SUSY ALGEBRA & TRF.

Lorentz Gp.

$$-c^2t'^2 + \vec{r}'^2 = -c^2t^2 + \vec{r}^2 \qquad X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu}$$

$$= X'^{\mu} X'_{\mu} \quad = X^{\mu} X_{\mu} = \eta_{\mu\nu} X^{\mu} X^{\nu},$$

$$= \eta_{\mu\nu} X'^{\mu} X'^{\nu} \qquad \underline{\eta_{\mu\nu} = \overset{\simeq \delta^{\lambda}_{\mu} + \delta\omega^{\lambda}{}_{\mu}}{\Lambda^{\lambda}{}_{\mu}}\, \eta_{\lambda\kappa}\, \overset{\simeq \delta^{\kappa}_{\nu} + \delta\omega^{\kappa}{}_{\nu}}{\Lambda^{\kappa}{}_{\nu}}}$$

For an infinitesimal trf,

$$\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}_{\nu} + \delta\omega^{\mu}{}_{\nu} \qquad \cancel{\eta_{\mu\nu}} \simeq \cancel{\eta_{\mu\nu}} + \underset{\delta\omega_{\mu\nu}}{\underset{\shortparallel}{\eta_{\mu\kappa}\delta\omega^{\kappa}{}_{\nu}}} + \underset{\delta\omega_{\nu\mu}}{\underset{\shortparallel}{\eta_{\lambda\nu}\delta\omega^{\lambda}{}_{\mu}}}$$

$$= \delta^{\mu}_{\nu} + \eta^{\mu\lambda}\eta^{\delta}{}_{\nu}\delta\omega_{\lambda\delta} \qquad \delta\omega_{\mu\nu} + \delta\omega_{\nu\mu} = 0 \quad \text{anti-sym.}$$

① vector

$$= \delta^{\mu}_{\nu} + \frac{i}{2}(M^{\lambda\delta})^{\mu}{}_{\nu}\delta\omega_{\lambda\delta} \qquad \searrow \ 6 \text{ d.o.f}$$

where $\underline{i(M^{\lambda\delta})^{\mu}{}_{\nu} = \eta^{\lambda\mu}\eta^{\delta}{}_{\nu} - \eta^{\delta\mu}\eta^{\lambda}{}_{\nu}}$ vec. rep.

M^{0i} : 3 boosts, M^{ij} : 3 spatial rots

$$\underline{[M^{\mu\nu}, M^{\lambda\delta}] = i(\eta^{\mu\lambda}M^{\nu\delta} - \eta^{\nu\lambda}M^{\mu\delta} - \eta^{\mu\delta}M^{\nu\lambda} + \eta^{\nu\delta}M^{\mu\lambda})}$$

$$\left(J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk} \qquad [J_i, J_j] = i\epsilon_{ijk}J_k\right) \qquad \boxed{\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\epsilon^{abc}\frac{\sigma^c}{2} \quad \text{e.g. In } SU(2),\ U = e^{i\theta^a T^a} \simeq 1 + i\theta^a T^a}$$

$\overset{\leftarrow}{\sigma^a/2}$

For a finite Lorentz trf, $\Lambda = e^{\frac{i}{2}\omega_{\lambda\delta}M^{\lambda\delta}} \in O(3,1)$

O(3,1) 4 sheets

$L^{\uparrow}_{+}$	$\det\Lambda = +1$	$\Lambda^0{}_0 \geq 1$	→ form $SO(3,1) \ni \Lambda$	
$L^{\uparrow}_{-}$	〃 -1	〃		$P\Lambda$
$L^{\downarrow}_{-}$	〃 -1	$\Lambda^0{}_0 \leq -1$		$T\Lambda$
$L^{\downarrow}_{+}$	〃 $+1$	〃		$PT\Lambda$

② spinor

$$\overset{\to \psi^{\dagger}\gamma^0}{\bar{\Psi}}\psi \qquad \bar{\Psi}\gamma^{\mu}\psi \qquad \bar{\Psi}\gamma^{[\mu}\gamma^{\nu]}\psi \qquad \{\gamma^{\mu}\gamma^{\nu}\} = 2\eta^{\mu\nu}$$

"Clifford algebra"

$$\psi_D(x) \longrightarrow \psi'_D(x') = S(\Lambda)\,\psi_D(\overset{\Lambda^{-1}x'}{\overset{\shortparallel}{x}})$$

$$\Sigma^{0i\dagger} = -\Sigma^{0i} \quad \{\Sigma^{0i}\gamma^0\} = 0$$

$$\Sigma^{ij\dagger} = +\Sigma^{ij}$$

$$[\Sigma^{ij}, \gamma^0] = 0$$

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$$

$$S(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}\underset{1/2}{\Sigma^{\mu\nu}}}, \quad \text{where} \quad \underline{\frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}\gamma^{\nu}]}$$

↑**Spinor rep.** satisfying the above Lie algebra

③ scalar field

$$\text{Lorentz trf: } \delta x^{\mu} = \overset{\to\text{anti-sym.}}{\varepsilon^{\mu}{}_{\nu}}x^{\nu}$$

$$\phi'(x') = \phi(\overset{\Lambda^{-1}x'}{\overset{\shortparallel}{x}}), \qquad \delta\phi(x) = \delta x^{\mu}\partial_{\mu}\phi = i\varepsilon^{\mu\nu}(-ix_{\nu}\partial_{\mu})\phi$$

$$= \frac{i}{2}\varepsilon^{\mu\nu}(-ix_{\nu}\partial_{\mu} + ix_{\mu}\partial_{\nu})\phi$$

$$\underline{M^{\mu\nu} = -i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})}$$

$$\left(J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \longrightarrow L_i\right)$$

Poincaré Gp.

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + \ell^\mu$$

① vec.

* transl. $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu = x^\mu + i\varepsilon^\lambda (P_\lambda)^\mu$

where $i(P_\lambda)^\mu = \delta^\mu_\lambda$

Together with $M^{\mu\nu}$ $(= -i\,\eta^\mu\,\eta^\nu + i\,\eta^\nu\,\eta^\mu)$,

$[P^\mu, P^\nu] = 0$, $[M^{\mu\nu}, P^\lambda] = i(\eta^{\mu\lambda}P^\nu - \eta^{\nu\lambda}P^\mu)$

$[M^{\mu\nu}, M^{\lambda\delta}] = i(\eta^{\mu\lambda}M^{\nu\delta} - \cdots)$

② spinor field

* transl. $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu$, $\delta x^\mu = \varepsilon^\mu$

$\psi(x) \to \psi'(x') = \psi(x) + \delta x^\mu \partial_\mu \psi(x)$ $\delta\psi(x) = \psi'(x') - \psi(x)$

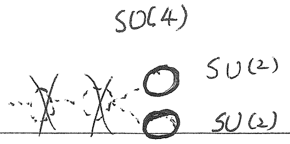
$= \delta x^\mu \partial_\mu \psi(x)$

$P_\mu = -i\partial_\mu$ transl. generator $\psi'(x') = e^{i\varepsilon^\mu P_\mu}\psi(x)$

③ scalar field

* transl. $\delta x^\mu = \varepsilon^\mu$, $\phi(x) \to \phi'(x) = \phi(x) + \delta x^\mu \partial_\mu \phi(x)$, $P_\mu = -i\partial_\mu$

$$SO(3,1) \sim SO(4) \sim SU(2)_L \times SU(2)_R$$



③ $$M^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu) \quad \left\{ \begin{array}{lll} M_{01} = K_1, & M_{02} = K_2, & M_{03} = K_3 \\ M_{12} = L_3 & M_{23} = L_1, & M_{31} = L_2 \end{array} \right.$$

$$(\vec{L} = -i\vec{r}\times\nabla,\ \underset{1}{L_j} = -i\,\epsilon_{jkl}\underset{23-32}{x_k\partial_l})$$

$$K_i = +i(x^0\partial_i + x^i\partial_0) \qquad L_i = -i\,\epsilon_{ijk}x^j\partial_k$$

$$[L_i\ L_j] = i\epsilon_{ijk}L_k, \quad [L_i\ K_j] = i\epsilon_{ijk}K_k, \quad [K_i\ K_j] = -i\epsilon_{ijk}L_k$$

$$K_i^\pm \equiv \frac{1}{2}(L_i \pm iK_i), \quad [K_i^+\ K_j^-] = 0, \quad \left\{ \begin{array}{ll} [K_i^+\ K_j^+] = i\epsilon_{ijk}K_k^+ & SU(2)_L \\ [K_i^-\ K_j^-] = i\epsilon_{ijk}K_k^- & SU(2)_R \end{array} \right.$$

② $$M^{\mu\nu} = \frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu\gamma^\nu] \equiv S,\ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \begin{array}{l} \sigma^\mu = (-1, \vec\sigma) \\ \bar\sigma^\mu = (-1, -\vec\sigma) \end{array}$$

$$\frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix} - (\mu \leftrightarrow \nu) = \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$

$$\frac{1}{2}\Sigma^{0i} = \frac{i}{2}\left(\begin{array}{c|c} \sigma^i & 0 \\ \hline 0 & -\sigma^i \end{array}\right) \qquad \frac{1}{2}\Sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\left(\begin{array}{c|c} \sigma^k & 0 \\ \hline 0 & \sigma^k \end{array}\right)$$

$$\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k + \delta^{ij}$$

$$\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$$

$$\epsilon^{ijk}\epsilon^{ijm} = 3\delta^{km} - \delta^{jm}\delta^{kj}$$
$$= 2\delta^{km}$$

$$\omega_{0i} \equiv \zeta_i \qquad \omega_{ij} \equiv \epsilon_{ijk}\omega^k$$

$$e^{\frac{i}{2}\omega_{\mu\nu}\overset{1/2}{\vee}\Sigma^{\mu\nu}} = \left[\begin{array}{c|c} e^{i(\vec\omega + i\vec\zeta)\cdot\frac{\vec\sigma}{2}} & 0 \\ 0 & e^{i(\vec\omega - i\vec\zeta)\cdot\frac{\vec\sigma}{2}} \end{array}\right]$$

Lorentz trf

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \begin{array}{l} L \\ R \end{array} \qquad \begin{array}{l} M_\alpha{}^\beta = e^{i(\vec\omega + i\vec\zeta)\cdot\frac{\vec\sigma}{2}} \\ M^{\dot\alpha}{}_{\dot\beta} = e^{i(\ -\ )\cdot\frac{\vec\sigma}{2}} \end{array} \qquad \begin{array}{l} \psi'_\alpha(x') = M_\alpha{}^\beta\,\psi_\beta(x) \\ \bar\chi^{\dot\alpha\prime}(x') = \bar M^{\dot\alpha}{}_{\dot\beta}\,\bar\chi^{\dot\beta}(x) \end{array}$$

$$\psi_\alpha\phi_\beta = -\phi_\beta\psi_\alpha, \qquad \bar\chi^{\dot\alpha}\bar\lambda^{\dot\beta} = -\bar\lambda^{\dot\beta}\bar\chi^{\dot\alpha}$$ Grassmann!

$$\varepsilon^{\alpha\beta}\psi_\alpha\phi_\beta, \quad \varepsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\alpha}\bar\lambda^{\dot\beta}$$ Lorentz scalars

