + X^4 + iX^3 & (x^2 + ix^1)\mathbf{1}_k + X^2 + iX^1 & H_+ \\ -(x^2 - ix^1)\mathbf{1}_k - (X^2 - iX^1) & (x^4 - ix^3)\mathbf{1}_k + X^4 - iX^3 & H_- \end{pmatrix}$$

Construction of U(N) k-instantons

$$D(x) \equiv \left(\bar{\mathbf{x}} - \bar{\mathbf{X}}, \mathbf{H}\right) \quad : \quad 2k \times (2k+N) \text{ matrix}$$

$$= \begin{pmatrix} (x^4 + ix^3)\mathbf{1}_k + X^4 + iX^3 & (x^2 + ix^1)\mathbf{1}_k + X^2 + iX^1 & H_+ \\ -(x^2 - ix^1)\mathbf{1}_k - (X^2 - iX^1) & (x^4 - ix^3)\mathbf{1}_k + X^4 - iX^3 & H_- \end{pmatrix}$$

Condition 1:

$$\mathrm{rank} D(x) = 2k. \quad (\forall x \in \mathbb{R}^4)$$

$D(x)D(x)^\dagger$ is a $2k \times 2k$ invertible matrix.

- Then there is a $(2k+N) \times N$ matrix $u(x)$ satisfying

$$D(x)u(x) = 0, \quad u(x)^\dagger u(x) = \mathbf{1}_N,$$

which is unique up to U(N) gauge rotations $u(x) \to u(x)g(x)$.

- Let us then define a U(N) gauge field by $A = iu^\dagger du$.

Construction of U(N) k-instantons

remember: $D(x)u(x) = 0, \quad u(x)^\dagger u(x) = \mathbf{1}_N$

$A = iu^\dagger du$ is anti-self-dual **under certain conditions.**

$$\begin{aligned} F \equiv dA - iA^2 &= idu^\dagger du + iu^\dagger du u^\dagger du \\ &= idu^\dagger (1 - uu^\dagger) du \\ &= idu^\dagger D^\dagger (DD^\dagger)^{-1} D du \\ &= iu^\dagger (dD^\dagger)(DD^\dagger)^{-1}(dD) u \end{aligned}$$

$$\begin{aligned} uu^\dagger &= \text{projection op. to Ker}D \\ (1 - uu^\dagger) &= \text{projection op. to } (\text{Ker}D)^\perp \\ &= D^\dagger (DD^\dagger)^{-1} D \end{aligned}$$

recall : $D = \left(\bar{\mathbf{x}} - \bar{\mathbf{X}}, \mathbf{H}\right), \quad dD = \left(d\bar{\mathbf{x}}, 0\right), \quad d\bar{\mathbf{x}} \equiv dx^a \bar{\sigma}^a \otimes \mathbf{1}_k.$

$$F = iu^\dagger \begin{pmatrix} d\mathbf{x} \\ 0 \end{pmatrix} (DD^\dagger)^{-1} \begin{pmatrix} d\bar{\mathbf{x}} & 0 \end{pmatrix} u$$

Condition 2: $\boxed{DD^\dagger \text{ commutes with } d\mathbf{x} \text{ or } d\bar{\mathbf{x}}.}$

Then $F = iu^\dagger \begin{pmatrix} d\mathbf{x}d\bar{\mathbf{x}}(DD^{\dagger\,-1}) & 0 \\ 0 & 0 \end{pmatrix} u, \qquad d\mathbf{x}d\bar{\mathbf{x}} = dx^m dx^n \sigma^{mn} \otimes \mathbf{1}_k.$

Construction of U(N) k-instantons

The condition 2 amounts to : $DD^\dagger$ commutes with $\boldsymbol{\tau}^a \otimes \mathbf{1}_k$ $(a = 1, 2, 3)$.

$$DD^\dagger = \underbrace{\bar{\mathbf{x}}\mathbf{x} - \bar{\mathbf{x}}\mathbf{X}}\, - \bar{\mathbf{X}}\mathbf{x} + \bar{\mathbf{X}}\mathbf{X} + \mathbf{H}\mathbf{H}^\dagger$$

$$= (x^a \bar{\sigma}^a \otimes \mathbf{1}_k)(x^b \sigma^b \otimes \mathbf{1}_k) - (x^a \bar{\sigma}^a \otimes \mathbf{1}_k)(\sigma^b \otimes X^b) - (\bar{\sigma}^b \otimes X^b)(x^a \sigma^a \otimes \mathbf{1}_k) + \cdots$$

$$= (x^a x^b \bar{\sigma}^a \sigma^b) \otimes \mathbf{1}_k - (x^a \bar{\sigma}^a \sigma^b + x^a \bar{\sigma}^b \sigma^a) \otimes X^b + \cdots$$

$$= \underbrace{x^2\, \mathbf{1}_2 \otimes \mathbf{1}_k - 2x^a\, \mathbf{1}_2 \otimes X^a}_{\text{commutes with } \boldsymbol{\tau}^a \otimes \mathbf{1}_k.} + \bar{\mathbf{X}}\mathbf{X} + \mathbf{H}\mathbf{H}^\dagger.$$

Denote $$\bar{\mathbf{X}} = \begin{pmatrix} X^4 + iX^3 & X^2 + iX^1 \\ -X^2 + iX^1 & X^4 - iX^3 \end{pmatrix} \equiv \begin{pmatrix} iB_2^\dagger & iB_1^\dagger \\ iB_1 & -iB_2 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \equiv \begin{pmatrix} J^\dagger \\ I \end{pmatrix}.$$

Then $$\bar{\mathbf{X}}\mathbf{X} + \mathbf{H}\mathbf{H}^\dagger = \begin{pmatrix} B_1^\dagger B_1 + B_2^\dagger B_2 + J^\dagger J & B_2^\dagger B_1^\dagger - B_1^\dagger B_2^\dagger + J^\dagger I^\dagger \\ B_1 B_2 - B_2 B_1 + IJ & B_1 B_1^\dagger + B_2 B_2^\dagger + II^\dagger \end{pmatrix}.$$

It commutes with $\boldsymbol{\tau}^a \otimes \mathbf{1}_k$ if $$\mu_{\mathbb{C}} := [B_1, B_2] + IJ = 0,$$

$$\mu_{\mathbb{R}} := [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0.$$

Construction of U(N) k-instantons

Let us show that $A \equiv iu^\dagger du$ has instanton number k.

- There is a $(2k+N)\times(2k)$ matrix v such that $(v\ u) \in U(2k+N)$,

  which is unique up to $U(2k)$ rotations.

- $A \equiv iu^\dagger du,\ \ A' \equiv iv^\dagger dv$ are connections on the bundles

$$E = \bigcup_{x\in\mathbb{R}^4} \mathrm{Ker} D(x), \quad E' = \bigcup_{x\in\mathbb{R}^4} \mathrm{Span} D(x)^\dagger.$$

- Since $E \oplus E'$ is a trivial bundle, one can show $\mathrm{Tr}(F^n) = -\mathrm{Tr}(F'^n)$.

  So the instanton number of A is minus the instanton number of A'.

$$(\text{instanton number}) = +\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F'\wedge F') = \frac{1}{8\pi^2}\int_{S^3_\infty} \mathrm{Tr}\left(A'dA' - \frac{2i}{3}A'^3\right)$$

Construction of U(N) k-instantons

Let us show that $A \equiv i u^\dagger du$ has instanton number k.

- Since $D(x) \equiv \left(\bar{\mathbf{x}} - \bar{\mathbf{X}}, \mathbf{H}\right) \stackrel{x\to\infty}{\longrightarrow} \left(\bar{\mathbf{x}}, 0\right)$ one can take

$$v(x)\Big|_{x\to\infty} = \frac{1}{|x|}\begin{pmatrix}\mathbf{x}\\ 0\end{pmatrix}, \qquad \text{where } \mathbf{x} \equiv x^a \sigma^a \otimes \mathbf{1}_k.$$

- Substitute $A' = i v^\dagger dv$ in

$$(\text{instanton number}) = \frac{1}{8\pi^2}\int_{S^3_\infty} \mathrm{Tr}\left(A' dA' - \frac{2i}{3}A'^3\right)$$

$$= -\frac{1}{24\pi^2}\int_{|x|=1} \mathrm{Tr}\left(\bar{\mathbf{x}} d\mathbf{x} d\bar{\mathbf{x}} d\mathbf{x}\right) = k.$$

Construction of U(N) k-instantons

Remark **1**: • For $U(N)$ gauge field on $\mathbb{R}^4$ which is anti-self-dual, the $U(1)$ part can always be gauged away.

We may assume A is an $SU(N)$ gauge field.

Remark **2**: • The equation $D(x)u(x)=0$ is invariant under

$$D \to (\mathbf{1}_2 \otimes U) D \begin{pmatrix} \mathbf{1}_2 \otimes U^{-1} & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad u \to \begin{pmatrix} \mathbf{1}_2 \otimes U & 0 \\ 0 & g \end{pmatrix} u,$$

where $U \in U(k),\ g \in U(N)$.

It acts on the ADHM data as

$$(B_1, B_2, I, J) \to (UB_1U^{-1}, UB_2U^{-1}, UIg^{-1}, gJU^{-1})$$

This leaves $A = iu^\dagger du$ invariant.

Construction of U(N) k-instantons

Remark **2'**: • On the other hand, at $x \to \infty$ one has

$$D(x) \longrightarrow \left(\bar{\mathbf{x}}, \mathbf{0}_{2k\times N}\right), \quad u \longrightarrow \begin{pmatrix} \mathbf{0}_{2k\times N} \\ g_\infty \end{pmatrix}.$$

g_∞ : the framing of the vector bundle E at infinity.

- If we require $g_\infty = \mathbf{1}_N$ as boundary condition,

  then the above transformation rule has to be modified.

$$D \to (\mathbf{1}_2 \otimes U) D \begin{pmatrix} \mathbf{1}_2 \otimes U^{-1} & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad u \to \begin{pmatrix} \mathbf{1}_2 \otimes U & 0 \\ 0 & g \end{pmatrix} u g^{-1}, \quad A \to gAg^{-1}.$$

With the above boundary condition on $u(x)$, the g in

$$(B_1, B_2, I, J) \to (UB_1U^{-1}, UB_2U^{-1}, UIg^{-1}, gJU^{-1})$$

corresponds to the constant U(N) gauge rotation.

ADHM Construction : Summary

SU(N) k-instanton solutions can be constructed from the ADHM data

$$B_1,\ B_2\ (k\times k),\quad I\ (k\times N),\quad J\ (N\times K)$$

$$\mu_{\mathbb{C}} := [B_1, B_2] + IJ = 0,$$

$$\mu_{\mathbb{R}} := [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0.$$

$$(B_1, B_2, I, J) \sim (UB_1U^{-1}, UB_2U^{-1}, UI, JU^{-1})$$

The moduli space of framed SU(N) k-instantons
= the space of ADHM data

Other facts:

- The moduli space is hyperKahler, so it is a symplectic manifold.
- It has bad singularities (corresponding to UV infinities), which can be smoothed by

$$\mu_{\mathbb{R}} = 0 \longrightarrow \mu_{\mathbb{R}} = \zeta \cdot \mathbf{1}_k. \quad \big(\zeta > 0\big)$$

Volume of
the Instanton Moduli Space

Volume of the Moduli Space

- We introduce the Omega deformation corresponding to

  - rotation of $\mathbb{R}^4$: $(B_1, B_2, I, J) \to (e^{i\epsilon_1}B_1, e^{i\epsilon_2}B_2, I, e^{i(\epsilon_1+\epsilon_2)}J)$

  - gauge rotation: $(B_1, B_2, I, J) \to (B_1, B_2, Ig^{-1}, gJ)$

  [vector field]

$$\delta\big(B_1\,,\ B_2\,,\ I\,,\ J\big) = \big(\epsilon_1 B_1\,,\ \epsilon_2 B_2\,,\ Ia\,,\ (\epsilon_1+\epsilon_2)J - aJ\big)$$

$$a = \mathrm{diag}(a_1, \cdots, a_N) \in U(N)$$

and compute the volume using the fixed point theorem.

- We need to

  ① find all the fixed points.

  ② compute the weights at each fixed point.

Fixed Points

Recall the ADHM data are subject to the $U(k)$ identification

$$(B_1, B_2, I, J) \sim (UB_1U^{-1}, UB_2U^{-1}, UI, JU^{-1})$$

So the condition for fixed points is

$$\delta\big(B_1\,,\ B_2\,,\ I\,,\ J\big) = \big(\epsilon_1 B_1\,,\ \epsilon_2 B_2\,,\ Ia\,,\ (\epsilon_1+\epsilon_2)J - aJ\big)$$

$$\big(a = \mathrm{diag}(a_1, \cdots, a_N) \in U(N)\big)$$

$$= \big([\varphi, B_1]\,,\ [\varphi, B_2]\,,\ \varphi I\,,\ -J\varphi\big)$$

for a $k \times k$ Hermite matrix φ.

We solve the following for a generic choice of $(\epsilon_1, \epsilon_2; a_1, \cdots, a_N)$.

$$\boxed{\begin{array}{lll} [\varphi, B_1] = \epsilon_1 B_1, & \varphi I = Ia, & [B_1, B_2] + IJ = 0, \\ [\varphi, B_2] = \epsilon_2 B_2, & J\varphi = aJ - (\epsilon_1+\epsilon_2)J, & [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \cdot \mathbf{1}_k. \end{array}}$$

We wish to show :

Each fixed point is labeled by *k* boxes forming *N* Young diagrams.

$$[\varphi, B_1] = \epsilon_1 B_1, \quad \varphi I = I a, \qquad [B_1, B_2] + IJ = 0,$$

$$[\varphi, B_2] = \epsilon_2 B_2, \quad J\varphi = aJ - (\epsilon_1 + \epsilon_2) J, \quad [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \cdot \mathbf{1}_k.$$

1. $I_i = (\text{the } i\text{-th column of } I),\ J_i = (\text{the } i\text{-th row of } J)$

are eigenvectors of φ.

$$\varphi I_i = a_i I_i, \quad J_i \varphi = (a_i - \epsilon_1 - \epsilon_2) J_i. \quad (i = 1, \cdots, N)$$

$J_i I_{i'} = 0$ since the eigenvalues of φ do not match.