* inv. spinor tensor $\varepsilon^{12} = -\varepsilon^{21} = 1 \qquad \varepsilon^{11} = \varepsilon^{22} = 0$

$$\varepsilon_{12} = -\varepsilon_{21} = -1 \qquad \varepsilon_{11} = \varepsilon_{22} = 0$$

$$\underset{\substack{\shortparallel \\ \varepsilon^{-1}}}{\varepsilon_{\alpha\beta}}\,\underset{\substack{\shortparallel \\ \varepsilon}}{\varepsilon^{\beta\gamma}} = \delta_\alpha^\gamma \qquad \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \varepsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\varepsilon_{\alpha\beta} \to M_\alpha{}^\gamma M_\beta{}^\delta\varepsilon_{\gamma\delta} = \varepsilon_{\alpha\beta} \qquad M^{-1} = \varepsilon M^T\varepsilon^{-1}$$

$$(M\varepsilon M^T)_{\alpha\beta} = (M\overbrace{\varepsilon M^T\varepsilon^{-1}}^{\overset{M^{-1}}{\shortparallel}}\varepsilon)_{\alpha\beta} = \varepsilon_{\alpha\beta}$$

* raising & lowering spinor indices

$$\psi^{\alpha} = \varepsilon^{\alpha\beta}\psi_{\beta}, \quad \psi_{\alpha} = \varepsilon_{\alpha\beta}\psi^{\beta}$$

$$\psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\psi^{\dot\beta}$$

$$\chi^{\alpha}\psi_{\alpha} = \psi^{\alpha}\chi_{\alpha} = \varepsilon_{\alpha\beta}\psi^{\alpha}\chi^{\beta}, \quad \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\alpha}\bar\chi^{\dot\beta}$$

* complex conjugate $(\psi_{\alpha})^{\dagger} = \bar\psi_{\dot\alpha}$, $(\chi_{\alpha}\psi_{\beta})^{\dagger} = \bar\psi_{\dot\beta}\bar\chi_{\dot\alpha}$

$$\sigma^{\mu}_{\alpha\dot\alpha} \quad \bar\sigma^{\mu\dot\alpha\alpha} \quad \bar\chi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\psi_{\alpha} = \bar\chi^{\dot\alpha}\sigma^{\mu}{}_{\alpha\dot\alpha}\psi^{\alpha} = -\psi^{\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\bar\chi^{\dot\alpha}$$

a vec. field

(0, 0) scalar
(½ 0) ψ_{α}
(0 ½) $\bar\psi^{\dot\alpha} \approx (\frac{1}{2}\ 0)^{*}$
(½ ½) (½ 0) ⊗ (0 ½) vec. $\bar\chi\bar\sigma^{\mu}\psi$ or $\psi\sigma^{\mu}\bar\chi$
(1, 0) $B_{\mu\nu} = -B_{\nu\mu}$ pseudo-scalar
(0, 1) $\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}B_{\rho\sigma} = \pm B^{\mu\nu}$ 〃
(1 ½) $\psi_{\mu\alpha}$ gravitino
(1, 1) $g_{\mu\nu}$ graviton

Coleman - Mandula Thm.

$$\left\{\begin{array}{llll} P^{\mu} & M^{\mu\nu} & Q_{\alpha} & \bar Q^{\dot\alpha} \\ \sigma^{\mu}_{\alpha\dot\alpha} & \bar\sigma^{\mu\dot\alpha\alpha} & \sigma^{\mu\nu}{}_{\alpha}{}^{\beta} & \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} \end{array}\right.$$

$$\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} \equiv \frac{1}{4}(\bar\sigma^{\mu}\sigma^{\nu} - \bar\sigma^{\nu}\sigma^{\mu})^{\dot\alpha}{}_{\dot\beta}$$

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \equiv \frac{1}{4}(\sigma^{\mu}\bar\sigma^{\nu} - \sigma^{\nu}\bar\sigma^{\mu})_{\alpha}{}^{\beta}$$

In local QFT (w/ massive ptl.)
only possible bosonic conserved quantities are
P_{μ}, $M_{\mu\nu}$, B_{a} (compact Lie gp. internal sym.).

Exceptions
1. if No mass gap, conformal charges
2. in D=1+1
3. if magnitude ∞, domain wall, string, ⋯
V 4. fermionic SUSY generators

① $[P^\mu, Q_\alpha] = a\,\sigma^\mu_{\alpha\dot\alpha}\bar Q^{\dot\alpha}$ ② $\left\{\begin{array}{l}\{Q_\alpha\, Q^\beta\} = b\,\sigma^{\mu\nu}{}_\alpha{}^\beta M_{\mu\nu}\\ \{Q_\alpha\, \bar Q_{\dot\alpha}\} = c\,\sigma^\mu_{\alpha\dot\alpha} P_\mu\end{array}\right.$

③ $[M^{\mu\nu}\, Q_\alpha] = d\,\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta$

$\sigma^{\mu *} = (\sigma^{\mu\dagger})^T = \sigma^{\mu T}$ $\qquad i\sigma^2 \qquad +i\sigma^2$

①* $[P^\mu, \bar Q_{\dot\alpha}] = a^*(\sigma^{\mu T})_{\dot\alpha\beta} Q^\beta$ $\qquad \dot\varepsilon \times$ ①* $\rightarrow$ $\varepsilon^{\dot\gamma\dot\alpha}(\sigma^{\mu T})_{\dot\alpha\beta}\varepsilon^{\beta\gamma}$

$\quad{}^{''}\varepsilon_{\dot\alpha\dot\beta}\bar Q^{\dot\beta} = \dot\varepsilon^{-1}\bar Q \qquad {}^{''}\varepsilon^{\beta\gamma}Q_\gamma = \varepsilon Q \qquad = -\sigma^2(\sigma^{\mu T})\sigma^2$

┌ $\sigma^2\sigma^\mu\sigma^2 = \bar\sigma^{\mu T} \quad \sigma^2\bar\sigma^\mu\sigma^2 = \sigma^{\mu T}$ $\qquad \therefore [P^\mu, \bar Q^{\dot\alpha}] = -a^*\bar\sigma^{\mu\dot\alpha\alpha}Q_\alpha$

$\sigma^2\sigma^{\mu\nu}\sigma^2 = -\sigma^{\mu\nu T} \quad \sigma^2\bar\sigma^{\mu\nu}\sigma^2 = -\bar\sigma^{\mu\nu T}$ ┘

Jacobi ID $\quad [\underset{P^\mu}{A}, [\underset{P^\nu}{B}, \underset{Q}{C}]] + [\underset{P^\nu}{B}, [\underset{Q}{C}, \underset{P^\mu}{A}]] + [\underset{Q}{C}, [\underset{P^\mu}{A}, \underset{P^\nu}{B}]] = 0$

$$[P^\mu, a\sigma^\nu_{\alpha\dot\alpha}\bar Q^{\dot\alpha}] - [P^\nu, a\sigma^\mu_{\alpha\dot\alpha}\bar Q^{\dot\alpha}] + 0 = 0$$

$$-|a|^2(\sigma^\mu\bar\sigma^\nu - \sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta Q_\beta = 0 \qquad a = 0$$

② $[P^\mu\; Q_\alpha] = 0 \qquad \left.\begin{array}{l}[M^{\mu\nu}, P^\lambda] = i(\eta^{\mu\lambda}P^\nu - \eta^{\nu\lambda}P^\mu) \neq 0\\ [\{Q_\alpha, Q^\beta\}, P^\lambda] = 0\end{array}\right\} \therefore b = 0$

But $[P^\mu\, P^\nu] = 0$, and so c can be non-zero. c should be $+$.

$\{Q_\alpha\,\bar Q_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu \qquad \underline{\mathrm{Tr}\,\bar\sigma^\mu\sigma^\nu = -2\eta^{\mu\nu}}$ $\qquad$ (normalized to 2)

$\bar\sigma^{\nu\dot\beta\alpha}\{Q_\alpha\,\bar Q_{\dot\beta}\} = 2\cdot\bar\sigma^{\nu\dot\beta\alpha}\sigma^\mu_{\alpha\dot\beta}P_\mu \qquad \sigma^\mu\bar\sigma^\nu = -\eta^{\mu\nu} + 2\sigma^{\mu\nu}$

$= -4P^\nu$

$\left(\begin{array}{l}\because (\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta = -2\eta^{\mu\nu}\delta_\alpha^\beta\\ (\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} = -2\eta^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta}\end{array}\right)$

$4P^0 = Q_1\bar Q_{\dot 1} + \bar Q_{\dot 1}Q_1 + Q_2\bar Q_{\dot 2} + \bar Q_{\dot 2}Q_2$

$= Q_\alpha Q_\alpha^* + Q_\alpha^* Q_\alpha \geq 0$

$\left(\begin{array}{l}\frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta \equiv \sigma^{\mu\nu}{}_\alpha{}^\beta\\ \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} \equiv \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\end{array}\right)$

③ $\frac{1}{2}\Sigma^{\mu\nu} = i\begin{pmatrix}\sigma^{\mu\nu} & 0\\ 0 & \bar\sigma^{\mu\nu}\end{pmatrix} \qquad \psi'_\alpha(x') = (1 - \frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu})_\alpha^\beta\psi_\beta(x)$

$U^\dagger(\Lambda)Q_\alpha U(\Lambda) \simeq Q_\alpha - \frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}, Q_\alpha]$

$\therefore \quad [M^{\mu\nu}, Q_\alpha] = -i\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta \qquad \therefore d = -i$

$[M^{\mu\nu}, \bar Q^{\dot\alpha}] = -i\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar Q^{\dot\beta}$

$[Q_\alpha, R] = Q_R \qquad [\bar Q_{\dot\alpha}\; R] = -\bar Q_{\dot\alpha}$

$\{ Q_\alpha , \bar{Q}_{\dot\alpha} \} = 2 \sigma^\mu_{\alpha\dot\alpha} P_\mu$

1. massless states $\quad P^\mu = (E, 0, 0, E)$

$$\{ Q_\alpha , \bar{Q}_{\dot\alpha} \} = -2\sigma^0_{\alpha\dot\alpha} E + 2\sigma^3_{\alpha\dot\alpha} E = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

define $a_{1,2} = \frac{Q_{1,2}}{2\sqrt{E}}$, $a^\dagger_{1,2} = \frac{\bar{Q}_{\dot1 \dot2}}{2\sqrt{E}}$ $\quad \{ a_1 a_1^\dagger \} = 1$, $\{ a_2 a_2^\dagger \} = \{ a_\alpha a_\beta \}$

$= \{ a^\dagger_\alpha a^\dagger_\beta \} = 0$

$a_1^\dagger | E, \overset{\swarrow \text{ helicity}}{\lambda} \rangle = ?$ $\quad W_\mu \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$: "Pauli-Lubanski

$W_\mu P^\mu = 0 \quad [W^\mu , P^\nu] = 0$ spin op."

$W_0 = \epsilon_{0312} E M^{12} = \overset{+\epsilon_{312} M^{12} \; \shortparallel}{P^3 J^3}$ $\quad W_\mu W^\mu$: 2nd Casimir of Poincaré gp.