2. $B_1^m B_2^n I_i,\ J_i B_1^m B_2^n$ are also φ-eigenvectors.

$$\varphi(B_1^m B_2^n I_i) = (a_i + m\epsilon_1 + n\epsilon_2) I_i,$$

$$(J_i B_1^m B_2^n)\varphi = \{a_i - (m+1)\epsilon_1 - (n+1)\epsilon_2\}\, J_i.$$

$\boxed{J_i \,(\text{any product of } B_1, B_2)\, I_{i'} = 0}$ because of eigenvalue mismatch.

$$[\varphi, B_1] = \epsilon_1 B_1, \quad \varphi I = I a, \quad [B_1, B_2] + IJ = 0,$$

$$[\varphi, B_2] = \epsilon_2 B_2, \quad J\varphi = aJ - (\epsilon_1 + \epsilon_2)J, \quad [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \cdot \mathbf{1}_k.$$

Assume hereafter $\zeta > 0$.

3. $J = 0$.

[proof] If not, there is a nonzero J_i.

There is a nonzero row vector of the form $\lambda = J_i \cdot (\text{product of } B_1, B_2)$

such that $\lambda B_1 = \lambda B_2 = 0$. Then

$$0 < \zeta \cdot \lambda\lambda^\dagger = \lambda\Big([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J\Big)\lambda^\dagger$$

$$= \lambda\Big(- B_1^\dagger B_1 - B_2^\dagger B_2 - J^\dagger J\Big)\lambda^\dagger < 0. \quad \textbf{(contradiction)}$$

$*$ $\lambda I = I^\dagger \lambda^\dagger = 0$ from the lemma **2.**

Now we also know: $[B_1, B_2] = 0$.

4. $I \neq 0$.

[proof] $\zeta \cdot \mathrm{Tr}(\mathbf{1}_k) = \mathrm{Tr}(II^\dagger)$.

$$[\varphi, B_1] = \epsilon_1 B_1, \quad \varphi I = Ia, \qquad [B_1, B_2] + IJ = 0,$$

$$[\varphi, B_2] = \epsilon_2 B_2, \quad J\varphi = aJ - (\epsilon_1 + \epsilon_2)J, \quad [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \cdot \mathbf{1}_k.$$

5. $B_1^\dagger I_i = B_2^\dagger I_i = 0.$

[proof] If not, there is a nonzero vector of the form

$$\lambda = (\text{product of } B_1^\dagger, B_2^\dagger) I_i, \ \lambda \neq I_i,$$

such that $B_1^\dagger \lambda = B_2^\dagger \lambda = 0.$ Then

$$0 < \zeta \cdot \lambda^\dagger \lambda = \lambda^\dagger \Big([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger \Big) \lambda = \cdots < 0.$$ **(contradiction)**

6. The vectors $B_1^m B_2^n I_j$ span the *k*-dimensional space.

[proof] If a row vector λ satisfies $\lambda B_1^m B_2^n I_j = 0 \ (\forall m, n, j),$

then any vector of the form $\lambda B_1^k B_2^\ell$ satisfies the same.

Choose λ which also satisfies $\lambda B_1 = \lambda B_2 = 0.$

Then $0 < \zeta \cdot \lambda \lambda^\dagger = \cdots < 0.$ **(contradiction)**

Fixed Points: Summary

$$[\varphi, B_1] = \epsilon_1 B_1, \quad \varphi I = I a, \quad [B_1, B_2] + IJ = 0, \quad (\zeta > 0)$$

$$[\varphi, B_2] = \epsilon_2 B_2, \quad J\varphi = aJ - (\epsilon_1 + \epsilon_2)J, \quad [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \cdot \mathbf{1}_k.$$

The solutions are summarized as follows.

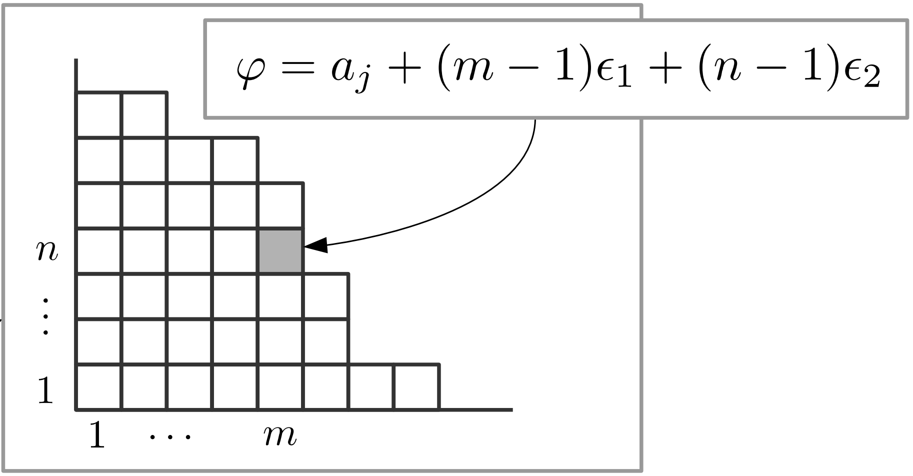
- $J = 0, \quad [B_1, B_2] = 0.$
- There are *k* linearly independent vectors of the form $B_1^m B_2^n I_j$,

  where $I_j = (j\text{-th column of } I)$.

  If $B_1^m B_2^n I_j$ is nonzero, it is an eigenvector for $\varphi = a_j + m\epsilon_1 + n\epsilon_2$.

- The eigenvalues of φ are described by *k* boxes forming *N* Young diagrams.

$$\vec{Y} \equiv (Y_1, \cdots, Y_j, \cdots, Y_N)$$



Weights and Character

At each fixed point, we need to study the action of the symmetry δ on the tangent space, and compute the product of weights

$$\prod_{i=1}^{2Nk} w_i.$$

Alternatively, one can consider the "**character**"

$$\mathrm{ch} \equiv \mathrm{Tr} e^{\delta} = \sum_{i=1}^{2Nk} e^{w_i}.$$

Weights and Character

Let $p = (B_1, B_2, I, J)$ be a fixed point and

(b_1, b_2, i, j) the coordinates for small variations.

$$e^{\delta}\begin{pmatrix} B_1 + b_1 \\ B_2 + b_2 \\ I + i \\ J + j \end{pmatrix} = \begin{pmatrix} B_1 + e^{\epsilon_1 - \varphi} b_1 e^{\varphi} \\ B_2 + e^{\epsilon_2 - \varphi} b_2 e^{\varphi} \\ I + e^{-\varphi} i e^{a} \\ J + e^{\epsilon_1 + \epsilon_2 - a} j e^{\varphi} \end{pmatrix}$$

The matrix elements of (b_1, b_2, i, j) transform under the symmetry δ with the weights

$$\begin{aligned} (b_1)_{k\ell} &: \; e^{\delta} = e^{\epsilon_1 - \varphi_k + \varphi_\ell} \\ (b_2)_{k\ell} &: \; e^{\delta} = e^{\epsilon_2 - \varphi_k + \varphi_\ell} \\ (i)_{k\ell} &: \; e^{\delta} = e^{-\varphi_k + a_\ell} \\ (j)_{k\ell} &: \; e^{\delta} = e^{\epsilon_1 + \epsilon_2 - a_k + \varphi_\ell} \end{aligned}$$

The character : $\mathrm{Tr}(e^{\delta}) = e^{\epsilon_1}\mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{\varphi}) + e^{\epsilon_2}\mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{\varphi})$

$$+\mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{a}) + e^{\epsilon_{1+2}}\mathrm{Tr}(e^{-a})\mathrm{Tr}(e^{\varphi})$$

$(2k^2 + 2Nk)$ weights is too many.

Correct Tangent Space

Tangent space coordinates (b_1, b_2, i, j) are subject to restrictions

and gauge equivalence.

- We need to restrict to variations preserving $\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0.$

$$\Delta\mu_{\mathbb{C}} = [b_1, B_2] + [B_1, b_2] + iJ + Ij \equiv \tau(b_1, b_2, i, j) = 0.$$

- We need to subtract variations generated by GL(k) gauge transformations.

$$\begin{pmatrix} b_1 \\ b_2 \\ i \\ j \end{pmatrix} = \delta_{GL(k)} \begin{pmatrix} B_1 \\ B_2 \\ I \\ J \end{pmatrix} = \begin{pmatrix} [\xi, B_1] \\ [\xi, B_2] \\ \xi I \\ -J\xi \end{pmatrix} \equiv \sigma(\xi)$$

- Note : $\tau \circ \sigma(\xi) = [[\xi, B_1], B_2] + [B_1, [\xi, B_2]] + \xi IJ + I(-J\xi) \equiv 0.$

- The correct tangent space is $\mathrm{Ker}\tau / \mathrm{Im}\sigma$.

Correct Character

The correct character is a difference of traces

$$\mathrm{ch} = \mathrm{Tr}(e^{\delta})\big|_{\mathrm{Ker}\tau/\mathrm{Im}\sigma} = \mathrm{Tr}(e^{\delta})\big|_{(b_1,b_2,i,j)} - \mathrm{Tr}(e^{\delta})\big|_{\Delta\mu_{\mathbb{C}}} - \mathrm{Tr}(e^{\delta})\big|_{\xi}$$

$$= e^{\epsilon_1}\mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{\varphi}) + e^{\epsilon_2}\mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{\varphi}) + \mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{a}) + e^{\epsilon_{1+2}}\mathrm{Tr}(e^{-a})\mathrm{Tr}(e^{\varphi})$$

$$-e^{\epsilon_1+\epsilon_2}\mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{\varphi}) - \mathrm{Tr}(e^{-\varphi})\mathrm{Tr}(e^{\varphi})$$

. . . should be a sum of *2Nk* terms.

Let us rewrite using $t_1 \equiv e^{\epsilon_1},\ t_2 \equiv e^{\epsilon_2}$ and substituting

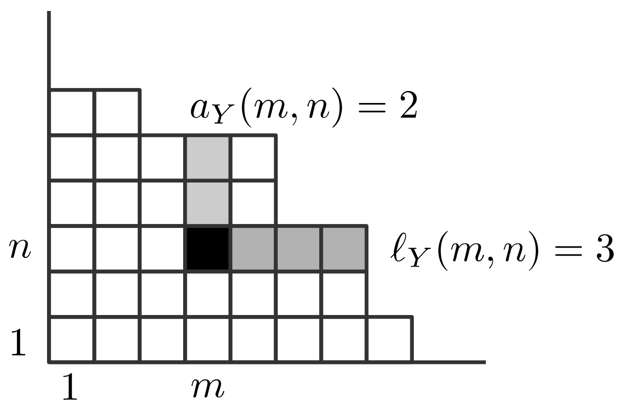
$$\mathrm{Tr}(e^{\pm a}) = \sum_{i=1}^{N} e^{\pm a_i}, \quad \mathrm{Tr}(e^{\pm\varphi}) = \sum_{i=1}^{N}\sum_{(m,n)\in Y_i} e^{\pm\{a_i+(m-1)\epsilon_1+(n-1)\epsilon_2\}}$$

Nakajima-Yoshioka's formula

$$\mathrm{ch} = \sum_{i,j=1}^{N} e^{a_i - a_j} \chi(Y_i, Y_j),$$

$$\chi(Y_i, Y_j) \equiv t_1 t_2 \sum_{(k,l)\in Y_i} t_1^{k-1} t_2^{l-1} + \sum_{(m,n)\in Y_j} t_1^{1-m} t_2^{1-n} - (1-t_1)(1-t_2) \sum_{(k,l)\in Y_i} \sum_{(m,n)\in Y_j} t_1^{k-m} t_2^{l-n}$$

$$= \sum_{(m,n)\in Y_i} t_1^{-\ell_{Y_j}(m,n)} t_2^{1+a_{Y_i}(m,n)} + \sum_{(m,n)\in Y_j} t_1^{1+\ell_{Y_i}(m,n)} t_2^{-a_{Y_j}(m,n)}.$$



Here $a_Y(m,n) = (\text{height of the } m\text{-th column of } Y) - n$

$\ell_Y(m,n) = (\text{length of the } n\text{-th row of } Y) - m$

Volume of the Moduli Space: Summary

- Omega-deformed volume of the U(N) k-instanton moduli space was defined using SO(4) rotation & constant gauge symmetry.

  Localization with a SUSY $\mathbf{Q}$ satisfying $\mathbf{Q}^2 = \epsilon_1 \mathbf{J}_{12} + \epsilon_2 \mathbf{J}_{34} + \mathbf{Gauge}(a)$

- the application of DH fixed point formula gives

$$\mathrm{vol}_{\epsilon_1,\epsilon_2;a}(\mathcal{M}_k) = \sum_{\vec{Y}} \frac{1}{e(\vec{Y})}.$$

- The fixed points are labeled by a set of N Young diagrams $\vec{Y}$, whose total number of boxes is k.

- Nakajima-Yoshioka's formula

$$e(\vec{Y}) = \prod_{i,j=1}^{N} n_{i,j}(\vec{Y}), \qquad n_{i,j}(\vec{Y}) = \prod_{s \in Y_i} \left(a_i - a_j - \epsilon_1 \ell_{Y_j}(s) + \epsilon_2 (a_{Y_i}(s) + 1)\right)$$
$$\cdot \prod_{s \in Y_j} \left(a_i - a_j + \epsilon_1 (\ell_{Y_i}(s) + 1) - \epsilon_2 a_{Y_j}(s)\right)$$

Comments:

- At the fixed points, what do the instanton solutions look like?

  . . . It is invariant under rotation (ϵ_1, ϵ_2) and constant gauge transformation a,
  so it is a sum of "point-like" "U(1) instantons" localized at the origin.

- There is a SUSY gauge theory whose path integral gives
  the generating function of the volume of moduli space.
  = Topological twisted $\mathcal{N} = 2$ SYM theory on the Omega background.

$$Z \ (\text{path integral}) = \sum_{k \geq 0} q^k \, \text{vol}_{\epsilon_1, \epsilon_2, a}(\mathcal{M}_k)$$

IV. Four-Sphere Partition Function

- Basics of N=2 SUSY theories
- Killing Spinors
- Construction of SUSY theories
- Topological Twist and Omega background
- Exact Partition Function via Localization
- N=4 SYM and Gaussian Matrix Models
- AGT relation and Ellipsoid Partition Function

Basics of N=2 SUSY theories

4D $\mathcal{N}$-extended SUSY

$Q_{\alpha A}$: chiral spinor $(\gamma^5 \equiv -i\gamma^{0123} = +1)$

$\bar{Q}^A_{\dot{\alpha}}$: anti-chiral spinor $(\gamma^5 = -1)$ $(A = 1, \cdots, \mathcal{N})$

$(Q_{\alpha A})^\dagger = \bar{Q}^A_{\dot{\alpha}}$

$$\boxed{\{Q_{\alpha A}, \bar{Q}^B_{\dot{\alpha}}\} = -2\delta^B_A (\sigma^m)_{\alpha\dot{\alpha}} P_m.}$$

R-symmetry $U(1) \times SU(\mathcal{N})$ rotates the supercharges preserving the anti-commutation relation.