$\shortparallel - W_3 \qquad \propto$ "helicity" $\qquad \overset{\shortparallel}{-m^2 J^2}$

$W_0 | E, \lambda \rangle = \lambda E | E, \lambda \rangle \qquad W_0$

$$W_0 (\bar{Q}_{\dot\alpha} | E, \lambda \rangle) = (\underbrace{[W_0 \bar{Q}_{\dot\alpha}]}_{iE(\bar\sigma_{12})^{\dot\beta}_{\dot\alpha} \bar{Q}_{\dot\beta}} + \lambda E \bar{Q}_{\dot\alpha}) | E, \lambda \rangle$$

from $[M_{\underset{12}{\mu\nu}} , \bar{Q}_{\dot\alpha}] = i \bar{Q}_{\dot\beta} \bar\sigma_{\underset{12}{\mu\nu}}{}^{\dot\beta}{}_{\dot\alpha}$

$i \bar\sigma_{12}{}^{\dot\beta}{}_{\dot\alpha} = \frac{i}{4} (\bar\sigma^1 \sigma^2 - \bar\sigma^2 \sigma^1) = \frac{1}{2} \sigma^3$

$$= (\frac{1}{2} \sigma^3 + \lambda)^{\dot\beta}_{\dot\alpha} E (\bar{Q}_{\dot\beta} | E \lambda \rangle)$$

$\lambda + \frac{1}{2}$ for $\dot\alpha = \dot\beta = 1$

$\lambda - \frac{1}{2}$ for $\dot\alpha = \dot\beta = 2$

For a ground state, $a_1 | G \rangle = a_2 | G \rangle = 0 \qquad | G \rangle \equiv a_1 a_2 | \psi \rangle$

$0 = \langle G | \{ a_2 a_2^\dagger \} | G \rangle = \langle G | a_2 a_2^\dagger | G \rangle \qquad a_2^\dagger | G \rangle = 0$

zero norm state

ϕ , ϕ^*

$\phi \rightarrow 2$ d.o.f $\leftarrow \phi^*$

complex scalar + Weyl fermion ↑

	$\vert G \rangle$	$a_1^\dagger \vert G \rangle$	+ CTP	$\sim \vert \lambda = -\frac{1}{2} \rangle$
Chiral multiplet	$\vert s=0 \rangle$	$\vert \lambda = \frac{1}{2} \rangle (= a_1^\dagger \vert \lambda = 0 \rangle)$	$\vert s=0 \rangle^*$	$(a_1^\dagger \vert \lambda = 0 \rangle)^*$
Vector "	$\vert \lambda = \frac{1}{2} \rangle$	$\vert \lambda = 1 \rangle$	$\vert \frac{1}{2} \rangle^*$ $\sim \vert \lambda = -\frac{1}{2} \rangle$	$\vert \lambda = 1 \rangle^*$

↙ Weyl fermion + vector

2. massive states $\qquad p^\mu = (M, \vec{0})$

$$\{ Q_\alpha \bar{Q}_{\dot\alpha} \} = 2\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} = 2M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a_\alpha \equiv \frac{Q_\alpha}{\sqrt{2M}}, \quad a_\alpha^\dagger \equiv \frac{\bar{Q}_{\dot\alpha}}{\sqrt{2M}}$$

$$\{a_\alpha, a_\beta^\dagger\} = \delta_{\alpha\beta}, \quad \{a_\alpha, a_\beta\} = \{a_\alpha^\dagger, a_\beta^\dagger\} = 0$$

For a ground state, $Q_\alpha |G\rangle = 0$, $\quad |G\rangle = Q_1 Q_2 |\psi\rangle$

$$\Gamma_\alpha \equiv a_\alpha + a_\alpha^\dagger, \quad \Gamma_{\alpha+2} \equiv (a_\alpha - a_\alpha^\dagger)$$

$$\{ \Gamma_\alpha \Gamma_\beta \} = 2\delta_{\alpha\beta} \quad (\alpha, \beta = 1,2,3,4)$$

"Clifford vacuum"

$$SO(4) \text{ generator} = \frac{i}{4} [\Gamma_\alpha \Gamma_\beta]$$

2 scalars
2 fermions
← + CTP

$\psi^c = \psi$

$M \chi^\alpha \lambda_\alpha = M \epsilon^{\alpha\beta} \lambda_\alpha \lambda_\beta$

Chiral multiplet $\quad |0\rangle \quad a_{1,2}^\dagger |0\rangle, \quad a_1^\dagger a_2^\dagger |0\rangle$

1 complex scalar + 1 ~~Weyl~~ fermion
Maj

$S_z \quad 0 \quad \pm\frac{1}{2} \quad 0$

massless χ multiplet $\qquad$ massless χ multiplet

Maj $\begin{pmatrix} \pm 1/2 \\ 0 \end{pmatrix}$ + CTP
or
Dirac $\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$ + CTP

„$|\alpha\rangle$

Vec. multiplet $\quad |\tfrac{1}{2}\rangle \quad a_{1,2}^\dagger |\tfrac{1}{2}\rangle \quad a_1^\dagger a_2^\dagger |\tfrac{1}{2}\rangle$

1 vec + 1 scalar + 2 Weyl fermions

$\pm\frac{1}{2} \quad \pm 1, 0, 0 \quad \pm\frac{1}{2}$

Majorana fermion → massless multiplet

Goldstone + superpartner
(Higgs)

$$\overset{\sim M^{12}}{W_3 = \epsilon_{3012}\, m J^3} = -m J^3 \qquad W_3 |m\, S_z\rangle = -m S_z |m\, S_z\rangle$$

$$W_3 (\bar{Q}_{\dot\alpha} |m S_z\rangle) = (\underbrace{[W_3\ \bar{Q}_{\dot\alpha}]}_{-m(i \underbrace{\bar{\sigma}_{12}}_{=\frac{1}{2}\sigma^3})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}} - m S_z \bar{Q}_{\dot\alpha}) |m\, S_z\rangle = -m \underbrace{(\tfrac{1}{2}\sigma^3 + S_z)}_{\substack{S_z + \frac{1}{2} \text{ for } \dot\alpha = \dot\beta = 1 \\ S_z - \frac{1}{2} \text{ for } \dot\alpha = \dot\beta = 2}} \bar{Q}_{\dot\alpha} |m S_z\rangle$$

** $Q_\alpha|f\rangle = |b\rangle, \quad Q_\alpha|b\rangle = |f\rangle$

$\bar{Q}^{\dot\alpha}|f\rangle = |b\rangle, \quad \bar{Q}^{\dot\alpha}|b\rangle = |f\rangle$

→ fermion number op.

$(-1)^F Q_\alpha = -Q_\alpha(-1)^F$

$\mathrm{Tr}\left[(-1)^F\{Q_\alpha \bar{Q}_{\dot\beta}\}\right] = \mathrm{Tr}\left[-Q_\alpha(-1)^F\bar{Q}_{\dot\beta} + \bar{Q}_{\dot\beta}Q_\alpha(-1)^F\right]$

↘ $\sum_{\text{states}}$ $= \mathrm{Tr}\left[\quad '' \quad + Q_\alpha(-1)^F\bar{Q}_{\dot\beta}\right] = 0$

$\therefore \quad 2\sigma^\mu_{\alpha\dot\alpha}\,\mathrm{Tr}\left[(-1)^F P_\mu\right] = 0$

for non-zero P_μ $\quad \mathrm{Tr}(-1)^F = n_B - n_F = 0$

** $P^2 = -m^2 \quad [P^2, Q] = 0$

$|\psi\rangle, \quad Q|\psi\rangle$ have the same mass.

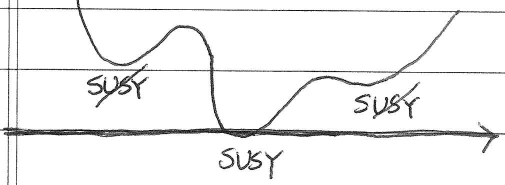
↖ superpartner ↗

** $4P^0 = Q_\alpha Q_\alpha^* + Q_\alpha^* Q_\alpha \geq 0$ ↗ Not a ground state but the Vacuum state

$\langle 0|P^0|0\rangle = 0 \Leftrightarrow Q_\alpha|0\rangle = 0 \quad \bar{Q}_{\dot\alpha}|0\rangle = 0$

E = 0 state $\qquad$ SUSY vacuum

(lowest energy)

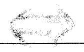


It is meaningless to consider vacuum energy in rigid SUSY as we can redefine the vacuum energy by the redefinition of phases.

$|E\rangle \to e^{ict}|E\rangle \qquad H|E\rangle = E|E\rangle \longrightarrow He^{ict}|E\rangle = (E-c)e^{ict}|E\rangle$

The statement implies that SUSY vacuum is the global minimum as it has the lowest energy compared to possible SUSY breaking vacua.

Consider the SUGRA scalar pot. to get $\cancel{\text{SUSY}}$ & $V_F = 0$.



SUSY trf

Grassmann para.

$$U(\xi) \approx 1 + i(\xi^\alpha Q_\alpha + \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha})$$

$$\{\xi^\alpha \xi^\beta\} = \{\xi^\alpha \bar\xi_{\dot\alpha}\} = 0,\quad \{\xi^\alpha Q_\beta\} = \{\xi^\alpha \bar Q^{\dot\beta}\} = \{\bar\xi_{\dot\alpha} Q_\beta\} = \{\bar\xi_{\dot\alpha} \bar Q^{\dot\beta}\} = 0$$

$$[\xi^\alpha, P^\mu] = [\xi^\alpha, M^{\mu\nu}] = 0$$

$$[P^\mu, \xi Q] = [P^\mu, \bar\xi\bar Q] = 0,\quad [M^{\mu\nu}, \xi Q] = -i(\xi\sigma^{\mu\nu}Q),\ [M^{\mu\nu}, \bar\xi\bar Q] = -i(\bar\xi\bar\sigma^{\mu\nu}\bar Q$$

$$\underline{[\xi Q, \eta Q] = [\bar\xi\bar Q, \bar\eta\bar Q] = 0},\quad \underline{[\xi Q, \bar\eta\bar Q] = 2(\xi\sigma^\mu\bar\eta)P_\mu}$$

$$\delta_\xi \hat{\Theta}(x) \equiv [-i(\xi Q + \bar\xi\bar Q), \hat\Theta(x)]$$

2 successive SUSY trf must close the algebra.

$$[\phi] = 1 \quad [\psi] = 3/2,\ [Q] = [\bar Q] = 1/2,\ [\xi] = [\bar\xi] = -1/2$$

$$\delta_\xi \begin{pmatrix}\phi\\ \psi\\ F\end{pmatrix} = +i(\xi Q + \bar\xi\bar Q)\begin{pmatrix}\phi\\ \psi\\ F\end{pmatrix}$$

$$\begin{cases} \delta_\xi\phi(x) = \overset{\sqrt{2}}{\overset{\shortparallel}{a}}\,\xi\psi(x) + \overset{0}{\overset{\shortparallel}{b}}\,\bar\xi\bar\psi(x) \\ \delta_\xi\psi_\alpha(x) = \underset{i\sqrt{2}}{\underset{\shortparallel}{c}}\,\sigma^\mu_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}\partial_\mu\phi(x),\quad \delta_\xi\bar\psi^{\dot\alpha}(x) = -c^*\bar\sigma^{\mu\dot\alpha\alpha}\xi_\alpha\partial_\mu\phi^*(x)\end{cases}$$

$$\delta_\eta\delta_\xi\phi = a\xi\delta_\eta\psi + b\bar\xi\delta_\eta\bar\psi = ac(\xi\sigma^\mu\bar\eta)\partial_\mu\phi - bc^*(\bar\xi\bar\sigma^\mu\eta)\partial_\mu\phi^*$$

$$[\delta_\eta\ \delta_\xi]\phi = \ldots$$

$$\boxed{\begin{matrix}ac = 2i\\ b = 0\end{matrix}}$$

$$\delta_\eta\delta_\xi\phi = [-i(\eta Q + \bar\eta\bar Q), [-i(\xi Q + \bar\xi\bar Q), \phi]]$$

$$[\delta_\eta, \delta_\xi]\phi \underset{\text{Jacobi ID}}{=} [[-i(\eta Q + \bar\eta\bar Q), -i(\xi Q + \bar\xi\bar Q)], \phi] = -2(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta)\underbrace{[P_\mu\ \phi]}_{= i\partial_\mu\phi}$$