We focus on the theories with $\mathcal{N} = 2$ SUSY.

4D $\mathcal{N} = 2$ field theories

Multiplets:

- vectormultiplet : gauge group G

vector	A_m	$(\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{0}}$
complex scalar	$\phi, \bar{\phi}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{\pm\mathbf{2}}$
chiral spinor	λ_A	$(\mathbf{2},\mathbf{1},\mathbf{2})_{\mathbf{1}}$
anti-chiral spinor	$\bar{\lambda}_A$	$(\mathbf{1},\mathbf{2},\mathbf{2})_{-\mathbf{1}}$
(auxiliary scalar)	D_{AB}	$(\mathbf{1},\mathbf{1},\mathbf{3})_{\mathbf{0}}$

- hypermultiplet : representation R

scalar	q_A	$(\mathbf{1},\mathbf{1},\mathbf{2})_{\mathbf{0}}$
chiral spinor	ψ	$(\mathbf{2},\mathbf{1},\mathbf{1})_{-\mathbf{1}}$
anti-chiral spinor	$\bar{\psi}$	$(\mathbf{1},\mathbf{2},\mathbf{1})_{\mathbf{1}}$
(auxiliary scalar)		

SUSY parameter: $\xi_A \; (\mathbf{2},\mathbf{1},\mathbf{2})_{\mathbf{1}} \quad , \quad \bar{\xi}_A \, (\mathbf{1},\mathbf{2},\mathbf{2})_{-\mathbf{1}}$

In Euclidean theory on $\mathbb{R}^4$, they transform under the symmetry

$$\underbrace{SU(2)_1 \times SU(2)_2}_{SO(4)} \times SU(2)_{\mathrm{R}} \times U(1)_{\mathrm{R}}.$$

Hypermultiplets in detail

- "r hypermultiplets" means $(q_{IA}, \psi_I, \bar{\psi}_I)$ $(I = 1, \cdots, 2r)$

- The scalars obey the reality condition

$$(q_{IA})^* = \Omega^{IJ}\epsilon^{AB}q_{BJ}, \qquad \begin{array}{lll} \Omega^{IJ} & : & \text{invariant tensor of } Sp(r) \\ \epsilon^{AB} & : & \text{invariant tensor of } Sp(1) = SU(2) \end{array}$$

- Gauge fields couple to them via

$$D_m q_{IA} \equiv \partial_m q_{IA} - iA_m^a (t^a)_I{}^J q_{JA},$$

where the representation matrix $(t^a)_I{}^J$ is Hermite and satisfies

$$\Omega^{KJ}(t^a)_K{}^I + \Omega^{IK}(t^a)_K{}^J = 0,$$

namely the gauge group has to be a subgroup of $Sp(r)$.

Hypermultiplets : Example

A hypermultiplet in the fundamental rep of $SU(n)$:

- $q_{IA}, \psi_I, \bar{\psi}_I$ are $2n$ component quantities.

  We use $(\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\Omega^{IJ}) = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$

- Choose the first *n* to be in the fundamental, the rest anti-fundamental.
- The reality condition is satisfied by $(q_{IA}) = (q_{I1}\ q_{I2}) = \begin{pmatrix} \boldsymbol{q} & -\tilde{\boldsymbol{q}}^* \\ \tilde{\boldsymbol{q}} & \boldsymbol{q}^* \end{pmatrix}.$

1 fundamental hypermultiplet

= 1 fundamental scalar $\boldsymbol{q}$, 1 anti-fundamental scalar $\tilde{\boldsymbol{q}}$ and superpartners.

- The $2n \times 2n$ representation matrix of $SU(n)$:

$$(t^a)_I{}^J = \begin{pmatrix} \hat{t}^a & 0 \\ 0 & -(\hat{t}^a)^* \end{pmatrix} \qquad \hat{t}^a : \text{ fundamental rep}$$

gives an embedding $SU(n) \subset Sp(n)$.

SUSY theory on R^4

Transformation rules

- vectormultiplet

$$
\begin{aligned}
\mathbf{Q}A_m &= i\xi^A\sigma_m\bar{\lambda}_A - i\bar{\xi}^A\bar{\sigma}_m\lambda_A, \\
\mathbf{Q}\phi &= -i\bar{\xi}^A\lambda_A, \\
\mathbf{Q}\bar{\phi} &= +i\bar{\xi}^A\bar{\lambda}_A, \\
\mathbf{Q}\lambda_A &= \tfrac{1}{2}\sigma^{mn}\xi_A F_{mn} + 2\sigma^m\bar{\xi}_A D_m\phi + 2i\xi_A[\phi,\bar{\phi}] + D_{AB}\xi^B, \\
\mathbf{Q}\bar{\lambda}_A &= \tfrac{1}{2}\bar{\sigma}^{mn}\bar{\xi}_A F_{mn} + 2\bar{\sigma}^m\xi_A D_m\bar{\phi} - 2i\bar{\xi}_A[\phi,\bar{\phi}] + D_{AB}\bar{\xi}^B, \\
\mathbf{Q}D_{AB} &= -2i\bar{\xi}_{(A}\bar{\sigma}^m D_m\lambda_{B)} + 2i\xi_{(A}\sigma^m D_m\bar{\lambda}_{B)} - 4[\phi,\bar{\xi}_{(A}\bar{\lambda}_{B)}] + 4[\bar{\phi},\xi_{(A}\lambda_{B)}].
\end{aligned}
$$

- hypermultiplet (on-shell)

$$
\begin{aligned}
\mathbf{Q}q_A &= -i\xi_A\psi + i\bar{\xi}_A\bar{\psi}, \\
\mathbf{Q}\psi &= 2\sigma^m\bar{\xi}_A D_m q^A - 4i\xi_A\bar{\phi}q^A, \\
\mathbf{Q}\bar{\psi} &= 2\bar{\sigma}^m\xi_A D_m q^A - 4i\bar{\xi}_A\phi q^A.
\end{aligned}
$$

Note:
There is no off-shell transformation rule for hypermultiplet
- with finitely many auxiliary fields
- realizing all the SUSY off-shell

SUSY theory on R^4

Invariant Lagrangian

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{g^2}\mathrm{Tr}\Big[\frac{1}{2}F_{mn}F_{mn} - 4D_m\bar{\phi}D_m\phi + i\lambda^A\sigma^m D_m\bar{\lambda}_A - 2i\lambda^A[\bar{\phi},\lambda_A] + 2\bar{\lambda}^A[\phi,\lambda_A]$$
$$+4[\phi,\bar{\phi}]^2 - \frac{1}{2}D^{AB}D_{AB}\Big] + \frac{i\theta}{32\pi^2}\varepsilon^{klmn}\mathrm{Tr}(F_{kl}F_{mn}),$$

$$\mathcal{L}_{\mathrm{FI}} = w^{AB}D_{AB},$$

* for U(1) group only, $w^{AB} = \text{constant.}$

$$\mathcal{L}_{\mathrm{mat}} = \frac{1}{2}D_m q^A D_m q_A - q^A\{\phi,\bar{\phi}\}q_A + \frac{i}{2}q^A D_{AB} q^B$$
$$-\frac{i}{2}\bar{\psi}\bar{\sigma}^m D_m\psi - \frac{1}{2}\psi\phi\psi + \frac{1}{2}\bar{\psi}\bar{\phi}\bar{\psi} - q^A\lambda_A\psi + \bar{\psi}\bar{\lambda}_A q^A.$$

* we have suppressed the indices *I, J,* . . . according to the rule

$$q^A q_A \equiv \epsilon^{AB}\Omega^{IJ}q_{IA}q_{JB}, \cdots .$$

- mass term for hypermultiplet can be introduced by turning on background vectormultiplets.

Killing Spinors

Killing spinors on round sphere

- On round sphere of any dimension with radius ℓ,

  the Killing spinor equation of the following form has solutions.

$$D_m\xi \equiv \left(\partial_m + \frac{1}{4}\gamma^{ab}\Omega_m ab\right)\xi = \pm\frac{i}{2\ell}\gamma_m\xi.$$

- For $\mathcal{N}=2$ SUSY theory on 4-sphere,

  the SUSY is generated by Killing spinors $\xi_A, \bar{\xi}_A$ satisfying

$$D_m\xi_A \equiv \left(\partial_m + \frac{1}{4}\sigma^{ab}\Omega_m^{ab}\right)\xi = -i\bar{\xi}'_A, \qquad D_m\xi'_A = -\frac{i}{\ell^2}\sigma_m\bar{\xi}_A,$$

$$D_m\bar{\xi}_A \equiv \left(\partial_m + \frac{1}{4}\bar{\sigma}^{ab}\Omega_m^{ab}\right)\bar{\xi} = -i\xi'_A, \qquad D_m\bar{\xi}'_A = -\frac{i}{\ell^2}\bar{\sigma}_m\xi_A.$$

1st order differential equation for 16 functions $(\xi_A, \bar{\xi}_A, \xi'_A, \bar{\xi}'_A)$.

The solutions correspond to the supercharges of 4D *N*=2 SCA.

Superconformal Theories on sphere

- 4D N=2 theories with no mass terms or FI terms are classically conformal.

  Such theories on sphere can be obtained from the theories on $\mathbb{R}^4$

  by a conformal map, because the round sphere is conformally flat.

$$ds^2_{S^4} = e^{2\sigma(x)}(dx_1^2 + \cdots + dx_4^2), \qquad e^{-\sigma(x)} = 1 + \frac{|x|^2}{4\ell^2}.$$

  Such theories are invariant under any of the 16 independent Killing spinors.

- Mass term or FI term break the superconformal invariance.

  They are invariant under a subset of the Killing spinors satisfying

$$D_m \xi_A = -\frac{i}{2\ell}\sigma_m \bar{\xi}_B \, t^B{}_A, \qquad D_m \bar{\xi}_A = -\frac{i}{2\ell}\bar{\sigma}_m \xi_B \, \bar{t}^B{}_A.$$

$t, \bar{t}$: constant traceless U(2) matrices satisfying $t\bar{t} = \bar{t}t = \mathbf{1}_2$.

Using R-symmetries one can set $t = \bar{t} = \boldsymbol{\tau}_3$.

Let us study a particular solution.

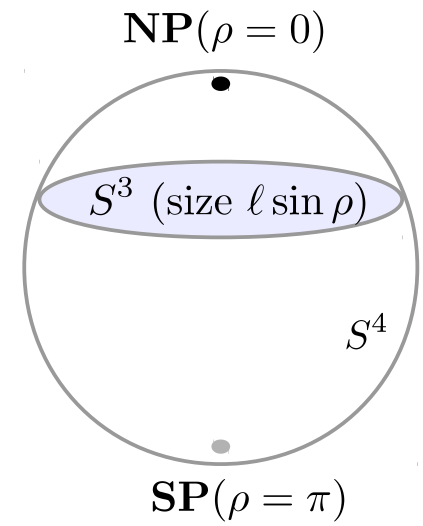
A coordinate system on the four-sphere

round sphere:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \ell^2.$$

coordinates:

$$x_0 = \ell \cos\rho$$
$$x_1 = \ell \sin\rho \cos\theta \cos\varphi$$
$$x_2 = \ell \sin\rho \cos\theta \sin\varphi$$
$$x_3 = \ell \sin\rho \sin\theta \cos\chi$$
$$x_4 = \ell \sin\rho \sin\theta \sin\chi$$



Note: φ, χ are the rotation angles of (x_1, x_2)-plane, (x_3, x_4)-plane.

The North & South poles are the fixed points.

metric: $$ds^2 = \ell^2 \Big\{ \sin^2\rho(\cos^2\theta d\varphi^2 + \sin^2\theta d\chi^2 + d\theta^2) + d\rho^2 \Big\}$$

vielbein: $$E^1 = \ell \sin\rho \cos\theta d\varphi, \quad E^3 = \ell \sin\rho d\theta,$$
$$E^2 = \ell \sin\rho \sin\theta d\chi, \quad E^4 = \ell d\rho.$$

A specific Killing spinor

$$\boxed{D_m\xi_A = -\frac{i}{2\ell}\sigma_m\bar{\xi}_B\, t^B{}_A, \quad D_m\bar{\xi}_A = -\frac{i}{2\ell}\bar{\sigma}_m\xi_B\, \bar{t}^B{}_A \quad (t = \bar{t} = \boldsymbol{\tau}_3)}$$

has the following solution on the round sphere of radius ℓ.

$$(\xi_A) = (\xi_1\ \xi_2) = \sin\frac{\rho}{2}\cdot(\kappa_+\ \kappa_-),$$

$$(\bar{\xi}_A) = (\bar{\xi}_1, \bar{\xi}_2) = \cos\frac{\rho}{2}\cdot(i\kappa_+, -i\kappa_-),$$

where $\kappa_\pm \equiv \frac{1}{2}\begin{pmatrix} e^{\frac{i}{2}(\pm\chi\pm\varphi-\theta)} \\ \mp e^{\frac{i}{2}\pm\chi\pm\varphi+\theta)} \end{pmatrix}$ are Killing spinors on the round S^3.

This satisfies $\xi^A\xi_A + \bar{\xi}_A\bar{\xi}^A = 1$ and

$$v \equiv 2\bar{\xi}^A\bar{\sigma}^m\xi_A\partial_m = \frac{1}{\ell}(\partial_\varphi + \partial_\chi) = \frac{1}{\ell}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_3}\right)$$

Near the north and the south poles, this is similar to the Omega-deformation

$$\epsilon_1 = \epsilon_2 = \frac{1}{\ell}.$$

Squashing

If the same $\xi_A, \bar{\xi}_A$ satisfy Killing spinor equation on the ellipsoid,

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1$$

coordinates:

$$x_0 = r\cos\rho$$
$$x_1 = \ell\sin\rho\cos\theta\cos\varphi$$
$$x_2 = \ell\sin\rho\cos\theta\sin\varphi$$
$$x_3 = \tilde{\ell}\sin\rho\sin\theta\cos\chi$$
$$x_4 = \tilde{\ell}\sin\rho\sin\theta\sin\chi$$

vielbein:

$$E^1 = \ell\sin\rho\cos\theta d\varphi,$$
$$E^2 = \tilde{\ell}\sin\rho\sin\theta d\chi,$$
$$E^3 = f(\theta)\sin\rho d\theta + h(\theta)d\rho,$$
$$E^4 = g(\rho,\theta)d\rho.$$

$$f(\theta) = \sqrt{\ell^2\sin^2\theta + \tilde{\ell}^2\cos^2\theta},$$
$$g(\rho,\theta) = \sqrt{r^2\sin^2\rho + \ell^2\tilde{\ell}^2 f^{-2}\cos^2\rho},$$
$$h(\theta) = \frac{\tilde{\ell}^2 - \ell^2}{f}\cos\rho\sin\theta\cos\theta,$$

Then we have $\quad v \equiv 2\bar{\xi}^A\bar{\sigma}^m\xi_A\partial_m = \frac{1}{\ell}\partial_\varphi + \frac{1}{\tilde{\ell}}\partial_\chi,$

Omega background with $\epsilon_1 = \frac{1}{\ell},\ \epsilon_2 = \frac{1}{\tilde{\ell}}.$

Generalized Killing spinor equation

We need to study whether $\xi_A, \bar{\xi}_A$ satisfy the Killing spinor equation

$$\mathbf{Q}\psi_{mA} = D_m\xi_A + T^{kl}\sigma_{kl}\sigma_m\bar{\xi}_A + i\sigma_m\bar{\xi}'_A = 0,$$

$$\mathbf{Q}\bar{\psi}_{mA} = D_m\bar{\xi}_A + \bar{T}^{kl}\bar{\sigma}_{kl}\bar{\sigma}_m\xi_A + i\bar{\sigma}_m\xi'_A = 0,$$

$$\mathbf{Q}\eta_A = 8\sigma^{mn}\sigma^l\bar{\xi}_A D_l T_{mn} + 16iT^{kl}\sigma_{kl}\xi'_A - 3\tilde{M}\xi_A + 2i\sigma^{mn}\xi_B (V_{mn})^B{}_A + 4i\sigma^{mn}\xi_A\tilde{V}_{mn} = 0,$$

$$\mathbf{Q}\bar{\eta}_A = 8\bar{\sigma}^{mn}\bar{\sigma}^l\xi_A D_l\bar{T}_{mn} + 16i\bar{T}^{kl}\bar{\sigma}_{kl}\bar{\xi}'_A - 3\tilde{M}\bar{\xi}_A + 2i\bar{\sigma}^{mn}\bar{\xi}_B (V_{mn})^B{}_A - 4i\bar{\sigma}^{mn}\bar{\xi}_A\tilde{V}_{mn} = 0,$$

where the covariant derivatives contain R-symmetry gauge fields,

$$D_m\xi_A \equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\sigma_{ab}\right)\xi_A + i\xi_B \underbrace{(V_m)^B{}_A}_{\textbf{SU(2)}} - i\underbrace{\tilde{V}_m}_{\textbf{U(1)}}\xi_A, \cdots$$

and $(V_{mn})^A{}_B, \tilde{V}_{mn}$ are their field strengths.

This Killing spinor equation originates from the off-shell 4D N=2 supergravity.

de Wit, van Holten, van Proeyen 1980, 1981; de Wit, van Holten, van Proeyen 1984,. . .

We solved this for the supergravity background fields

$$T_{kl}, \bar{T}_{kl}, (V_m)^A{}_B, V_m, M.$$

Construction of SUSY theories

SUSY theory on curved backgrounds

Transformation rules (vectormultiplet)

$$\mathbf{Q}A_m = i\xi^A\sigma_m\bar\lambda_A - i\bar\xi^A\bar\sigma_m\lambda_A,$$

$$\mathbf{Q}\phi = -i\xi^A\lambda_A,$$

$$\mathbf{Q}\bar\phi = +i\bar\xi^A\bar\lambda_A,$$

$$\mathbf{Q}\lambda_A = \frac{1}{2}\sigma^{mn}\xi_A(F_{mn} + 8\bar\phi T_{mn}) + 2\sigma^m\bar\xi_A D_m\phi + \sigma^m D_m\bar\xi_A\phi + 2i\xi_A[\phi,\bar\phi] + D_{AB}\xi^B,$$

$$\mathbf{Q}\bar\lambda_A = \frac{1}{2}\bar\sigma^{mn}\bar\xi_A(F_{mn} + 8\phi\bar T_{mn}) + 2\bar\sigma^m\xi_A D_m\bar\phi + \bar\sigma^m D_m\xi_A\bar\phi - 2i\bar\xi_A[\phi,\bar\phi] + D_{AB}\bar\xi^B,$$

$$\mathbf{Q}D_{AB} = -2i\bar\xi_{(A}\bar\sigma^m D_m\lambda_{B)} + 2i\xi_{(A}\sigma^m D_m\bar\lambda_{B)} - 4[\phi,\bar\xi_{(A}\bar\lambda_{B)}] + 4[\bar\phi,\xi_{(A}\lambda_{B)}].$$

Square of SUSY

$$\mathbf{Q}^2 = i\mathcal{L}_v + \mathbf{Gauge}(\Phi) + \mathbf{R}_{SU(2)}(\Theta_{AB}) + \cdots ,$$

where $v^m \equiv 2\bar\xi^A\bar\sigma^m\xi_A,$

$$\Phi \equiv 2\phi\bar\xi^A\bar\xi_A - 2\bar\phi\xi^A\xi_A + v^m A_m,$$

$$\Theta_{AB} \equiv -i\xi_{(A}\sigma^m D_m\bar\xi_{B)} + iD_m\xi_{(A}\sigma^m\bar\xi_{B)} + v^m V_{mAB}.$$

SUSY theory on curved backgrounds

Off-shell transformation rules (hypermultiplet)

Note: we only have to realize off-shell a single supercharge $\mathbf{Q}$ corresponding to a specific choice of $(\xi_A, \bar{\xi}_A)$.

Introduce an auxiliary field $F_{\check{A}}$ and auxiliary spinors $\eta_{\check{A}}, \bar{\eta}_{\check{A}}$, $(\check{A} = 1, 2)$

$$\mathbf{Q} q_A = -i\xi_A \psi + i\bar{\xi}_A \bar{\psi},$$

$$\mathbf{Q}\psi = 2\sigma^m \bar{\xi}_A D_m q^A + \sigma^m D_m \bar{\xi}_A q^A - 4i\xi_A \bar{\phi} q^A + 2\eta_{\check{A}} F^{\check{A}},$$

$$\mathbf{Q}\bar{\psi} = 2\bar{\sigma}^m \xi_A D_m q^A + \bar{\sigma}^m D_m \xi_A q^A - 4i\bar{\xi}_A \phi q^A + 2\bar{\eta}_{\check{A}} F^{\check{A}},$$

$$\begin{aligned}\mathbf{Q} F_{\check{A}} = & \, i\eta_{\check{A}} \sigma^m D_m \bar{\psi} - 2\eta_{\check{A}} \phi \psi - 2\eta_{\check{A}} \lambda_B q^B + 2i\eta_{\check{A}} (\sigma^{kl} T_{kl}) \psi \\ & -i\bar{\eta}_{\check{A}} \bar{\sigma}^m D_m \psi + 2\bar{\eta}_{\check{A}} \bar{\phi} \bar{\psi} + 2\bar{\eta}_{\check{A}} \bar{\lambda}_B q^B - 2i\bar{\eta}_{\check{A}} (\bar{\sigma}^{kl} \bar{T}_{kl}) \bar{\psi} \, .\end{aligned}$$

The added terms are invariant under an $SU(2)$ acting on the indices $\check{A}, \check{B}, \cdots$.

We choose $\eta_{\check{A}}, \bar{\eta}_{\check{A}}$ to satisfy

$$\boxed{\begin{aligned} &\xi_A \eta_{\check{B}} - \bar{\xi}_A \bar{\eta}_{\check{B}} = 0, && \bar{\xi}^A \bar{\xi}_A + \eta^{\check{A}} \eta_{\check{A}} = 0, \\ &\xi^A \xi_A + \bar{\eta}^{\check{A}} \bar{\eta}_{\check{A}} = 0, && \xi^A \sigma^m \bar{\xi}_A + \eta^{\check{A}} \sigma^m \bar{\eta}_A = 0. \end{aligned}}$$

SUSY theory on curved backgrounds

Off-shell transformation rules (hypermultiplet)

$$\mathbf{Q}q_A = -i\xi_A\psi + i\bar{\xi}_A\bar{\psi},$$

$$\mathbf{Q}\psi = 2\sigma^m\bar{\xi}_A D_m q^A + \sigma^m D_m\bar{\xi}_A q^A - 4i\xi_A\bar{\phi}q^A + 2\check{\xi}_{\check{A}}F^{\check{A}},$$

$$\mathbf{Q}\bar{\psi} = 2\bar{\sigma}^m\xi_A D_m q^A + \bar{\sigma}^m D_m\xi_A q^A - 4i\bar{\xi}_A\phi q^A + 2\bar{\check{\xi}}_{\check{A}}F^{\check{A}},$$

$$\begin{aligned}\mathbf{Q}F_{\check{A}} = {}& i\check{\xi}_{\check{A}}\sigma^m D_m\bar{\psi} - 2\check{\xi}_{\check{A}}\phi\psi - 2\check{\xi}_{\check{A}}\lambda_B q^B + 2i\check{\xi}_{\check{A}}(\sigma^{kl}T_{kl})\psi \\ & -i\bar{\check{\xi}}_{\check{A}}\bar{\sigma}^m D_m\psi + 2\bar{\check{\xi}}_{\check{A}}\bar{\phi}\bar{\psi} + 2\bar{\check{\xi}}_{\check{A}}\bar{\lambda}_B q^B - 2i\bar{\check{\xi}}_{\check{A}}(\bar{\sigma}^{kl}\bar{T}_{kl})\bar{\psi}\,.\end{aligned}$$

$$\xi_A\eta_{\check{B}} - \bar{\xi}_A\bar{\eta}_{\check{B}} = 0, \qquad \bar{\xi}^A\bar{\xi}_A + \eta^{\check{A}}\eta_{\check{A}} = 0,$$
$$\xi^A\xi_A + \bar{\eta}^{\check{A}}\bar{\eta}_{\check{A}} = 0, \qquad \xi^A\sigma^m\bar{\xi}_A + \eta^{\check{A}}\sigma^m\bar{\eta}_A = 0.$$

Then the following is realized off-shell.

$$\mathbf{Q}^2 = i\mathcal{L}_v + \mathbf{Gauge}(\Phi) + \mathbf{R}_{SU(2)}(\Theta_{AB}) + \check{\mathbf{R}}_{SU(2)}(\check{\Theta}_{AB}) + \cdots\,,$$

$$\begin{aligned}\check{\Theta}_{\check{A}B} = {}& 2i\eta_{(\check{A}}\sigma^m D_m\bar{\eta}_{\check{B})} - 2iD_m\eta_{(\check{A}}\sigma^m\bar{\eta}_{\check{B})} \\ & +4i\eta_{(\check{A}}\sigma^{kl}T_{kl}\eta_{\check{B})} - 4i\bar{\eta}_{(\check{A}}\bar{\sigma}^{kl}\bar{T}_{kl}\bar{\eta}_{B)} + v^m\check{V}_{m\check{A}\check{B}}\end{aligned}$$

An explicit choice of the auxiliary spinors

We chose $\quad (\xi_A) = (\xi_1\ \xi_2) = \sin\frac{\rho}{2}\cdot(\kappa_+\ \kappa_-),$

$$(\bar{\xi}_A) = (\bar{\xi}_1, \bar{\xi}_2) = \cos\frac{\rho}{2}\cdot(i\kappa_+, -i\kappa_-), \quad \kappa_\pm \equiv \frac{1}{2}\begin{pmatrix} e^{\frac{i}{2}(\pm\chi\pm\varphi-\theta)} \\ \mp e^{\frac{i}{2}\pm\chi\pm\varphi+\theta)} \end{pmatrix}$$

and need to solve

$$\boxed{\begin{aligned} &\xi_A\eta_{\breve{B}} - \bar{\xi}_A\bar{\eta}_{\breve{B}} = 0, && \bar{\xi}^A\bar{\xi}_A + \eta^{\breve{A}}\eta_{\breve{A}} = 0, \\ &\xi^A\xi_A + \bar{\eta}^{\breve{A}}\bar{\eta}_{\breve{A}} = 0, && \xi^A\sigma^m\bar{\xi}_A + \eta^{\breve{A}}\sigma^m\bar{\eta}_A = 0. \end{aligned}}$$

An explicit choice is $\quad (\eta_{\breve{A}}) = (\eta_1\ \eta_2) = \cos\frac{\rho}{2}\cdot(\kappa_+\ \kappa_-),$

$$(\bar{\eta}_{\breve{A}}) = (\bar{\eta}_1, \bar{\eta}_2) = -\sin\frac{\rho}{2}\cdot(i\kappa_+, -i\kappa_-).$$

SUSY theory on curved backgrounds

Lagrangians

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{g^2}\mathrm{Tr}\Big(\tfrac{1}{2}F_{mn}F^{mn} + 16F_{mn}(\bar{\phi}T^{mn} + \phi\bar{T}^{mn}) + 64\bar{\phi}^2 T_{mn}T^{mn} + 64\phi^2\bar{T}_{mn}\bar{T}^{mn}$$
$$-4D_m\bar{\phi}D^m\phi + 2M\bar{\phi}\phi - 2i\lambda^A\sigma^m D_m\bar{\lambda}_A - 2\lambda^A[\bar{\phi},\lambda_A] + 2\bar{\lambda}^A[\phi,\bar{\lambda}_A]$$
$$+4[\phi,\bar{\phi}]^2 - \tfrac{1}{2}D^{AB}D_{AB}\Big) + \frac{i\theta}{32\pi^2}\mathrm{Tr}\Big(\varepsilon^{klmn}F_{kl}F_{mn}\Big)\,.$$

$$\mathcal{L}_{\mathrm{mat}} = \tfrac{1}{2}D_m q^A D^m q_A - q^A\{\phi,\bar{\phi}\}q_A + \tfrac{i}{2}q^A D_{AB}q^B + \tfrac{1}{8}(M+R)q^A q_A$$
$$-\tfrac{i}{2}\bar{\psi}\bar{\sigma}^m D_m\psi - \tfrac{1}{2}\psi\phi\psi + \tfrac{1}{2}\bar{\psi}\bar{\phi}\bar{\psi} + \tfrac{i}{2}\psi\sigma^{kl}T_{kl}\psi - \tfrac{i}{2}\bar{\psi}\bar{\sigma}^{kl}\bar{T}_{kl}\bar{\psi}$$
$$-q^A\lambda_A\psi + \bar{\psi}\bar{\lambda}_A q^A - \tfrac{1}{2}F^{\check{A}}F_{\check{A}}.$$

SUSY theory on curved backgrounds

Lagrangians

$$\mathcal{L}_{\rm FI} = \zeta \left\{ w^{AB} D_{AB} - M(\phi + \bar{\phi}) - 64\phi T^{kl} T_{kl} - 64\bar{\phi}\bar{T}^{kl}\bar{T}_{kl} - 8F^{kl}(T_{kl} + \bar{T}_{kl}) \right\} ,$$

where $w^{AB} = w^{BA}$ is an $SU(2)_{\rm R}$-triplet background field satisfying

$$w^{AB}\xi_B = \tfrac{1}{2}\sigma^n D_n \bar{\xi}^A + 2T_{kl}\sigma^{kl}\xi^A,$$
$$w^{AB}\bar{\xi}_B = \tfrac{1}{2}\bar{\sigma}^n D_n \xi^A + 2\bar{T}_{kl}\bar{\sigma}^{kl}\bar{\xi}^A.$$

The matter mass can be introduced by turning on a background vectormultiplet with $\phi = \bar{\phi} = -\dfrac{im}{2}, \quad D_{AB} = -imw_{AB}.$

Note: w^{AB} here is related to $t^A{}_B, \bar{t}^A{}_B$ in the discussion of Killing spinors on S^4.