In the same way, $\delta_\eta\delta_\xi\psi_\alpha = ca\,\sigma^\mu_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}\eta\partial_\mu\psi$

$$\overset{\text{F. ID①}}{=} -\frac{1}{2}ca\,\sigma^\mu_{\alpha\dot\alpha}(\eta\sigma^\nu\bar\xi)(\partial_\mu\psi\sigma_\nu)_{\dot\beta}\epsilon^{\dot\alpha\dot\beta}$$

$$-ca[(\eta\sigma_\nu\bar\xi)\partial^\nu\psi_\alpha - (\bar\xi\bar\sigma^\mu\partial_\mu\psi)\eta_\alpha] \overset{\text{F. ID②}}{=} \frac{-1}{2}ca(\eta\sigma_\nu\bar\xi)[2\partial^\nu\psi_\alpha + (\sigma^\nu\bar\sigma^\mu)_\alpha{}^\delta\partial_\mu\psi_\delta]$$

$$\therefore [\delta_\eta\ \delta_\xi]\psi_\alpha = -2(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta)i\partial_\mu\psi \quad \overset{\shortparallel}{0}\ \text{by e.o.m.}$$

$$+2i(\eta\bar\xi\bar\sigma^\mu\partial_\mu\psi - \xi\bar\eta\bar\sigma^\mu\partial_\mu\psi) = 0$$

Fiertz ID

$$\begin{cases} ①\ \eta\psi\bar\xi_{\dot\alpha} = \frac{-1}{2}\eta\sigma_\mu\bar\xi(\psi\sigma^\mu)_{\dot\alpha} \\ ②\ \bar\eta\bar\psi\xi_\alpha = \frac{-1}{2}\xi\sigma_\mu\bar\eta(\sigma^\mu\bar\psi)_\alpha \end{cases}$$

$$\begin{cases} \partial_\mu\psi^\gamma\sigma_{\nu\gamma\dot\beta}\epsilon^{\dot\alpha\dot\beta} = \partial_\mu\psi_\delta\epsilon^{\delta\gamma}\sigma_{\nu\gamma\dot\beta}\epsilon^{\dot\beta\dot\alpha} \\ = \epsilon^{\dot\alpha\dot\beta}(\sigma_\nu^T)_{\dot\beta\gamma}\epsilon^{\gamma\delta}\partial_\mu\psi_\delta = -\bar\sigma_\nu^{\dot\alpha\delta}\partial_\mu\psi_\delta \\ \searrow i\sigma^2 \swarrow \\ \sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\delta} = -2\eta^{\mu\nu}\delta_\alpha^\delta - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\delta}\end{cases}$$

$$\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi^* + i\bar\psi\bar\sigma^\mu\partial_\mu\psi \qquad \{+\sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F)\}$$

$$\delta_\xi\mathcal{L} = \partial_\mu[\ldots]^\mu \qquad a = ic^* \quad \therefore |c|^2 = 2$$

On-shell realization

$$\begin{cases}\delta_\xi\phi = \sqrt{2}\xi\psi \\ \delta_\xi\psi = i\sqrt{2}\sigma^\mu\bar\xi\partial_\mu\phi\end{cases}$$

$$(\bar\sigma^\mu\partial_\mu\psi = 0)$$

Off-shell realization

$$\begin{cases}\delta_\xi\phi = \ldots \\ \delta_\xi\psi = \ldots + \sqrt{2}\xi F \\ \delta_\xi F = i\sqrt{2}\bar\xi\bar\sigma^\mu\partial_\mu\psi\end{cases}$$

total deriv.

auxiliary field

$$\begin{pmatrix}F^* + m\phi = 0\\ F + m\phi^* = 0\end{pmatrix}$$

$$\mathcal{L}_m = m(\phi F + \phi^* F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\bar\psi\bar\psi) \quad \mathcal{L}_0 = -|\partial\phi|^2 + i\bar\psi\bar\sigma^\mu\partial_\mu\psi + FF^*$$

※ transl. $x' = x + \epsilon$

$$\hat{\mathcal{O}}(x') = \hat{\mathcal{O}}(x+\epsilon) = \hat{\mathcal{O}}(x) + \epsilon^{\mu}\partial_{\mu}\hat{\mathcal{O}}(x)$$

$$= U^{\dagger}(\epsilon)\,\hat{\mathcal{O}}(x)\,U(\epsilon), \quad \text{where} \quad U(\epsilon) = e^{i\epsilon^{\mu}P_{\mu}}$$

$$\approx \hat{\mathcal{O}}(x) + [-i\epsilon\cdot P, \hat{\mathcal{O}}(x)]$$

$$\rightarrow i\partial_{\mu}\hat{\mathcal{O}}(x) = [P_{\mu}, \hat{\mathcal{O}}(x)]$$

$$\therefore \delta_{\epsilon}\hat{\mathcal{O}}(x) = \hat{\mathcal{O}}(x') - \hat{\mathcal{O}}(x) = -[i\epsilon\cdot P, \hat{\mathcal{O}}(x)]$$

$$\psi(x) = \langle\psi|\,\hat{\mathcal{O}}(x)\,|0\rangle \qquad |\psi'\rangle = U(\epsilon)|\psi\rangle \approx (1 + i\epsilon\cdot P)|\psi\rangle$$

$$\psi'(x') = \langle\psi'|\,\hat{\mathcal{O}}(x)\,|0\rangle = \langle\psi|\,U^{\dagger}(\epsilon)\hat{\mathcal{O}}(x)U(\epsilon)U^{\dagger}(\epsilon)|0\rangle \qquad U^{\dagger}|0\rangle = |0\rangle$$

Vac. is inv.

$$= \langle\psi|U^{\dagger}(\epsilon)\,\hat{\mathcal{O}}(x)U(\epsilon)|0\rangle$$

$$\approx \psi(x) + \langle\psi|[-i\epsilon\cdot P, \hat{\mathcal{O}}(x)]|0\rangle$$

$$\psi'(x') = S\,\psi(x) = S\langle\psi|\,\hat{\mathcal{O}}(x)|0\rangle, \qquad U^{\dagger}(\epsilon)\,\hat{\mathcal{O}}(x)\,U(\epsilon) = S\,\hat{\mathcal{O}}(x)$$

SUPERFIELDS

- supersp. $\{x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha}\}$ super Poincaré gp/Lorentz gp = super transl. gp $\ni e^{i(-x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha})}$

$$\theta\theta = \theta^\alpha\theta_\alpha, \quad \bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$$

$$(\theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha})^\dagger = \bar Q_{\dot\alpha}\bar\theta^{\dot\alpha} + Q^\alpha\theta_\alpha$$

$\therefore$ Hermitian $= \theta Q + \bar\theta\bar Q$

$$\begin{cases} \theta^\alpha\theta^\beta = \frac{-1}{2}\varepsilon^{\alpha\beta}\theta\theta \\ \theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta \end{cases} \qquad \theta^1\theta^2 = \frac{-1}{2}\overset{-\theta^2}{\overset{\shortparallel}{(\theta^1\theta_1+\theta^2\theta_2)}} \quad \begin{matrix}\varepsilon^{12}=+1\\ \varepsilon_{12}=-1\end{matrix}$$

$$\begin{cases} \theta\eta = \eta\theta \\ \theta\sigma^\mu\bar\eta = -\bar\eta\bar\sigma^\mu\theta \end{cases}$$

$$\begin{cases} \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \\ \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = \frac{-1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \end{cases} \qquad \begin{cases} (\theta\phi)(\theta\psi) = \frac{-1}{2}(\phi\psi)(\theta\theta) \\ (\bar\theta\bar\phi)(\bar\theta\bar\psi) = \frac{-1}{2}(\bar\phi\bar\psi)(\bar\theta\bar\theta) \end{cases}, \quad (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{-1}{2}\theta\theta\bar\theta\bar\theta\eta^{\mu\nu}$$

* θ-deriv.

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \qquad \partial_\alpha\theta^\beta = \delta_\alpha^\beta, \qquad \partial^\alpha = \frac{\partial}{\partial\theta_\alpha} = -\varepsilon^{\alpha\beta}\partial_\beta \qquad \partial^1\theta_1 = 1 = -\varepsilon^{12}\partial_2\theta^1 = \partial_2\theta^2$$

$$\bar\partial^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} \qquad \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta} \qquad \bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} = -\varepsilon_{\dot\alpha\dot\beta}\bar\partial^{\dot\beta}$$

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \delta_\alpha^\gamma\theta^\beta \qquad \partial_\alpha(\theta\theta) = \varepsilon_{\beta\gamma}\partial_\alpha(\theta^\beta\theta^\gamma) = 2\theta_\alpha$$

$$\partial^2 = \partial^\alpha\partial_\alpha \qquad \partial^2\theta^2 = 2\partial^\alpha\theta_\alpha = 4$$

* θ-integ.

$$\int d\theta\,\theta = 1, \quad \int d\theta\,1 = 0 \qquad f(\theta) = f_0 + \theta f_1 \qquad \int d\theta f(\theta) = f_1$$

$$\int d\theta f(\theta+\xi) = f_1 \quad \text{(transl. inv.)}$$

$$\delta(\theta) = \theta \qquad \int d\theta\,\delta(\theta) = 1 \qquad \int d\theta\,\delta(\theta)f(\theta) = f_0 = f(\theta=0)$$

$$\int d\theta\,\delta(\theta-\xi)f(\theta) = \int d\theta(\theta-\xi)f(\theta) = f_0 + \xi f_1 = f(\theta=\xi)$$

$$d^2\theta = \frac{1}{2}d\theta^1 d\theta^2 \quad (\theta^2 = \theta^\alpha\theta_\alpha = -2\theta^1\theta^2 = 2\theta^2\theta^1) \qquad d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta_\alpha, \quad d^2\bar\theta = -\frac{1}{4}d\bar\theta_{\dot\alpha}d\bar\theta^{\dot\alpha}$$

$$d^4\theta = d^2\theta d^2\bar\theta$$

$$\int d^2\theta\,\theta^2 = 1 \qquad \int d^2\bar\theta\,\bar\theta^2 = 1$$

- superfield $\mathbb{F}(x,\theta,\bar\theta)$ $\boxed{\text{scalar field}}$

expand in $\theta, \bar\theta$

$$\mathbb{F}(x,\theta,\bar\theta) = f(x) + \theta\phi(x) + \bar\theta\bar\phi(x) + \overset{(\theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta^2)}{\theta\theta}\, m(x) + \bar\theta\bar\theta\,\bar m(x)$$
$$+ \theta\sigma^\mu\bar\theta\, v_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\lambda(x) + \theta\theta\bar\theta\bar\theta d(x)$$