Summary: 4D $\mathcal{N}=2$ SUSY theories on curved spaces

On flat space

- There are vectormultiplet and hypermultiplets.
- Invariant Lagrangians: $\mathcal{L}_{\rm YM}.\ \mathcal{L}_{\rm mat},\ \mathcal{L}_{\rm FI}$ and the matter mass term.
- Difficulty in writing off-shell SUSY for hypermultiplets.

On sphere

- Killing spinor equation has 16 solutions (= supercharges in *N*=2 SCA)
- Straightforward to write superconformal theories.

  For theories with mass / FI terms, Killing spinors satisfy stronger conditions $(t^A{}_B, \bar{t}^A{}_B)$

On ellipsoid

- One needs to turn on supergravity backgrounds $(T_{mn}, \bar{T}_{mn}, V_m, M)$
- One can write the off-shell SUSY for hypermultiplets

  since there is only one supercharge. We introduced $F_{\check{A}}; \eta_{\check{A}}, \bar{\eta}_{\check{A}}$.

Topological Twists

and Omega Backgrounds

Topologically Twisted Gauge Theory **Witten 1988**

It is known that

4D N=2 SUSY gauge theories can be put on any curved spaces

preserving a single “scalar SUSY”, by a procedure called “topological twist”.

Let us rephrase this within the supergravity framework.

Recall the Killing spinor equations:

$$D_m \xi_A + T^{kl}\sigma_{kl}\sigma_m \bar{\xi}_A + i\sigma_m \bar{\xi}'_A = 0,$$
$$D_m \bar{\xi}_A + \bar{T}^{kl}\bar{\sigma}_{kl}\bar{\sigma}_m \xi_A + i\bar{\sigma}_m \xi'_A = 0, \cdots$$

$\bar{\xi}^{\dot{\alpha}}_A = \delta^{\dot{\alpha}}_A,\ \xi_A = \bar{\xi}'_A = \xi'_A = 0$ solves these equations if $T_{mn} = \bar{T}_{mn} = 0$ and

$$D_m \bar{\xi}^{\dot{\alpha}}_A = \frac{1}{4}\Omega^{ab}_m (\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}}_A + i\bar{\xi}^{\dot{\alpha}}_B (V_m)^B{}_A = 0,$$

namely if $V_m = \dfrac{i}{4}\Omega^{ab}_m \bar{\sigma}^{ab}$ as 2 x 2 matrices.

Topologically Twisted Gauge Theory

- If one chooses $\bar{\xi}^{\dot{\alpha}}_A = \delta^{\dot{\alpha}}_A$ and $V_m = \frac{i}{4}\Omega_m^{ab}\bar{\sigma}^{ab}$,

  then the doublets of SU(2) R-symmetry behave the same way as

  anti-chiral spinors under parallel transport.

- Let us look at a SUSY transformation rule

$$\begin{aligned}\mathbf{Q}A_m &= i\xi^A\sigma_m\bar{\lambda}_A - i\bar{\xi}^A\bar{\sigma}_m\lambda_A \\ &= -i\delta^A_{\dot{\alpha}}(\bar{\sigma}_m)^{\dot{\alpha}\alpha}\lambda_{\alpha A} = -i(\bar{\sigma}_m)^{A\alpha}\lambda_{\alpha A} \equiv \psi_m.\end{aligned}$$

  Here we defined ψ_m by simply "renaming" the components of $\lambda_{\alpha A}$.

- By a similar renaming we obtain $\bar{\lambda}^{\dot{\alpha}}_A \to \{\chi^+_{mn}, \eta\}$, $D_{AB} \to D^+_{mn}$.

- The SUSY $\mathbf{Q}$ acts on them as

$$\begin{aligned}&\mathbf{Q}A_m = \psi_m, \qquad \mathbf{Q}\chi^+_{mn} = D^+_{mn}, \\ &\mathbf{Q}\bar{\varphi} = \eta, \qquad \mathbf{Q}\phi = 0, \qquad \text{and } \mathbf{Q}^2 = \mathbf{Gauge}(\varphi).\end{aligned}$$

Topologically Twisted Gauge Theory

$$\mathbf{Q}A_m = \psi_m, \quad \mathbf{Q}\chi^+_{mn} = D^+_{mn},$$
$$\mathbf{Q}\bar{\varphi} = \eta, \quad \mathbf{Q}\varphi = 0, \quad \mathbf{Q}^2 = \mathbf{Gauge}(\varphi),$$

Under the above scalar SUSY, the Yang-Mills action is almost exact.

$$S_{\mathrm{YM}} = 2\pi i\tau \cdot \frac{1}{8\pi^2}\int \mathrm{Tr}F \wedge F + \mathbf{Q}(\cdots).$$

This theory is therefore a model to compute the generating function of integrals over instanton moduli spaces,

$$\int \mathcal{D}(\text{fields})e^{-S_{\mathrm{YM}}} = \sum_{k\geq 0} e^{2\pi i\tau k} \cdot (\text{integral over } \mathcal{M}_k),$$

$$\mathcal{M}_k \equiv \frac{(\text{gauge s.t. } F^+_{mn} = 0, \text{ instanton number } k)}{(\text{gauge})}.$$

* the pair χ^+_{mn}, D^+_{mn} serve as the Lagrange multipler to put the constraint $F^+_{mn} = 0$.

Omega background **Nekrasov, 2002**

- To regularize the integral over the moduli spaces,
  Nekrasov considered a deformation of topologically twisted theories on $\mathbb{R}^4$
  such that
$$\mathbf{Q}^2 = \epsilon_1 J_{12} + \epsilon_2 J_{34} + \mathbf{Gauge}(\varphi)$$
  If $\langle\varphi\rangle = a$, after Pestun's gauge fixing this becomes
$$\mathbf{Q}^2 = \epsilon_1 J_{12} + \epsilon_2 J_{34} + \mathbf{Gauge}(a).$$

- The path integral of topological twisted SYM would then give
$$Z(\epsilon_1, \epsilon_2, a, \tau) = \sum_{k\geq 0} e^{2\pi i \tau k} \cdot \mathrm{vol}_{\epsilon_1,\epsilon_2,a}(\mathcal{M}_k).$$
  This is called “Nekrasov's instanton partition function”.

- On the other hand, Nekrasov also argued that the path integral should give
$$Z(\epsilon_1, \epsilon_2, a, \tau) \overset{\epsilon_1,\epsilon_2\to 0}{=} \exp\left(\frac{\mathcal{F}_{\mathrm{inst}}(a,\Lambda) + \mathcal{O}(\epsilon_1,\epsilon_2)}{\epsilon_1\epsilon_2}\right)$$

Omega background

Let us reinterpret the Omega-background within the supergravity framework.

Choose the Killing spinor on $\mathbb{R}^4$ as

$$\bar{\xi}^{\dot{\alpha}}_A = \frac{1}{\sqrt{2}}\delta^{\dot{\alpha}}_A, \quad \xi_{\alpha A} = -v_m(\sigma^m)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}_A, \qquad v^m \equiv (-\epsilon_1 x_2, \epsilon_1 x_1, -\epsilon_2 x_4, \epsilon_2 x_3)$$

so that $\mathbf{Q}^2 = \epsilon_1 J_{12} +_2 J_{34} + \mathbf{Gauge}(\cdots)$.

This satisfies the Killing spinor equation

$$D_m \xi_A + T^{kl}\sigma_{kl}\sigma_m \bar{\xi}_A + i\sigma_m \bar{\xi}'_A = 0,$$
$$D_m \bar{\xi}_A + \bar{T}^{kl}\bar{\sigma}_{kl}\bar{\sigma}_m \xi_A + i\bar{\sigma}_m \xi'_A = 0, \cdots$$

if $V_m = \bar{T}_{kl} = 0$ and $T_{kl} = -\frac{1}{8}D_{[k}v^{-}_{l]}$, or more explicitly

$$\frac{1}{2}T_{kl}dx^k dx^l = \frac{\epsilon_2 - \epsilon_1}{16}(dx^1 dx^2 - dx^3 dx^4).$$

Polar regions of the ellipsoid

Recall we chose the Killing spinor on the ellipsoid as

$$(\xi_A) = (\xi_1\ \xi_2) = \sin\tfrac{\rho}{2}\cdot(\kappa_+, \kappa_-),$$

$$(\bar\xi_A) = (\bar\xi_1\ \bar\xi_2) = \cos\tfrac{\rho}{2}\cdot(i\kappa_+, -i\kappa_-),$$ where $\kappa_\pm$ are Killng spinors on S^3.

Near the north pole $\rho = 0$, ξ_A vanishes linearly but $\bar\xi_A$ remains finite.

By a local Lorentz and SU(2) R-symmetry rotation one can bring them into the form

$$\bar\xi^{\dot\alpha}_A = \frac{1}{\sqrt{2}}\delta^{\dot\alpha}_A, \quad \xi_{\alpha A} = -v_m(\sigma^m)_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}_A, \quad v^m = (-\frac{x_2}{\ell}, \frac{x_1}{\ell}, -\frac{x_4}{\tilde\ell}, \frac{x_3}{\tilde\ell})$$

Therefore, on ellipsoid background we need $T_{mn} \sim \partial_{[m}v^-_{n]} \neq 0.$

Exact Partition Function via Localization

Saddle Points

- On the ellipsoid we have only one supercharge $\mathbf{Q}$.

  We use it for the localization program.

- The usual saddle point condition $\boxed{\mathbf{Q}\Psi = 0 \text{ for all the fermions } \Psi}$

  turned out too complicated to solve.

- On the round sphere, Pestun found a nicer set of saddle-point conditions

$$(\mathbf{Q}\mathcal{V})\big|_{\text{bosonic}} = \sum_{\Psi} |\mathbf{Q}\Psi|^2$$

$$= (\text{total derivative}) \; + \mathrm{Tr}\Big[(D_m\phi_1)^2 - [\phi_1, \phi_2]^2 + \frac{1}{2}(D_{AB} + i\phi_1 w_{AB})^2$$

$$+ (D_m\phi_2)^2 + \xi^A\xi_A(F^-_{mn} + 4\phi_2 S^-_{mn})^2 + \bar{\xi}_A\bar{\xi}^A(F^+_{mn} - 4\phi_2 S^+_{mn})^2\Big]$$

where $\quad \phi_1 = i(\phi + \bar{\phi}),\ \phi_2 = \phi - \bar{\phi},$

$S_{mn} = (\text{a nonzero background 2-form}),$

w_{AB} was introduced at the discussion of FI and mass terms.

Saddle Points (for sphere)

$$\mathrm{Tr}\Big[(D_m\phi_1)^2 - [\phi_1,\phi_2]^2 + \frac{1}{2}(D_{AB} + i\phi_1 w_{AB})^2 \\ +(D_m\phi_2)^2 + \xi^A\xi_A(F^-_{mn} + 4\phi_2 S^-_{mn})^2 + \bar{\xi}_A\bar{\xi}^A(F^+_{mn} - 4\phi_2 S^+_{mn})^2\Big] = 0$$

The above saddle point condition is solved by

$$\phi_1 = a,\ D_{AB} = -iaw_{AB},\ \phi_2 = A_m = 0.$$

But recall

- $\xi_A = 0$ at the north pole
- $\bar{\xi}_A = 0$ at the south pole

$\Rightarrow$

F^-_{mn} can be nonzero at the north pole,

F^+_{mn} can be nonzero at the south pole.

So we need to include the effect of

- point-like instantons localized at the north pole

  -- topologically twisted theory on Ω-background = Nekrasov's partition function

- point-like anti-instantons localized at the south pole

  -- anti topologically twisted theory

Saddle Points (for ellipsoid)

$$\mathrm{Tr}\Big[(D_m\phi_1)^2 - [\phi_1,\phi_2]^2 + \frac{1}{2}(D_{AB} + i\phi_1 w_{AB})^2$$
$$+(D_m\phi_2)^2 + \xi^A\xi_A(F^-_{mn} + 4\phi_2 S^-_{mn})^2 + \bar{\xi}_A\bar{\xi}^A(F^+_{mn} - 4\phi_2 S^+_{mn})^2\Big] = 0$$

For ellipsoid, we could not find such a nice “re-completion” into squares as the above.

So we *assumed* the same (following) saddle-point condition and proceeded.

$$\phi_1 = a,\ D_{AB} = -iaw_{AB},\ \phi_2 = A_m = 0.$$

For hypermultiplets, $q_A = F_{\check{A}} = 0.$

Partition Function

$$Z = \int d^r a e^{-S_{\mathrm{cl}}(a)} Z_{\text{1-loop}}(a, m, \epsilon_1, \epsilon_2) Z_{\mathrm{inst}}(a, m, \epsilon_1, \epsilon_2, q) Z_{\mathrm{inst}}(a, m, \epsilon_1, \epsilon_2, \bar{q})$$

where $\epsilon_1 = \ell^{-1}, \ \epsilon_2 = \tilde{\ell}^{-1}.$

- Nekrasov partition functions Z_{inst}

  express the instanton contributions at N,S poles

- The classical action $S(a)$ is a sum of

$$S_{\mathrm{YM}}(a) = \frac{8\pi^2}{g^2}\ell\tilde{\ell}\mathrm{Tr}(a^2), \quad S_{\mathrm{FI}}(a) = -16i\pi^2\ell\tilde{\ell}\xi a, \quad S_{\mathrm{mat}}(a) = 0.$$

- It remains to compute $Z_{\text{1-loop}}$.

One-Loop Determinant

We compute the determinant using cohomological variables.

- **Vectormultiplet (plus ghosts)**

$$\{A_m, \phi_1, \phi_2, \lambda_{\alpha A}, \bar{\lambda}^{\dot{\alpha}}_A, D_{AB}, c, \bar{c}, B\} \qquad \langle\phi_1\rangle = a$$

consists of 10 Grassmann even fields, 10 Grassmann odd fields.

One can rearrange them into

$$\mathbf{X} \equiv \begin{pmatrix} A_m \\ \phi_2 \end{pmatrix}$$

[5 bosons]

$$\widehat{\mathbf{Q}}\mathbf{X} = \begin{pmatrix} i\xi^A\sigma_m\bar{\lambda}_A - i\bar{\xi}^A\bar{\sigma}_m\lambda_A + D_m c \\ -i\xi^A\lambda_A - i\bar{\xi}^A\bar{\lambda}_A + i[c, \phi_2] \end{pmatrix}$$

[5 fermions]

$$\mathbf{\Xi} \equiv \begin{pmatrix} \Xi_{AB} \equiv 2\bar{\xi}_{(A}\bar{\lambda}_{B)} - 2\xi_{(A}\lambda_{B)} \\ \bar{c} \\ c \end{pmatrix}$$

[5 fermions]

$$\widehat{\mathbf{Q}}\mathbf{\Xi} = \begin{pmatrix} D_{AB} + \cdots \\ B \\ a - \Phi + icc \end{pmatrix}$$

[5 bosons]

$$\Phi \equiv 2i\bar{\xi}^A\xi_A\bar{\phi} - 2i\bar{\xi}^A\bar{\xi}_A\phi - 2i\bar{\xi}^A\bar{\sigma}^m\xi_A A_m$$

One-Loop Determinant (vectormultiplet)

- $Z_{\text{1-loop}}$ is the ratio of determinants of $\widehat{\mathbf{H}} \equiv \widehat{\mathbf{Q}}^2$.

$$Z_{\text{1-loop}} = \frac{\mathrm{Det}(\widehat{\mathbf{H}})_{\boldsymbol{\Xi}}}{\mathrm{Det}(\widehat{\mathbf{H}})_{\mathbf{X}}},$$

$$\begin{aligned}\widehat{\mathbf{H}} \equiv i\epsilon_1 \partial_\varphi + i\epsilon_2 \partial_\chi + i\mathbf{Gauge}(a) + \mathbf{R}_{SU(2)}[-\tfrac{1}{2}(\epsilon_1+\epsilon_2)\boldsymbol{\tau}_3] \\ + \check{\mathbf{R}}_{SU(2)}[\cdots],\end{aligned}$$

$$\left(\epsilon_1 \equiv \frac{1}{\ell}, \epsilon_2 \equiv \frac{1}{\tilde{\ell}}\right)$$

One can find a differential operators mapping between the space of wave functions of $\mathbf{X}$ and $\boldsymbol{\Xi}$,

but the following argument does not rely much on its explicit form.

Atiyah-Bott Fixed Point Theorem

- Now consider, instead of the ratio of determinant, the **difference of traces,**

$$\mathrm{Str}[e^{-it\widehat{\mathbf{H}}}] = \mathrm{Tr}_{\mathbf{X}}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\boldsymbol{\Xi}}[e^{-it\widehat{\mathbf{H}}}].$$

- Here is a nice argument when computing the traces. Recall

$$\begin{aligned}\widehat{\mathbf{H}} \equiv i\epsilon_1\partial_\varphi + i\epsilon_2\partial_\chi + i\mathbf{Gauge}(a) + \mathbf{R}_{SU(2)}[-\tfrac{1}{2}(\epsilon_1+\epsilon_2)\boldsymbol{\tau}_3] \\ + \check{\mathbf{R}}_{SU(2)}[\cdots],\end{aligned} \qquad \left(\epsilon_1 \equiv \frac{1}{\ell}, \epsilon_2 \equiv \frac{1}{\tilde{\ell}}\right)$$

Then $e^{-it\widehat{\mathbf{H}}}$ involves a finite diffeomorphism

$$e^{-it\widehat{\mathbf{H}}}: \ \tilde{\varphi} = \varphi + \epsilon_1 t, \quad \tilde{\chi} = \chi + \epsilon_2 t.$$

It has two fixed points, the north and south poles.

Near the north pole one can introduce local complex coordinates z_1, z_2 such that

$$\begin{aligned} e^{-it\widehat{\mathbf{H}}}: \ \tilde{z}_1 &= z_1 e^{i\epsilon_1 t} \equiv z_1 q_1, \\ \tilde{z}_2 &= z_2 e^{i\epsilon_2 t} \equiv z_2 q_2. \end{aligned}$$

Atiyah-Bott Fixed Point Theorem

- For linear operators expressible in terms of integration kernels,

$$\hat{\mathcal{O}} f(x) \equiv \int dy K_{\mathcal{O}}(x,y) f(y),$$

the trace is given by $\mathrm{Tr}\hat{\mathcal{O}} = \int dx K_{\mathcal{O}}(x,x).$

- $e^{-it\widehat{\mathbf{H}}}$ is a finite diffeomorphism operator, so it is expressed using the kernel

$$K_{e^{-it\widehat{\mathbf{H}}}} = (\cdots) \cdot \delta^2(z_1 - \tilde{z}_1)\delta^2(z_2 - \tilde{z}_2).$$

Note: $\displaystyle\int d^2 z_1 d^2 z_2 \delta^2(z_1 - q_1 z_1)\delta^2(z_2 - q_2 z_2) = \frac{1}{|1-q_1|^2|1-q_2|^2}.$

- The trace therefore localizes onto the two poles where $z_1 = \tilde{z}_1, z_2 = \tilde{z}_2.$

$$\left[\mathrm{Tr}_{\mathbf{X}}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\boldsymbol{\Xi}}[e^{-it\widehat{\mathbf{H}}}]\right]_{\mathrm{NP}} = \frac{\mathrm{Tr}_{\mathbf{X}(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\boldsymbol{\Xi}(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}]}{|1-q_1|^2|1-q_2|^2}$$

Atiyah-Bott Fixed Point Theorem

Let us compute the enumerator of

$$\Big[\mathrm{Tr}_{\mathbf{X}}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\mathbf{\Xi}}[e^{-it\widehat{\mathbf{H}}}]\Big]_{\mathrm{NP}} = \frac{\mathrm{Tr}_{\mathbf{X}(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\mathbf{\Xi}(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}]}{|1-q_1|^2|1-q_2|^2},$$

with $\mathbf{X} \equiv (A_m, \phi_2)$, $\mathbf{\Xi} \equiv (\Xi_{AB}, \bar{c}, c)$ and

$$\widehat{\mathbf{H}} \equiv i\epsilon_1\partial_\varphi + i\epsilon_2\partial_\chi + i\mathbf{Gauge}(a) + \mathbf{R}_{SU(2)}[-\tfrac{1}{2}(\epsilon_1+\epsilon_2)\boldsymbol{\tau}_3]$$

$e^{-it\widehat{\mathbf{H}}}$ rotates

- the four components of A_m by phases $q_1, q_2, \bar{q}_1, \bar{q}_2$
- the 3 components of Ξ_{AB} by phases $q_1 q_2, 1, \bar{q}_1\bar{q}_2$

$$\Big[\mathrm{Tr}_{\mathbf{X}}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\mathbf{\Xi}}[e^{-it\widehat{\mathbf{H}}}]\Big]_{\mathrm{NP}}$$

$$= \mathrm{Tr}_{\mathrm{adj}}(e^{ta}) \cdot \frac{(q_1+q_2+\bar{q}_1+\bar{q}_2+1) - (q_1q_2+1+\bar{q}_1\bar{q}_2+1+1)}{|1-q_1|^2|1-q_2|^2}$$

$$= -\mathrm{Tr}_{\mathrm{adj}}(e^{ta}) \cdot \frac{1+q_1q_2}{(1-q_1)(1-q_2)}$$

From Index to Determinant

we obtained

$$\mathrm{Tr}_{\mathbf{X}}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\mathbf{\Xi}}[e^{-it\widehat{\mathbf{H}}}]$$

$$= \mathrm{Tr}_{\mathrm{adj}}(e^{ta}) \cdot \left\{ \left[- \frac{1+q_1q_2}{(1-q_1)(1-q_2)} \right]_{\mathrm{NP}} + \left[- \frac{1+q_1q_2}{(1-q_1)(1-q_2)} \right]_{\mathrm{SP}} \right\}$$

To translate this into determinant, we need to series expand it in q_1, q_2.

By a suitable "regularization" one finds the correct expansion is

$$= \mathrm{Tr}_{\mathrm{adj}}(e^{ta}) \sum_{m,n\geq 0} \left\{ -q_1^m q_2^n - q_1^{m+1} q_2^{n+1} - q_1^{-m} q_2^{-n} - q_1^{-m-1} q_2^{-n-1} \right\}$$

A term $(+/-) q_1^{-m} q_2^{-n} e^{t\boldsymbol{\alpha}\cdot\boldsymbol{a}}$ corresponds to an eigenvalue

$\widehat{\mathbf{H}} = m\epsilon_1 + n\epsilon_2 + i\boldsymbol{a}\boldsymbol{\alpha}$ in the denominator / enumerator.

Determinant from vectormultiplet

$$Z_{\text{1-loop}}^{\text{vec}} = (\text{const}) \cdot \prod_{\boldsymbol{\alpha}\in\Delta} \prod_{m,n\geq 0} \big(m\epsilon_1 + n\epsilon_2 + i\boldsymbol{a}\boldsymbol{\alpha}\big)\big((m+1)\epsilon_1 + (n+1)\epsilon_2 + i\boldsymbol{a}\boldsymbol{\alpha}\big)$$

This can be expressed using **Upsilon function**

$$\Upsilon(x) = \text{const} \cdot \prod_{m,n\geq 0} (x + mb + nb^{-1})(Q - x + mb + nb^{-1})$$

$$(Q \equiv b + b^{-1})$$

One-Loop Determinant (hypermultiplet)

Let us move from the original set of fields $q_A,\ \psi_\alpha,\ \bar\psi^{\dot\alpha},\ F_{\check A}$

to cohomological variables.

$q_A,$	$\Psi_A \equiv \widehat{\mathbf{Q}} q_A = -i\xi_A\psi + i\bar\xi_A\bar\psi + icq_A,$
$\Xi_{\check A} \equiv \eta_{\check A}\psi - \bar\eta_{\check A}\bar\psi,$	$f_{\check A} \equiv \widehat{\mathbf{Q}}\Xi_{\check A} = F_{\check A} + \cdots .$

The equivariant index localizes onto the north, south poles.

The north-pole contribution is

$$\Big[\mathrm{Tr}_{q_A}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\Xi_{\check A}}[e^{-it\widehat{\mathbf{H}}}]\Big]_{\mathrm{NP}} = \frac{\mathrm{Tr}_{q_A(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\Xi_{\check A}(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}]}{|1-q_1|^2|1-q_2|^2},$$

The enumerator can be computed most easily in the gauge

$$\bar\xi^{\dot\alpha}_{A} \simeq \frac{1}{\sqrt2}\delta^{\dot\alpha}_{A},\quad \xi_{\alpha A} \simeq -\frac{1}{2\sqrt2}v^m(\sigma_m)_{\alpha A} \sim 0,$$

$$\eta^{\check A}_{\alpha} \simeq \frac{1}{\sqrt2}\delta^{\check A}_{\alpha},\quad \bar\eta^{\dot\alpha\check A} \simeq \frac{1}{2\sqrt2}v^m(\bar\sigma_m)^{\dot\alpha\check A} \sim 0.$$

[identification of indices]

$$A \Longleftrightarrow \dot\alpha,\ \check A \Longleftrightarrow \alpha$$

One-Loop Determinant (hypermultiplet)

$e^{-it\widehat{\mathbf{H}}}$ rotates

- the 2 components of $\overbrace{\textbf{anti-chiral spinors}}^{\textbf{and } q_A}$ at NP by phases $q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}, q_1^{-\frac{1}{2}} q_2^{-\frac{1}{2}}$
- the 2 components of $\underbrace{\textbf{chiral spinors}}_{\textbf{and } \Xi_{\check{A}}}$ at NP by phases $q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}}, q_1^{-\frac{1}{2}} q_2^{\frac{1}{2}}$

Why? Notice that $dz_1 dz_2, d\bar{z}_1 d\bar{z}_2$ are anti-self-dual 2-forms, and so are $(\bar{\sigma}_{mn})_{\dot{\alpha}}{}^{\dot{\beta}}$.

$$\Big[\mathrm{Tr}_{q_A}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\Xi_{\check{A}}}[e^{-it\widehat{\mathbf{H}}}]\Big]_{\mathrm{NP}} = \frac{\mathrm{Tr}_{q_A(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\Xi_{\check{A}}(\mathrm{NP})}[e^{-it\widehat{\mathbf{H}}}]}{|1-q_1|^2|1-q_2|^2}$$

$$= \mathrm{Tr}_{R\oplus\bar{R}}(e^{ta}) \cdot \frac{q_1^{\frac{1}{2}} q_2^{\frac{1}{2}} + q_1^{-\frac{1}{2}} q_2^{-\frac{1}{2}} - q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}} - q_1^{-\frac{1}{2}} q_2^{\frac{1}{2}}}{|1-q_1|^2|1-q_2|^2}$$

$$= \mathrm{Tr}_{R\oplus\bar{R}}(e^{ta}) \cdot \frac{q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}}{(1-q_1)(1-q_2)}$$

One-Loop Determinant (hypermultiplet)

$$
\begin{aligned}
&\mathrm{Tr}_{q_A}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\Xi_{\check{A}}}[e^{-it\widehat{\mathbf{H}}}] \\
&\qquad = \mathrm{Tr}_{R\oplus\bar{R}}(e^{ta}) \cdot \left\{ \left[\frac{q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}}{(1-q_1)(1-q_2)} \right]_{\mathrm{NP}} + \left[\frac{q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}}{(1-q_1)(1-q_2)} \right]_{\mathrm{SP}} \right\} \\
&\qquad = \mathrm{Tr}_{R\oplus\bar{R}}(e^{ta}) \sum_{m,n\geq 0} \left\{ q_1^{m+\frac{1}{2}} q_2^{n+\frac{1}{2}} + q_1^{-m-\frac{1}{2}} q_2^{-n-\frac{1}{2}} \right\}
\end{aligned}
$$

Notes:

- The eigenvalues appear in pairs of opposite sign.
- The one-loop determinant is the square-root of the product of eigenvalues, because the hypermultiplet fields obey reality condition.