* $G \equiv \exp[i(-x^\mu P_\mu + \theta Q + \bar\theta\bar Q)]$

$$G(x^\mu\,\theta\,\bar\theta)\,G(a^\mu\,\xi\,\bar\xi) = G(x^\mu + a^\mu - i\xi\sigma^\mu\bar\theta + i\theta\sigma^\mu\bar\xi,\ \theta+\xi,\ \bar\theta+\bar\xi)$$

Hausdorff formula

$$e^A e^B = \exp[A+B+\frac{1}{2}[A,B]+\cdots)$$

- transl. $\mathbb{P}_\mu$ (Nöther)

$$e^{-iy^\mu \mathbb{P}_\mu} \mathbb{F}(x,\theta,\bar\theta) e^{iy^\mu \mathbb{P}_\mu} = \mathbb{F}(x+y,\theta,\bar\theta)$$

$$= e^{iy^\mu P_\mu} \mathbb{F}(x,\theta,\bar\theta) \qquad iP_\mu = \partial_\mu \text{ or } P_\mu = -i\partial_\mu$$

- supertransl.

$$e^{-i\eta\mathbb{Q}} \mathbb{F}(x,\theta,\bar\theta) e^{i\eta\mathbb{Q}} = \mathbb{F}(x^\mu - i\eta\sigma^\mu\bar\theta, \theta+\eta, \bar\theta)$$

$$= e^{i\eta Q} \mathbb{F}(x,\theta,\bar\theta) \qquad i\eta Q = \eta^\alpha\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\right)$$

$$e^{-i\bar\eta\bar{\mathbb{Q}}} \mathbb{F}(x,\theta,\bar\theta) e^{i\bar\eta\bar{\mathbb{Q}}} = \mathbb{F}(x^\mu + i\theta\sigma^\mu\bar\eta, \theta, \bar\theta+\bar\eta)$$

$$= e^{i\bar\eta\bar Q} \mathbb{F}(x,\theta,\bar\theta) \qquad i\bar\eta\bar Q = \bar\eta_{\dot\alpha}\left(\frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\varepsilon^{\dot\beta\dot\alpha}\partial_\mu\right)$$

$\therefore$ SUSY generators
(a rep.)

$$\begin{cases} iQ_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \\ i\bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \end{cases} \qquad \bar Q^{\dot\beta} = \varepsilon^{\dot\beta\dot\alpha}\bar Q_{\dot\alpha}$$

$$\partial_{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\partial^{\dot\beta} \qquad = \varepsilon^{\dot\beta\dot\alpha}(-\partial_{\dot\alpha} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu)$$

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}(-i\partial_\mu) \qquad = +\partial^{\dot\beta} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\dot\alpha\dot\beta}\partial_\mu$$

$$\delta_\eta\mathbb{F} = [-i(\eta\mathbb{Q} + \bar\eta\mathbb{Q}), \mathbb{F}], \qquad [\delta_\eta\ \delta_\xi]\mathbb{F} = -2(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta)i\partial_\mu\mathbb{F}$$

$$\delta_\eta\mathbb{F} = i(\eta Q + \bar\eta\bar Q)\mathbb{F}$$

- cov. deriv.

$$\begin{cases} D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \\ \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \end{cases} \qquad \{D_\alpha\ \bar D_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

$$\{D\ D\} = \{\bar D\ \bar D\} = 0 \qquad \{\xi^\alpha D_\beta\} = \{\xi^\alpha \bar D_{\dot\beta}\} = \{\bar\xi_{\dot\alpha} D_\beta\} = \{\bar\xi_{\dot\alpha}\bar D_{\dot\beta}\} = 0$$

$$\{D_\alpha \bar Q_{\dot\beta}\} = \{D_\alpha Q_\beta\} = \{\bar D_{\dot\alpha} Q_\beta\} = \{\bar D_{\dot\alpha}\bar Q_{\dot\beta}\} = 0$$

$$y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar\theta \qquad y^{\mu\dagger} = x^\mu - i\theta\sigma^\mu\bar\theta$$

$$\bar D_{\dot\alpha} y^\mu = -i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha} = 0, \quad \bar D_{\dot\alpha}\theta^\alpha = 0, \quad \bar D_{\dot\alpha}\bar\theta^{\dot\beta} = -\delta^{\dot\beta}_{\dot\alpha}$$

D_α, $\bar D_{\dot\alpha}$ are inv. under SUSY trf. $\because [D_\alpha, \xi Q] = [D_\alpha, \bar\xi\bar Q]$
$= [\bar D_{\dot\alpha}, \xi Q] = [\bar D_{\dot\alpha}, \bar\xi\bar Q] = 0$

$$\left(\begin{aligned} \theta'^{\alpha} &= \theta^{\alpha} + \xi^{\alpha} \\ \bar{\theta}'_{\dot{\alpha}} &= \bar{\theta}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} \\ x'^{\mu} &= x^{\mu} - i\xi\sigma^{\mu}\bar{\theta} + i\theta\sigma^{\mu}\bar{\xi} \end{aligned} \right.$$

$$\partial_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} = \frac{\partial x'^{\nu}}{\partial\theta^{\alpha}}\partial'_{\nu} + \frac{\partial\theta'^{\beta}}{\partial\theta^{\alpha}}\partial'_{\beta} + \frac{\partial\bar{\theta}'_{\cancel{\dot{\beta}}}}{\partial\theta^{\alpha}}\bar{\partial}'^{\dot{\beta}}$$

$$= i\sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial'_{\nu} + \partial'_{\alpha}$$

$$,\quad \bar{\partial}_{\dot{\alpha}} = +i\xi^{\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial'_{\nu} + \bar{\partial}'_{\dot{\alpha}}$$

$$\partial_{\mu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\partial'_{\nu} + \frac{\partial\theta'^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial\theta'^{\alpha}} + \frac{\partial\bar{\theta}'_{\dot{\alpha}}}{\partial x^{\mu}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} = \partial_{\mu}$$

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}$$

$$= i\sigma^{\mu}_{\alpha\dot{\alpha}}\cancel{\bar{\xi}^{\dot{\alpha}}}\partial'_{\mu} + i\sigma^{\mu}_{\alpha\dot{\alpha}}(\bar{\theta}'^{\dot{\alpha}} - \cancel{\bar{\xi}^{\dot{\alpha}}})\partial'_{\mu} \qquad = -i\cancel{\xi^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial'_{\mu}} - \bar{\partial}'_{\dot{\alpha}} - i(\theta'^{\alpha} - \cancel{\xi^{\alpha}})\sigma^{\mu}_{\alpha\dot{\alpha}}\partial'_{\mu}$$

$$= D'_{\alpha} \quad + \partial'_{\alpha} \qquad = \bar{D}'_{\dot{\alpha}}$$

$\mathbb{F}(x\theta\bar\theta)$ is highly reducible rep. of SUSY.
We can impose covariant constraints to reduce the rep.

$\bar D_\alpha \mathbb{F} = 0$ ← chiral or scalar multiplet $(\phi\ \psi_\alpha\ F)$
$\mathbb{F} = \mathbb{F}^\dagger$ ← vec. multiplet $(v_\mu\ \lambda_\alpha\ D)$

— Chiral superfield

$$\bar D_{\dot\alpha}\Phi = \left[-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\right]\Phi(x\theta\bar\theta) = 0$$
$$\leftrightarrow \Phi = \Phi(y,\theta)$$

$$\Phi(y,\theta) = \phi(y) + \sqrt2\,\theta\psi(y) + \theta\theta F(y)$$
$$= \phi(x) + i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi(x) + \frac12 \underbrace{i\theta\sigma^\mu\bar\theta\, i\theta\sigma^\nu\bar\theta}_{}\partial_\mu\partial_\nu\phi(x)$$

$\overset{..}{=} \frac12\theta^2\bar\theta^2\eta^{\mu\nu}$

$(\theta\phi)(\theta\psi) = \frac{-1}{2}(\phi\psi)(\theta\theta)$
$(\bar\theta\bar\phi)(\bar\theta\bar\psi) = \frac{-1}{2}(\bar\phi\bar\psi)(\bar\theta\bar\theta)$

$$+ \sqrt2\,\theta\psi(x) + \underbrace{\sqrt2\, i\theta\sigma^\mu\bar\theta\,\theta\partial_\mu\psi(x)}_{} + \theta\theta F(x)$$

$= \frac{i}{\sqrt2}\theta^2\bar\theta\bar\sigma^\mu\partial_\mu\psi = -\frac{i}{\sqrt2}\theta^2\partial_\mu\psi\sigma^\mu\bar\theta$

anti-chiral field $\bar\Phi$ $\quad D_\alpha\bar\Phi = 0 \leftrightarrow \bar\Phi = \bar\Phi(y^\dagger\bar\theta)$

under $(x^\mu, \theta, \bar\theta) \to (y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta, \theta, \bar\theta)$
$$iQ_\alpha = \frac{\partial}{\partial\theta^\alpha}, \quad i\bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu}$$

$$\delta_\eta\Phi = i(\eta Q + \bar\eta\bar Q)\Phi = i(\eta^\alpha Q_\alpha - \bar\eta^{\dot\alpha}\bar Q_{\dot\alpha})\Phi$$
$$= \left(\eta^\alpha\frac{\partial}{\partial\theta^\alpha} + \cancelto{0}{\bar\eta^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}} + 2i\theta\sigma^\mu\bar\eta\frac{\partial}{\partial y^\mu}\right)\Phi$$
$$= \sqrt2\,\eta\psi + 2\eta\theta F + 2i\theta\sigma^\mu\bar\eta\frac{\partial}{\partial y^\mu}\phi + 2\sqrt2\, i\theta\sigma^\mu\bar\eta\,\theta\frac{\partial}{\partial y^\mu}\psi$$
$$= \sqrt2\,\eta\psi(y) + \sqrt2\,\theta\left(\sqrt2\,\eta F + \sqrt2\, i\sigma^\mu\bar\eta\frac{\partial}{\partial y^\mu}\phi\right) - \sqrt2\, i\theta\theta\frac{\partial}{\partial y^\mu}\psi\sigma^\mu\bar\eta$$
$$= \delta_\eta\phi + \sqrt2\,\theta\delta_\eta\psi + \theta\theta\delta_\eta F$$

$\therefore$
$$\begin{cases}\delta_\eta\phi = \sqrt2\,\eta\psi \\ \delta_\eta\psi = i\sqrt2\,\sigma^\mu\bar\eta\,\partial_\mu\phi + \sqrt2\,\eta F \\ \delta_\eta F = i\sqrt2\,\bar\eta\bar\sigma^\mu\partial_\mu\psi\end{cases}$$
$(= \partial_\mu(i\sqrt2\,\bar\eta\bar\sigma^\mu\psi)$ total deriv.$)$

(For a General Superfield $\mathbb{F}$, $\delta_\eta\mathbb{F}|_{\theta\theta\bar\theta\bar\theta}$ = total deriv.
For a chiral superfield Φ, $\delta_\eta\Phi|_{\theta\theta}$ = total deriv.