Partition Function: Summary

$$Z = \int d^r a e^{-S_{\mathrm{cl}}(a)} Z_{\text{1-loop}}(a, m, \epsilon_1, \epsilon_2) Z_{\text{inst}}(a, m, \epsilon_1, \epsilon_2, q) Z_{\text{inst}}(a, m, \epsilon_1, \epsilon_2, \bar{q})$$

where $\epsilon_1 = \ell^{-1},\ \epsilon_2 = \tilde{\ell}^{-1}.$

- Nekrasov partition functions Z_{inst}

  express the instanton contributions at N,S poles

- The classical action $S(a)$ is a sum of

$$S_{\mathrm{YM}}(a) = \frac{8\pi^2}{g^2}\ell\tilde{\ell}\mathrm{Tr}(a^2), \quad S_{\mathrm{FI}}(a) = -16i\pi^2\ell\tilde{\ell}\xi a, \quad S_{\mathrm{mat}}(a) = 0.$$

- One-loop determinant: $Z^{\mathrm{vec}}_{\text{1-loop}} = \prod_{\boldsymbol{\alpha}\in\Delta} \Upsilon(i\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha}), \quad Z^{\mathrm{hyp}}_{\text{1-loop}} = \prod_{\boldsymbol{w}\in R} \Upsilon(\tfrac{Q}{2} + i\hat{\boldsymbol{a}}\cdot\boldsymbol{w})^{-1}$

$$\Upsilon(x) = \mathrm{const}\cdot \prod_{m,n\geq 0}(x + mb + nb^{-1})(Q - x + mb + nb^{-1}) \qquad (Q \equiv b + b^{-1})$$

A closer look at the "regularization"

How to justify our prescription of expanding $\dfrac{1}{(1-q_1)(1-q_2)}$ into series?

Let us recall the problem of equivariant index for hypermultiplet,

$$\mathrm{Str}[e^{-it\widehat{\mathbf{H}}}] \equiv \mathrm{Tr}_{q_A}[e^{-it\widehat{\mathbf{H}}}] - \mathrm{Tr}_{\Xi_{\check{A}}}[e^{-it\widehat{\mathbf{H}}}]$$

One can compute it using fixed point theorem, or one can compute it as an index of a differential operator $\mathcal{D}$,

$$\mathrm{Str}[e^{-it\widehat{\mathbf{H}}}] \equiv \mathrm{Tr}[e^{-it\widehat{\mathbf{H}}}]_{\mathrm{Ker}\mathcal{D}} - \mathrm{Tr}[e^{-it\widehat{\mathbf{H}}}]_{\mathrm{Ker}\mathcal{D}^\dagger}$$

$$\Xi^{\check{A}}(\mathcal{D})_{\check{A}B}q^B \equiv \Xi^{\check{A}}(i\bar{\eta}_{\check{A}}\bar{\sigma}^m\xi_B - i\eta_{\check{A}}\sigma^m\bar{\xi}_B)D_m q^B.$$

The index depends only on the terms of highest order in the derivative in $\mathcal{D}$.

But the behavior of each zeromode of $\mathcal{D}$, $\mathcal{D}^\dagger$ depends on the subleading terms.

A closer look at the “regularization”

Witten proposed to use a vector field v and deform $\mathcal{D}$ so that the zeromodes localize to v-fixed points. **Witten, “Holomorphic Morse Inequalities,” 1984**

$$\Xi^{\check{A}}(\mathcal{D})_{\check{A}B}q^B \equiv \Xi^{\check{A}}(i\bar{\eta}_{\check{A}}\bar{\sigma}^m\xi_B - i\eta_{\check{A}}\sigma^m\bar{\xi}_B)(D_m - 2isv_m)q^B, \quad s \in \mathbb{R}$$

Let us see what kind of zeromodes localize near the NP.

In the gauge $\bar{\xi}^{\dot{\alpha}}_A \sim \delta^{\dot{\alpha}}_A,\ \eta^{\check{A}}_\alpha \sim \delta^{\check{A}}_\alpha,\ \mathcal{D}$ takes the form

$$\mathcal{D} \sim \frac{1}{2}\sigma^m(i\partial_m + 2sv_m) = \begin{pmatrix} \partial_{\bar{z}_2} + s\epsilon_2 z_2 & \partial_{z_1} - s\epsilon_1 z_1 \\ \partial_{\bar{z}_1} + s\epsilon_1\bar{z}_1 & -\partial_{z_2} + s\epsilon_2 z_2 \end{pmatrix}$$

For $\epsilon_1, \epsilon_2 > 0,\ \Xi_{\check{A}}$ has no zeromodes but q_A has the following zeromodes.

$(s > 0)$

$$q^A = \begin{pmatrix} z_1^m z_2^n e^{-s\epsilon_1|z_1|^2 - s\epsilon_2|z_2|^2} \\ 0 \end{pmatrix}$$

$$e^{-it\widehat{\mathbf{H}}} = e^{at}\cdot q_1^{m+\frac{1}{2}} q_2^{n+\frac{1}{2}}$$

$(s < 0)$

$$q^A = \begin{pmatrix} 0 \\ \bar{z}_1^m \bar{z}_2^n e^{s\epsilon_1|z_1|^2 + s\epsilon_2|z_2|^2} \end{pmatrix}$$

$$e^{-it\widehat{\mathbf{H}}} = e^{at}\cdot q_1^{-m-\frac{1}{2}} q_2^{-n-\frac{1}{2}}$$

N=4 SYM

and Gaussian Matrix Models

Wilson loops in N=4 SYM

- A conjecture by Drukker-Gross, Erickson-Semenoff-Zarembo

> The expectation value of supersymmetric circular Wilson loops in N=4 SYM is given by a Gaussian matrix integral.

Pestun succeeded in showing this using localization.

N=2* theory

- The theory of a vectormultiplet and an adjoint hypermultiplet.
- Labeled by the mass parameter m for the hypermultiplet
- a special value of mass $m = m_*$ corresponds to the N=4 theory.

Partition function for N=2* SYM

$$Z = \int d^r\hat{a}\, e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(\hat{\boldsymbol{a}}^2)} |Z_{\mathrm{inst}}|^2 \underbrace{\prod_{\boldsymbol{\alpha}\in\Delta_+} \frac{\Upsilon(i\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha})\Upsilon(-i\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha})}{\Upsilon(\frac{Q}{2}+i\hat{m}+i\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha})\Upsilon(\frac{Q}{2}+i\hat{m}-i\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha})}}_{Z_{1\text{-loop}}}$$

<u>Special values of m:</u>

$$m = \pm iQ/2 \longrightarrow \quad Z_{1\text{-loop}} = 1 \qquad (\text{cf. } \Upsilon(x) = \Upsilon(Q-x))$$

$$Z_{\mathrm{inst}} = \prod_{k\geq 1}(1-q^k)^{-N} \quad (q \equiv e^{2\pi i\tau})$$

$$m = \pm\frac{i}{2}(b^{-1}-b) \longrightarrow \quad Z_{1\text{-loop}} = \prod_{\boldsymbol{\alpha}\in\Delta_+}(\hat{\boldsymbol{a}}\cdot\boldsymbol{\alpha})^2,$$

$$Z_{\mathrm{inst}} = 1.$$

For U(N) the partition function can be written as $Z = \int d^{N^2}M e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(M^2)}$

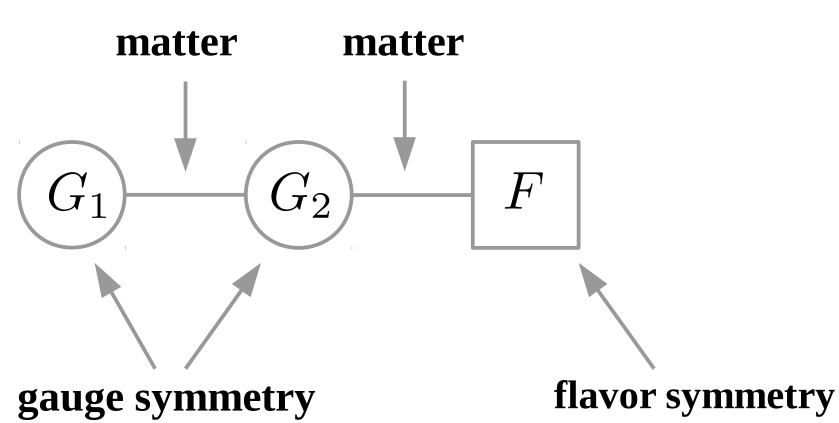
AGT Relation and
Ellipsoid Partition Function

Dynamics of Wrapped M5-branes

- **M5-brane** is another fundamental object in M-theory, which has (5+1)-dimensional worldvolume.
- It is believed that the worldvolume of multiplet M5-branes support a nontrivial 6d SCFT called **(2,0) theory.**
- When N M5-branes are wrapped on a Riemann surface Σ, it gives rise to a 4D $\mathcal{N}=2$ superconformal gauge theory $T_N(\Sigma)$, which is a kind of quiver theory.

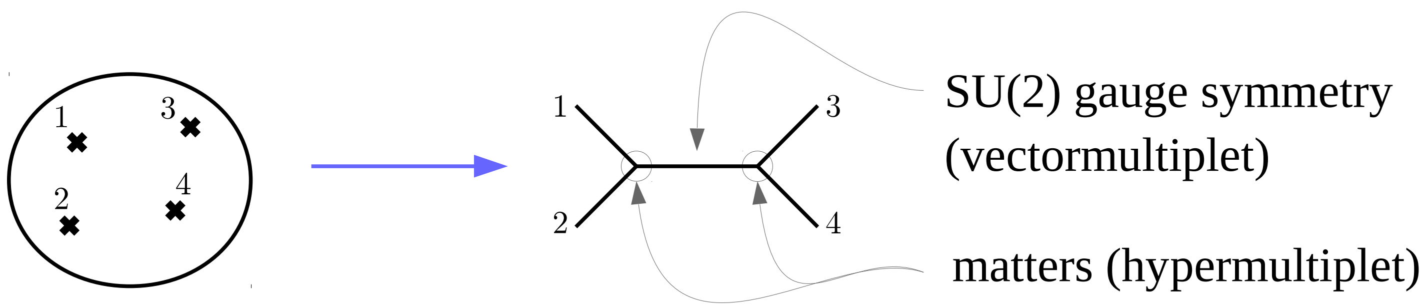
Let us look at examples of $T_2(\Sigma)$ (two M5-branes) for different Σ.

[example of quiver diagram]



Examples of 4D SCFTs

[1] 4-punctured sphere = SU(2) SQCD with $N_{\mathrm{F}} = 4$



Facts:

- The shape (positions of punctures) determines gauge coupling.

  The arrow $\longrightarrow$ corresponds to the weak coupling limit.

- each puncture (external leg) correspond to an SU(2) flavor symmetry.

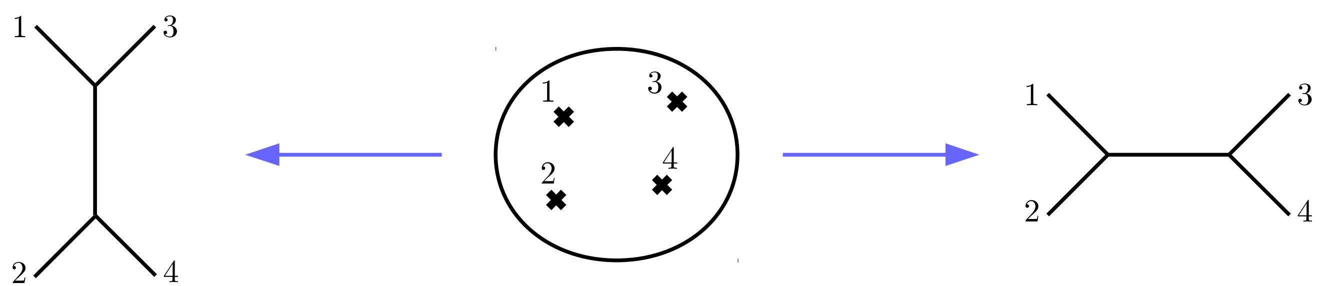
  The SQCD has flavor symmetry

$$SO(8) \supset SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4$$

- One can associate a mass parameter to each puncture.

S-duality

SU(2) SQCD[2] | 4-punctured sphere | SU(2) SQCD[1]



Facts:

- There are more than one weak-coupling limits.

  They give different Lagrangian descriptions for a single "theory".

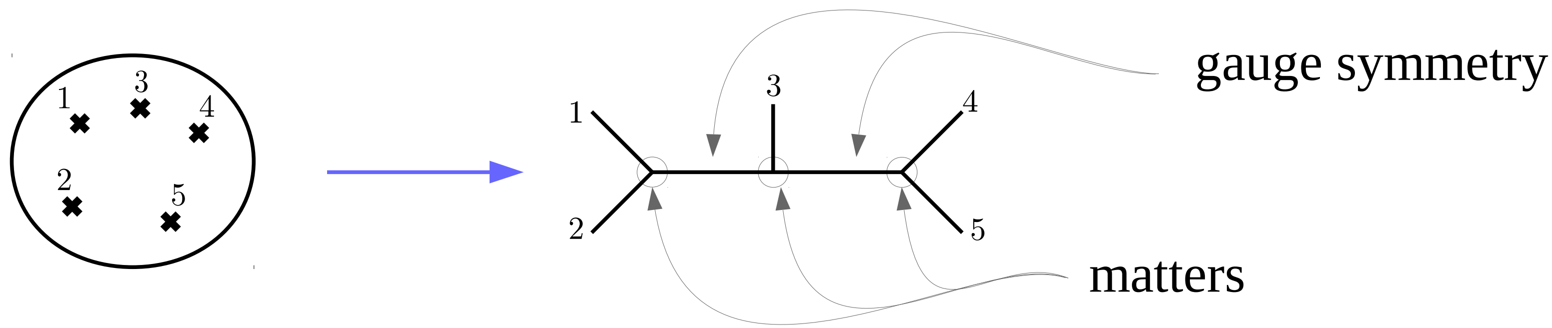
SQCD[2] weak coupling ⟷ SQCD[1] strong coupling

SQCD[2] strong coupling ⟷ SQCD[1] weak coupling

"S-duality"