Φ_1, Φ_2 are chiral → $\Phi_1\Phi_2$ is chiral.
$\Phi\Phi^\dagger$, $\Phi + \Phi^\dagger$ → not chiral but still superfield

$$\begin{cases} iQ_\alpha = \dfrac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu & i\bar{Q}_{\dot\alpha} = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \\ D_\alpha = \dfrac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu & \bar{D}_{\dot\alpha} = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \end{cases}$$

$$(x^\mu, \theta, \bar\theta) \to (\underset{x^\mu + i\theta\sigma^\mu\bar\theta}{\underset{\parallel}{y^\mu}}, \overset{\theta}{\overset{\parallel}{\theta'}}, \overset{\bar\theta}{\overset{\parallel}{\bar\theta'}})$$

$$\begin{cases} \partial_\alpha = \dfrac{\partial\theta'^\beta}{\partial\theta^\alpha}\dfrac{\partial}{\partial\theta'^\beta} + \dfrac{\partial y^\mu}{\partial\theta^\alpha}\dfrac{\partial}{\partial y^\mu} \\ \quad = \partial'_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\dfrac{\partial}{\partial y^\mu} \\ \bar\partial_{\dot\alpha} = \bar\partial'_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\dfrac{\partial}{\partial y^\mu} \end{cases}$$

$$\begin{cases} iQ_\alpha = \partial_\alpha & i\bar{Q}_{\dot\alpha} = -\bar\partial_{\dot\alpha} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\dfrac{\partial}{\partial y^\mu} \\ D_\alpha = \partial_\alpha + 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\dfrac{\partial}{\partial y^\mu}, & \bar{D}_{\dot\alpha} = -\bar\partial_{\dot\alpha} \end{cases}$$

$$\Phi^\dagger_i(y^\dagger\bar\theta)\,\Phi_j(y,\theta) \Rightarrow \Phi^\dagger_i(x\theta\bar\theta)\,\Phi_j(x\theta\bar\theta)\Big|_{\theta^2\bar\theta^2} \qquad i=1,2,\cdots,n$$

$$= (-i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi_i^*)(i\theta\sigma^\nu\bar\theta\,\partial_\nu\phi_j) \qquad \rightarrow -\tfrac{1}{2}\partial_\mu\phi_i^*\partial^\mu\phi_j$$

$$+\theta\theta\bar\theta\bar\theta\,F_i^*F_j$$

$$+\tfrac{1}{2}(i\theta\sigma^\mu\bar\theta)(i\theta\sigma^\nu\bar\theta)(\phi_i^*\partial_\mu\partial_\nu\phi_j + \partial_\mu\partial_\nu\phi_i^*\phi_j)$$

$$+2(\bar\theta\bar\psi_i\,i\theta\sigma^\mu\bar\theta\,\theta\partial_\mu\psi_j + \theta\psi_j(-i\theta\sigma^\mu\bar\theta)\bar\theta\partial_\mu\bar\psi_i)$$

$$=\theta\theta\bar\theta\bar\theta\Big[F_i^*F_j - \tfrac{1}{2}\partial_\mu\phi_i^*\partial^\mu\phi_j + \tfrac{1}{4}(\phi_i^*\Box\phi_j + \Box\phi_i^*\phi_j)$$
$$-\tfrac{i}{2}\bar\psi_i\sigma^\mu\partial_\mu\psi_j + \tfrac{i}{2}\partial_\mu\bar\psi_i\bar\sigma^\mu\psi_j\Big)$$

$$\mathcal{L}_{\text{kinetic}} = \Phi_i^\dagger\Phi_i\Big|_{\theta\theta\bar\theta\bar\theta} = -\partial_\mu\phi_i^*\partial^\mu\phi_i + F_i^*F_i + i\underbrace{\partial_\mu\bar\psi_i\bar\sigma^\mu\psi_i}_{\text{Hermitian}}$$

"superpotential"

$$W[\Phi_i] = +\Big[\lambda_i\Phi_i + \tfrac{1}{2}m_{ij}\Phi_i\Phi_j + \tfrac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \cdots\Big]$$

$$\mathcal{L}_{\text{pot}} = +W\big|_{\theta\theta} + \overline{W}\big|_{\bar\theta\bar\theta}$$

$$= \lambda_iF_i + \tfrac{1}{2}m_{ij}(\phi_iF_j + \phi_jF_i - \psi_i\psi_j)$$
$$+\tfrac{1}{3}g_{ijk}(F_i\phi_j\phi_k + F_j\phi_k\phi_i + F_k\phi_i\phi_j$$
$$-\psi_i\psi_j\phi_k - \psi_j\psi_k\phi_i - \psi_k\psi_i\phi_j) + \text{h.c.}$$

$\rightarrow 2\phi_iF_j \times m_{ij}$

$\rightarrow 3\phi_i\phi_jF_k \times g_{ijk}$

$\rightarrow -3\psi_i\psi_j\phi_k \times g_{ijk}$

SUSY action $$S = \int d^4x\Big[\int d^4\theta\,\Phi_i^\dagger\Phi_i + \Big(\int d^2\theta\,W(\Phi) + \text{h.c.}\Big)\Big]$$

$$\partial\mathcal{L}/\partial F_k^* = F_k + (\overbrace{\lambda_k + m_{ik}\phi_i + g_{ijk}\phi_i\phi_j}^{\boxed{\frac{\partial}{\partial\phi_k}W}})^* = 0$$

$$\partial\mathcal{L}/\partial F_k = F_k^* + (\qquad ''\qquad '' \qquad) = 0$$

$$= \frac{\partial W}{\partial\phi_k}$$

$$\underline{V_F = \sum_k |F_k|^2 = \sum_k\left|\frac{\partial W}{\partial\phi_k}\right|^2}, \qquad \underline{-\mathcal{L}_{\text{Yukawa}} = \frac{1}{2}\sum_{ij}\frac{\partial^2W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j + \text{h.c.}}$$

$$\geq 0$$

$$\Phi_i\Phi_j = \phi_i(y)\phi_j(y) + \sqrt{2}\theta\big[\psi_i(y)\phi_j(y) + \phi_i(y)\psi_j(y)\big] + \theta\theta[\cdots]$$

chiral superfields

$$\Phi_i\Phi_j\Phi_k = \phi_i(y)\phi_j(y)\phi_k(y) + \sqrt{2}\theta\big[\psi_i\phi_j\phi_k + \psi_j\phi_k\phi_i + \psi_k\phi_i\phi_j\big]$$
$$+\theta\theta[\cdots]$$

$$\delta_\xi V = i(\xi Q + \bar\xi\bar Q)V \Rightarrow \begin{cases} \delta_\xi \lambda_\alpha = -iD\xi_\alpha - \frac{1}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta \xi_\beta v_{\mu\nu} \\ \delta_\xi v^\mu = i(\xi\sigma^\mu\bar\lambda - \lambda\bar\sigma^\mu\bar\xi) - \partial^\mu(\xi\chi + \bar\xi\bar\chi) \\ \delta_\xi D = \partial_\mu(-\xi\sigma^\mu\bar\lambda + \lambda\sigma^\mu\bar\xi) \longrightarrow \text{total deriv.!} \end{cases}$$

$$\left[\delta_\xi v^{\mu\nu} = i\partial^\mu(\xi\sigma^\nu\bar\lambda - \lambda\sigma^\nu\bar\xi) - (\mu\leftrightarrow\nu)\right]$$

\- vector superfield $V^\dagger = V$

$$\begin{aligned} V(x,\theta,\bar\theta) = {} & C(x) + i\theta\chi - i\bar\theta\bar\chi + \frac{i}{2}\theta^2(M+iN) - \frac{i}{2}\bar\theta^2(M-iN) \\ & - \theta\sigma^\mu\bar\theta v_\mu + i\theta^2\bar\theta\left[\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi\right] \\ & - i\bar\theta^2\theta\left[\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi\right] \\ & + \frac{1}{2}\theta^2\bar\theta^2\left[D + \frac{1}{2}\Box C\right] \end{aligned}$$

$\{C\ M\ N\ D\ v_\mu\}$ real

* SUSY generalization of the gauge trf

a chiral superfield Λ, $\bar D_{\dot\alpha}\Lambda = 0$, $i\Lambda = \phi(\overset{x+i\theta\sigma\bar\theta}{\overset{\shortparallel}{y}}) + \sqrt{2}\theta\psi + \theta^2 F$

$$\begin{aligned} i(\Lambda - \Lambda^\dagger) = {} & \phi(x) + \phi^* + \sqrt{2}(\theta\psi + \bar\theta\bar\psi) + \theta^2 F + \bar\theta^2 F^* \\ & + \partial_\mu\phi\, i\theta\sigma^\mu\bar\theta - \partial_\mu\phi^*\, i\theta\sigma^\mu\bar\theta \\ & + \frac{i}{\sqrt{2}}\theta^2\bar\theta\bar\sigma^\mu\partial_\mu\psi + \frac{i}{\sqrt{2}}\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\psi \\ & + \frac{1}{4}\theta^2\bar\theta^2\Box(\phi + \phi^*) \end{aligned}$$

local $U(1)$ gauge trf $V \longrightarrow V' = V + i(\Lambda - \Lambda^\dagger)$

$$\begin{aligned} & C \to C + \phi + \phi^* && v_\mu \longrightarrow v_\mu - i\partial_\mu(\phi - \phi^*) \\ & \chi \to \chi - i\sqrt{2}\psi && \lambda \longrightarrow \lambda \\ & M+iN \to M+iN-2iF && D \longrightarrow D \end{aligned}$$

- Wess - Zumino gauge $C = \chi = M+iN = 0$ gauge

$V_{WZ}^2 = \frac{-1}{2}\theta^2\bar\theta^2 v_\mu v^\mu$

$V_{WZ}^3 = 0$

$$\begin{aligned} V_{WZ} & = -\theta\sigma^\mu\bar\theta v_\mu(x) + i\theta^2\bar\theta\bar\lambda(x) - i\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2 D(x) \\ & = -\theta\sigma^\mu\bar\theta v_\mu(y) + i\theta^2\bar\theta\bar\lambda(y) - i\bar\theta^2\theta\lambda(y) + \frac{1}{2}\theta^2\bar\theta^2\left[D(y) - i\partial_\mu v^\mu(y)\right] \end{aligned}$$

$$\begin{cases} W_\alpha \equiv -\frac{1}{4}\bar D^2 D_\alpha V_{WZ} = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha \overset{\partial_\mu v_\nu - \partial_\nu v_\mu}{\overset{\shortparallel}{v_{\mu\nu}}}(y) + \theta^2(\sigma^\mu\partial_\mu\bar\lambda(y))_\alpha \\ \bar W_{\dot\alpha} \equiv -\frac{1}{4}D^2\bar D_{\dot\alpha} V_{WZ} = i\bar\lambda_{\dot\alpha}(y^\dagger) + \bar\theta_{\dot\alpha}D(y^\dagger) + \frac{i}{2}(\bar\sigma^\nu\sigma^\mu\bar\theta)_{\dot\alpha} v_{\mu\nu}(y^\dagger) - \bar\theta^2(\bar\sigma^\mu\partial_\mu\lambda(y))_{\dot\alpha} \end{cases}$$