Examples of 4D SCFTs

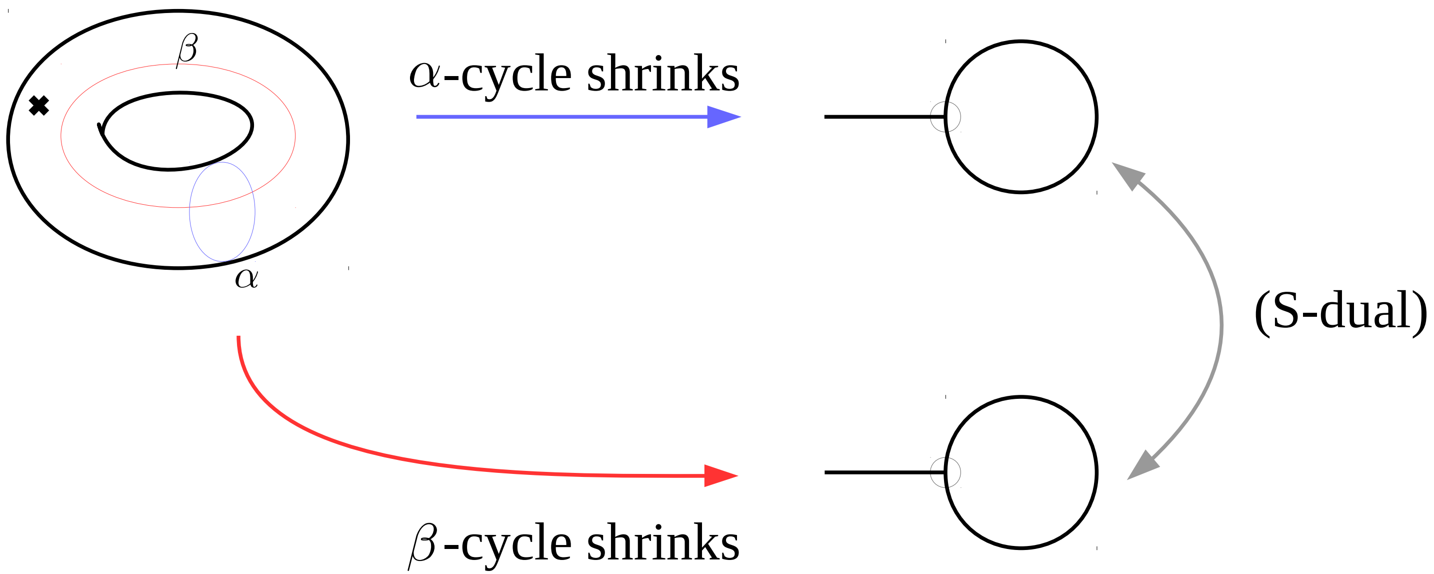
[2] 5-punctured sphere = an $SU(2) \times SU(2)$ gauge theory



Examples of 4D SCFTs

[3] 1-punctured torus = SU(2) theory with an adjoint matter

$(\mathcal{N} = 2^* \text{ theory})$



AGT relation

Alday, Gaiotto, Tachikawa, 2009

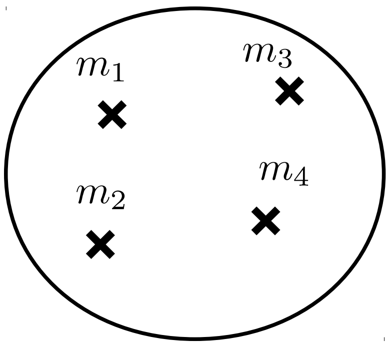
A surprising agreement was found between

- the partition function of $T_2(\Sigma)$ on the round 4-sphere
- the correlator of Liouville theory with c = 25 on Σ.

$$\langle \textstyle\prod_i V_{m_i} \rangle_\tau^{(\text{Liouville})} = Z_{S^4}^{(\text{gauge})}(m, \tau)$$

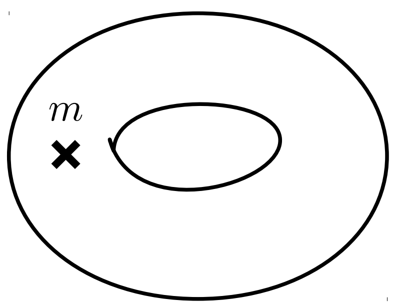
Liouville: sphere 4-point function

Gauge: SU(2) SQCD with 4 doublet matters



Liouville: torus 1-point function

Gauge: SU(2) theory with a triplet matter.



Generalization to $N > 2$ is also known.

Liouville Theory

Liouville theory is an interacting 2D CFT with a coupling constant b.

It has 2 copies of Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_{\mathrm{L}}}{12}(m^3 - m)\delta_{m+n,0}$$

with central charge $c_{\mathrm{L}} = 1 + 6(b + b^{-1})^2$.

It also exhibits strong-weak coupling duality $(b \leftrightarrow 1/b)$.

Z_{S^4} of gauge theory corresponds to Liouville correlators at $b = 1$.

What gauge theories correspond to the Liouville correlators for general b ?

$\longrightarrow$ There should be a SUSY deformation of round sphere.

The SUSY deformation (squashing) was first found for 3-sphere.

The squashing of the 4-sphere turned out much more difficult.

Conformal blocks

Liouville correlator $\langle \prod_i V_{m_i} \rangle_\tau^{(\text{Liouville})}$ satisfies a set of

- holomorphic differential equation in τ
- holomorphic differential equation in $\bar{\tau}$

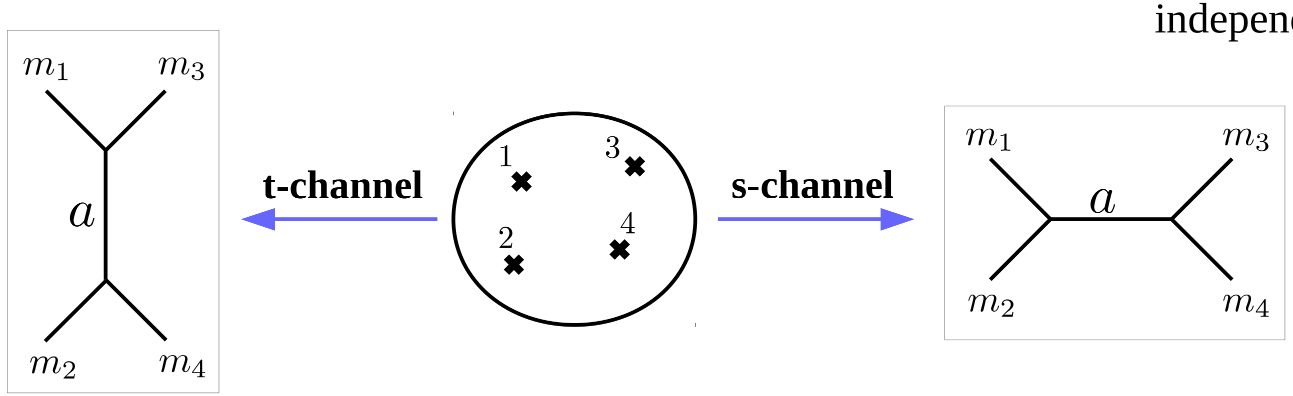
that follow from Virasoro symmetry.

The solutions (holomorphic functions in τ or $\bar{\tau}$) are called **conformal blocks**.

One can find a complete set of conformal blocks for each **channel.**

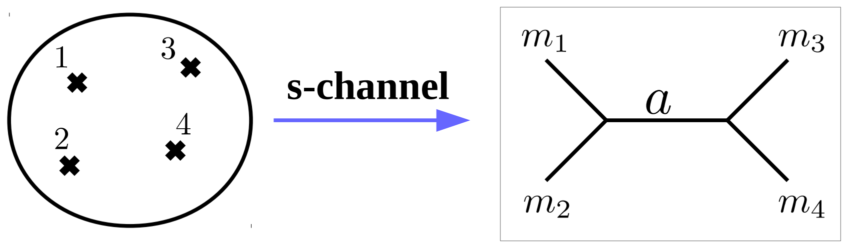
Example: sphere 4-point function

a parametrizes independent solutions

$F^{[t]}_{\{m_i\},a}(\tau) =$  $= F^{[s]}_{\{m_i\},a}(\tau)$

Construction of Correlators

Example: sphere 4-point function



$$= F^{[s]}_{\{m_i\},a}(\tau)$$

$$\langle \textstyle\prod_i V_{m_i} \rangle_\tau^{(\text{Liouville})} = \int da \cdot \Delta(\{m_i\}, a) \left| F^{[s]}_{\{m_i\},a}(\tau) \right|^2$$

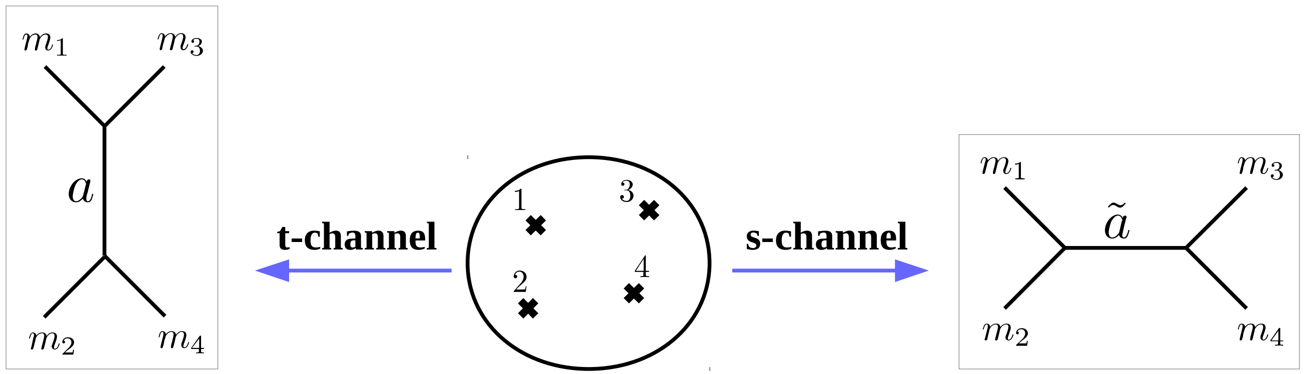
Here $\Delta(\{m_i\}, a)$ has to be chosen so that

the correlator is well-defined and channel-independent.

(The 4-sphere partition function of gauge theories takes the same structure.)

Change of basis

[example] sphere 4-point block in two channels.



$F^{[t]}_{\{m_i\},a}(\tau)$ $\qquad$ $F^{[s]}_{\{m_i\},\tilde{a}}(\tau)$

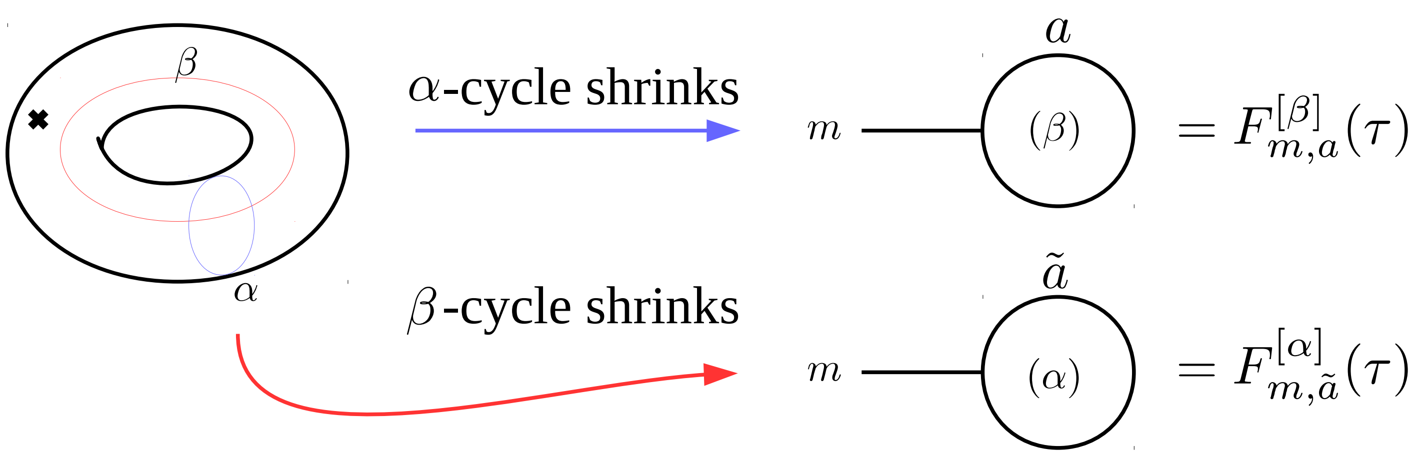
- Recall they are solutions to the same differential equation.

  They are therefore related by a linear transform

$$F^{[t]}_{\{m_i\},a}(\tau) = \int d\tilde{a} G(\{m_i\}, a, \tilde{a}) \cdot F^{[s]}_{\{m_i\},\tilde{a}}(\tau)$$

Change of basis

[example] torus 1-point block in two channels.



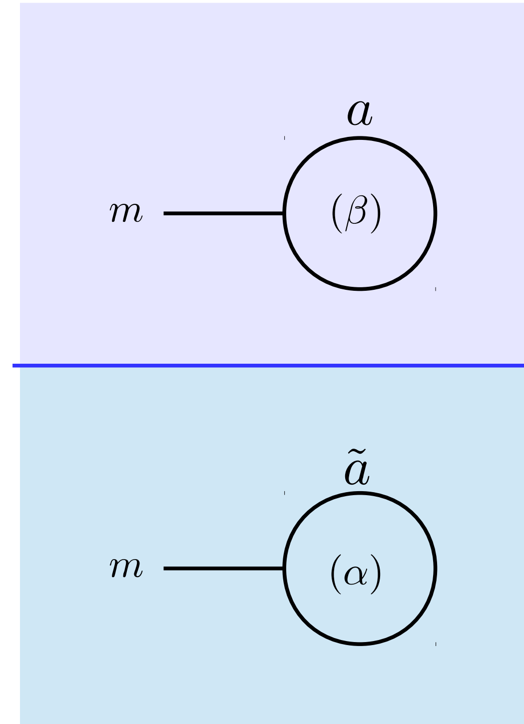
$$F^{[\beta]}_{m,a}(\tau) = \int d\tilde{a} G(m, a, \tilde{a}) \cdot F^{[\alpha]}_{m,a}(\tau)$$

The coefficient $G(m, a, \tilde{a})$ can in principle be computed using only the representation theory of Virasoro algebra.

$$\boxed{G(m, a, \tilde{a}) = \frac{2^{3/2}}{s_b(m)} \int d\sigma \frac{s_b(\tilde{a} + \sigma + \frac{1}{2}m + \frac{iQ}{4}) s_b(\tilde{a} - \sigma + \frac{1}{2}m + \frac{iQ}{4})}{s_b(\tilde{a} + \sigma - \frac{1}{2}m + \frac{iQ}{4}) s_b(\tilde{a} - \sigma - \frac{1}{2}m + \frac{iQ}{4})} e^{4\pi i a \sigma}.}$$

3d ellipsoid partition function?

S-duality wall



We now notice the similarity between

- different channels in which to construct the basis of conformal blocks
- different but mutually S-dual Lagrangians describing the same 4D theory

[A corollary of AGT relation] **Drukker, Gaiotto, Gomis '10**

- When two mutual S-dual theories meet along a 3D wall, a 3D $\mathcal{N}=2$ SUSY gauge theory is induced on it.
- The ellipsoid partition function of the wall theory should agree with $G(m, a, \tilde{a})$.

Identification of the wall theory : for torus 1-point **KH-Lee-Park '10**

for sphere 4-point **Le Floch, '1512**