$= (W_\alpha)^\dagger$

$$D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha} = 0$$

$$W_\alpha : \begin{cases} \text{Chiral} & \bar D_{\dot\beta} W_\alpha = 0 \\ \text{superfield} & \delta_\eta W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha \delta_\eta V \\ \text{gauge inv.} & D_\alpha\Phi^\dagger = 0,\ \bar D^2 D_\alpha\Phi = -\bar D^{\dot\alpha}\bar D_{\dot\alpha} D_\alpha\Phi = -\bar D^{\dot\alpha}\cancelto{0}{\{\bar D_{\dot\alpha} D_\alpha\}}\Phi + \bar D^{\dot\alpha} D_\alpha \cancelto{0}{\bar D_{\dot\alpha}\Phi} = 0 \end{cases}$$

$v_{\mu\nu} = \partial_{[\mu} v_{\nu]} + \frac{i}{2}[v_\mu v_\nu] \xrightarrow{\quad} +i[v_\mu v_\nu]$

$\mathbb{D}_\mu \lambda = \partial_\mu \lambda + i[\frac{v_\mu}{2}, \lambda] \xrightarrow{V \to +2V} +i[v_\mu \lambda]$

$\mathbb{D}_\mu = (\partial_\mu + i v_\mu^a T^a)$

$$W^\alpha W_\alpha|_{\theta^2} = -2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{2}v_{\mu\nu}v^{\mu\nu} + D^2 + \frac{i}{4}\epsilon_{\mu\nu\rho\sigma}v^{\mu\nu}v^{\rho\sigma}$$

$$\mathcal{L} = \frac{1}{4}(W^\alpha W_\alpha|_{\theta^2} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta^2}) = -\frac{1}{4}v_{\mu\nu}v^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2$$

Introduce a complex coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} = \frac{\theta}{2\pi} + \frac{i}{\alpha_g}$

$$S = \frac{1}{8\pi}\mathrm{Im}\left[\int d^4x \int d^2\theta\, W^\alpha W_\alpha\right]$$
$$= \int d^4x\, \frac{1}{g^2}\left[-\frac{1}{4}v_{\mu\nu}v^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2\right] + \frac{\theta}{32}v_{\mu\nu}\overset{\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}v_{\rho\sigma}}{\tilde{v}^{\mu\nu}}$$

Only in $U(1)$ $\quad \mathcal{L}_{FI} = \int d^4\theta\, 2\kappa V = \kappa D \quad \delta D =$ total deriv. under SUSY trf

[∵ D is gauge-inv. in $U(1)$.] "Fayet-Illiopoulos term"

matter coupling $\Phi_+' = e^{+ig\Lambda}\Phi_+ \quad \Phi_-' = e^{-ig\Lambda}\Phi_- \quad W = m\Phi_+\Phi_-$

$$\mathcal{L}_{kin} = \int d^4\theta\,(\Phi_+^\dagger e^{gV}\Phi_+ + \Phi_-^\dagger e^{-gV}\Phi_-)$$

In Non-Abelian SUSY gauge th.

$V_{ij} = 2g T^a_{ij} V^a$

$\Phi' = e^{+i\Lambda}\Phi, \quad \Phi'^\dagger = \Phi^\dagger e^{-i\Lambda^\dagger} \qquad \Lambda_{ij} = 2gT^a_{ij}\Lambda^a$

$[T^a T^b] = if^{abc}T^c$

$e^{V'} = e^{-i\Lambda^\dagger}e^V e^{i\Lambda}, \quad V' = V + i(\Lambda - \Lambda^\dagger) + \cdots \qquad \mathrm{Tr}\,T^aT^b = k\delta^{ab}$

$(\delta V = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V, \Lambda + \Lambda^\dagger] + \mathcal{O}(\Lambda^2))$

$\delta V_{WZ} = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V_{WZ}, \Lambda + \Lambda^\dagger]$

$\bar{W}_{\dot\alpha} \to e^{-i\Lambda^\dagger}\bar{W}_{\dot\alpha}e^{i\Lambda^\dagger}$

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V \qquad W_\alpha \longrightarrow W_\alpha' = e^{-i\Lambda}W_\alpha e^{i\Lambda}$$

$W^a_\alpha = -\frac{1}{4}\bar{D}\bar{D}[D_\alpha V^a_{WZ} + igf^{abc}(D_\alpha V^b_{WZ})V^c_{WZ}] = \lambda_\alpha(y) + D^a\theta_\alpha - (\sigma^{\mu\nu}\theta)_\alpha v^a_{\mu\nu} + i\theta\theta\sigma^\mu_{\alpha\dot\beta}\mathbb{D}_\mu\bar\lambda^a$

$$\mathcal{L} = \frac{1}{16kg^2}\mathrm{Tr}(W^\alpha W_\alpha|_{\theta^2} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta^2}) + \Phi_i^\dagger e^V \Phi_i|_{\theta^4}$$
$$+ \left[\left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\right)\Big|_{\theta^2} + h.c.\right]$$

$$\to -\frac{1}{4}v^a_{\mu\nu}v^{a\mu\nu} - i\bar\lambda^a\bar\sigma^\mu\mathbb{D}_\mu\lambda^a + \frac{1}{2}D^aD^a$$
$$- \mathbb{D}_\mu\phi^\dagger\mathbb{D}^\mu\phi - i\bar\psi\bar\sigma^\mu\mathbb{D}_\mu\psi + F^\dagger F$$
$$+ i\sqrt{2}g(\phi^\dagger T^a\psi\lambda^a - \bar\lambda^a\bar\psi T^a\phi)$$
$$+ gD^a\phi^\dagger T^a\phi$$

$v^a_{\mu\nu} \equiv \partial_\mu v^a_\nu - \partial_\nu v^a_\mu - gf^{abc}v^b_\mu v^c_\nu$

$\mathbb{D}_\mu\phi \equiv \partial_\mu\phi + igv^a_\mu T^a\phi$

$\mathbb{D}_\mu\psi \equiv \partial_\mu\psi + igv^a_\mu T^a\psi$

$\mathbb{D}_\mu\lambda^a \equiv \partial_\mu\lambda^a - gf^{abc}v^b_\mu\lambda^c$

$(\mathbb{D}_\mu\lambda \equiv \partial_\mu\lambda - i[v_\mu\lambda])$

$$V_D = \frac{1}{2}D^aD^a = \frac{g^2}{2}|\Sigma\phi^\dagger T^a\phi|^2 \geqslant 0 \quad (\because \partial\mathcal{L}/\partial D^a = D^a + g\Sigma\phi^\dagger T^a\phi)$$

* SUSY QED

$$\begin{cases} \Phi_\pm \rightarrow e^{\pm \overset{2e}{\vee} i\Lambda}\,\Phi_\pm \\ 2eV \rightarrow 2eV + i \overset{2e}{\vee}(\Lambda - \Lambda^\dagger) \end{cases} \qquad W = m\Phi_+\Phi_-$$

$$\mathcal{L}_{QED} = \frac{1}{4}\left(W^2\big|_{\theta^2} + \overline{W}^2\big|_{\bar\theta^2}\right) + \Phi_+^\dagger e^{2eV}\Phi_+ + \Phi_-^\dagger e^{-2eV}\Phi_-$$
$$+ m\left(\Phi_+\Phi_-\big|_{\theta^2} + \Phi_+^\dagger\Phi_-^\dagger\big|_{\bar\theta^2}\right) + 2\kappa V\big|_{\theta^2\bar\theta^2}$$

In W-Z gauge, V, V^2 $(V^3 = 0)$

$$= -\frac{1}{4}v_{\mu\nu}v^{\mu\nu} - i\partial_\mu\bar\lambda\bar\sigma^\mu\lambda + \frac{1}{2}D^2 + \kappa D$$
$$+ |\mathbb{D}_\mu\phi_+|^2 + |\mathbb{D}_\mu\phi_-|^2 + |F_+|^2 + |F_-|^2$$
$$+ i\mathbb{D}_\mu\bar\psi_+\bar\sigma^\mu\psi_+ + i\mathbb{D}_\mu\bar\psi_-\bar\sigma^\mu\psi_- + eD(|\phi_+|^2 - |\phi_-|^2)$$
$$- i\sqrt{2}e(\phi_+\bar\lambda\bar\psi_+ - \phi_+^*\lambda\psi_+ - \phi_-\bar\lambda\bar\psi_- + \phi_-^*\lambda\psi_-)$$
$$+ m(\phi_+F_- + \phi_-F_+ + \phi_+^*F_-^* + \phi_-^*F_+^* - \psi_+\psi_- - \bar\psi_+\bar\psi_-)$$

$$\mathbb{D}_\mu\begin{pmatrix}\phi_\pm \\ \psi_\pm\end{pmatrix} = (\partial_\mu \pm ie v_\mu)\begin{pmatrix}\phi_\pm \\ \psi_\pm\end{pmatrix} \qquad \Psi_D = \begin{pmatrix}\psi_{+\alpha} \\ \bar\psi_-^{\dot\alpha}\end{pmatrix}$$

$$F_+^* = -m\phi_- \qquad \bar\Psi_D = (-\psi_-^\alpha, -\bar\psi_{+\dot\alpha})$$
$$D = -e(|\phi_+|^2 - |\phi_-|^2) - \kappa$$

$$\mathcal{L}_m = -m\bar\Psi_D\Psi_D$$
$$= m(\psi_+\psi_- + \bar\psi_+\bar\psi_-)$$

$$V_F = m^2(|\phi_+|^2 + |\phi_-|^2) = \sum_{i=\pm}\left|\frac{\partial W}{\partial\phi_i}\right|^2$$
$$V_D = \frac{1}{2}\left|e(|\phi_+|^2 - |\phi_-|^2) + \kappa\right|^2$$

$$V = V_F + V_D$$

SUSY BREAKING

Spontaneous $\cancel{\text{SUSY}}$

$Q_\alpha |0\rangle \neq 0$ or $\bar{Q}_{\dot\alpha}|0\rangle \neq 0$ At SUSY min.,

$$H = P^0 = \frac{1}{4}\sum_{k=1,2}\left(Q_k \bar{Q}_{\dot k} + \bar{Q}_{\dot k} Q_k\right) \geq 0 \qquad E = 0$$

$$\langle 0|\delta\psi_\alpha|0\rangle = \sqrt{2}\,\xi_\alpha \underbrace{\langle 0|F|0\rangle}_{\neq 0} + i\sqrt{2}\,\partial_\mu\phi\,\sigma^\mu\bar{\xi}$$

preserving Lorentz sym.

($V_F > 0$, so Need SUGRA)

$\langle 0|F_i|0\rangle \neq 0 \Rightarrow$ spontaneous $\cancel{\text{SUSY}}$ ($m_{3/2} \neq 0$)

$\exists$ Goldstone fermion ψ_i $\quad (\phi_i\ \psi_i\ \overset{\neq 0\ \cancel{\text{SUSY}}}{F_i})$ $\quad F_i^* = -\dfrac{\partial W}{\partial \phi_i}$

(eaten by the **gravitino** in $\underset{\underset{\text{Local SUSY}}{\parallel}}{\text{SUGRA}}$)

* O'Raifeartaigh model ($\in$ F-term breaking)

$$W(\Phi_1\,\Phi_2\,\Phi_3) = \lambda\,\Phi_1(\Phi_3^2 - M^2) + \mu\,\Phi_2\Phi_3 \qquad M^2 \ll \mu^2/2\lambda^2$$

$$-F_1^* = \lambda(\phi_3^2 - M^2), \quad -F_2^* = \mu\phi_3, \quad -F_3^* = 2\lambda\phi_1\phi_3 + \mu\phi_2$$

$$V = \sum|F_i|^2 = \lambda^2|\phi_3^2 - M^2|^2 + \mu^2|\phi_3|^2 + |\mu\phi_2 + 2\lambda\phi_1\phi_3|^2$$

$\langle\phi_2\rangle = \langle\phi_3\rangle = 0$ $\quad\langle\phi_1\rangle$ undetermined

$F_1^* = \lambda M^2 \quad F_2^* = F_3^* = 0 \qquad V = \lambda^2 M^2 > 0$

ψ_1 is the Goldstone fermion.

$$\mathcal{L}_m^F = -\mu\psi_2\psi_3 - \lambda\langle\phi_1\rangle\psi_3\psi_3 + h.c.$$

ψ_1 is massless.

Suppose $\langle\phi_1\rangle = 0$. $\quad\phi_i \equiv \frac{1}{\sqrt{2}}(a_i + i b_i)$

$\boxed{\text{lighter than } \psi_2, \psi_3} \leftarrow m_{a_3}^2 = \mu^2 - 2\lambda^2 M^2, \quad m_{b_3}^2 = \mu^2 + 2\lambda^2 M^2 \rightarrow$ heavier than ψ_2, ψ_3

$\therefore\ m_{a_3}^2 + m_{b_3}^2 = 2\mu^2 = 2m_\psi^2$

$$\text{STr}M^2 = \sum_J (-1)^{2J}(2J+1)\,m_J^2 = 0 \qquad M_S^2 = \lambda M^2$$

* D-term breaking

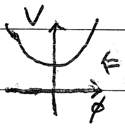
$$\langle 0|\delta\lambda|0\rangle = -i\xi\langle 0|D|0\rangle \neq 0 \qquad \text{Note } V_D = \tfrac{1}{2}DD \geq 0$$

$$\mathcal{L} \ni \kappa D \ (\kappa>0),\ D = -(\kappa + \Sigma q_i \phi_i^\dagger \phi_i)$$

heavier than $\psi_\pm$

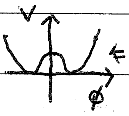
$$V_D = \tfrac{1}{2}(\kappa + e|\phi_+|^2 - e|\phi_-|^2)^2 \qquad m_{\phi_\pm}^2 = m^2 \pm e\kappa,\quad m_{\psi_\pm}^2 = m^2$$

lighter than $\psi_\pm$



if $\nexists\ \phi_-$ (for an Anomalous $U(1)$)

$V_D = \frac{1}{2}(\kappa + e|\phi_+|^2)^2$, $m_{\phi_+}^2 = e\kappa$, $\cancel{\text{SUSY}}$, λ is the Goldstino,



if $\nexists\ \phi_+$ (for an anomalous $U(1)$)

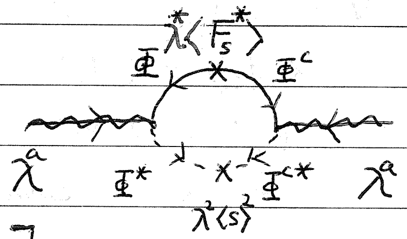
$V_D = \frac{1}{2}(\kappa - e|\phi_-|^2)^2$ SUSY unbroken but $\cancel{U(1)}$

$$\mathrm{STr}M^2 = \sum_J (-1)^{2J}(2J+1)m_J^2 = 0 \qquad \text{tree level result}$$

$\therefore\ \exists$ scalars lighter than fermions i.e $\exists$ superpartners lighter than SM fermions $\rightarrow$ X

$\Rightarrow$ Need to introduce a **hidden sector** where SUSY is broken originally.

* Gauge Mediated $\cancel{\text{SUSY}}$



MSSM gauginos

$\cancel{\text{SUSY}}$ source, i.e. $\langle F_s$

$$W_{mess} = \sum_i \lambda_i S \Phi_i \Phi_i^c$$

messenger fields which carry SM quantum numbers

$$\tilde{m}^2 = \begin{pmatrix} |\lambda_i\langle s\rangle|^2 & \lambda_i\langle F_s\rangle \\ \lambda_i^*\langle F_s^*\rangle & |\lambda_i\langle s\rangle|^2 \end{pmatrix}$$

Superpartners of the SM gauge fields

Only MSSM gauginos get masses. (not SM gauge fields)

$$M_a \approx \frac{g_a^2}{16\pi^2}\left|\frac{\langle F_s\rangle}{\langle S\rangle}\right|$$

superpartners of the SM fermions

Sfermions get masses at two-loop level.

$$m_{\tilde{f},H}^2 \approx 2\left|\frac{\langle F_s\rangle}{\langle S\rangle}\right|^2 \sum_\alpha \left(\frac{g_a^2}{16\pi^2}\right)^2 C_\alpha \qquad C_\alpha \equiv \sum_a (T^a T^a)_\alpha \qquad (T^a T^a)^i_j = C_a(i)\delta^i_j$$

* Gravity Mediated $\cancel{\text{SUSY}}$ (at tree level)

$$m_{3/2} \sim \frac{\langle F_s\rangle}{M_P} \sim m \sim \text{TeV}$$

$\cancel{\text{SUSY}}$ source $\quad \langle F_s\rangle \sim mM_P \sim (10^{10}\text{GeV})^2$

$$\mathcal{L} \supset \frac{1}{M_P^2} S^\dagger S \Phi_i^\dagger \Phi_i\Big|_{\theta^2\bar\theta^2} + \left(\frac{S}{M_P} W^\alpha W_\alpha\Big|_{\theta^2} + \frac{S}{M_P}\Phi_i\Phi_j\Phi_k\Big|_{\theta^2} + \text{h.c.}\right)$$

sfermions' masses $\leftarrow$ $\frac{|\langle F_s\rangle|^2}{M_P^2}\phi_i^\dagger\phi_i \sim m^2|\phi_i|^2$ gauginos' masses $\leftarrow$ $\frac{\langle F_s\rangle}{M_P}\lambda^a\lambda^a \sim m\lambda^a\lambda^a$ $\quad \frac{\langle F_s\rangle}{M_P}\phi_i\phi_j\phi_k \sim m\phi_i\phi_j\phi_k$ "A-term"

SUGRA

- LOCAL SUSY w/ the Gravity Multiplet ($g_{\mu\nu}$ or e_μ^m , ψ_μ)
- described w/ K or $G \equiv K + \ln|W|^2$, W , f_{ab}

(set $\frac{1}{8\pi G} = M_p^2 \equiv 1$) $\quad \overset{\shortparallel}{K}$ + $\qquad$ W, f_{ab} ↘ ↙ holomorfic

- $F_i = e^{G/2} (G^{-1})_i^j G_j + \frac{1}{4} f^*_{ab,j} (G^{-1})_i^j \lambda^a \lambda^b + \cdots$

$$\delta\psi_i = \cdots - \sqrt{2}\, e^{G/2} (G^{-1})_i^j G_j \xi - \frac{1}{8} f^*_{ab,j} (G^{-1})_i^j \lambda^a \lambda^b \xi + \cdots$$

where $G_i = \frac{\partial G}{\partial \phi^i} = \frac{\partial K}{\partial \phi^{i*}} + \frac{1}{W^*}\frac{\partial W^*}{\partial \phi^{i*}}$ $\quad \left(D_i W = \frac{\partial W^*}{\partial \phi^{i*}} + \frac{\partial K}{\partial \phi^{i*}} W^* \right)$

$$(G^{-1})_i^j = \frac{\partial^2 G^{-1}}{\partial \phi_j \partial \phi^{i*}} = \frac{\partial^2 K^{-1}}{\partial \phi_j \partial \phi^{i*}}$$

$\boxed{\text{S\!\!/USY if } G_i \neq 0 \text{ or } D_i W \neq 0}$

if $D_i W \neq 0$, $V_F = 0$ then $W \neq 0$ $\qquad \overset{(D_j W)^*}{\shortparallel}$

- $V_F = e^G [G_i (G^{-1})_j^i G^j - 3] = e^K [D_i W (K^{-1})_j^i D^j W - 3|W|^2]$

$$e^{-1}\mathcal{L}_F \supset e^{G/2} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu + \frac{1}{4} e^{G/2} G^i (G^{-1})_i^j f^*_{ab,j} \lambda^a \lambda^b + \frac{1}{32} (G^{-1})_i^j f^i_{ab} f^*_{cdj} \lambda^a \lambda^b \lambda^c \lambda^d + \cdots + \text{h.c.}$$

$(e^{K/2} |W| \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu)$
gravitino mass term induced if $W \neq 0$

gaugino mass term induced if $D_i W \neq 0$ & $f^*_{ab,j} \equiv \partial f^*_{ab} / \partial \phi i^* \neq 0$

- For example, in mSUGRA $\qquad K = \sum_i |\phi_i|^2$ (∴ canonical kinetic terms) $\searrow (K^{-1})_j^i = 1$

$W = W_0(\phi_i) + W_H(Z_i)$

assume $\langle Z_i \rangle = b_i M_p \quad \langle \partial W_H / \partial Z_i \rangle = a_i^* m M_p$, $\langle W_H \rangle = m M_p^2$

Keeping M_p^2,

$$V_F = \exp\left(\frac{|Z_i|^2 + |\phi_i|^2}{M_p^2}\right) \left[\left| \frac{\partial W_H}{\partial Z_i} + \frac{Z_i^*}{M_p^2} W \right|^2 + \left| \frac{\partial W_0}{\partial \phi_i} + \frac{\phi_i^*}{M_p^2} W \right|^2 - \frac{3}{M_p^2} |W|^2 \right]$$

$$= \quad \shortparallel \quad \left[\left| (a_i^* + b_i^*) m M_p + \frac{b_i^*}{M_p} W_0 \right|^2 + \left| \frac{\partial W_0}{\partial \phi_i} + m \phi_i^* + \frac{\phi_i^*}{M_p^2} W_0 \right|^2 - 3 \left| m M_p + \frac{W_0}{M_p} \right|^2 \right.$$

$$\simeq e^{|b_i|^2} \left(1 + \frac{|\phi_i|^2}{M_p^2} \right) \left[(|a_i + b_i|^2 - 3) m^2 M_p^2 + \left| \frac{\partial W_0}{\partial \phi_i} \right|^2 + m^2 |\phi_i|^2 + m \left\{ \phi_i \frac{\partial W_0}{\partial \phi_i} + (b_i^* (a_i + b_i) - 3) W_0 + \text{h.c.} \right\} + \mathcal{O}\left(\frac{1}{M_p^2}\right) \right]$$

* $\sum_i |a_i + b_i|^2 = 3 \qquad$ Zero C.C.

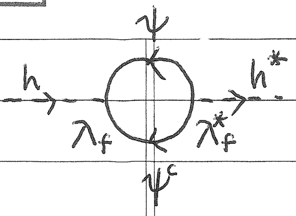
* $m_{3/2} = e^{K/2} |W| = e^{|b_i|^2/2} m$, $\qquad e^{|b_i|^2/2} W_0 \longrightarrow W_0$ rescaling

* $\left| \frac{\partial W_0}{\partial \phi_i} \right|^2$: V_F in the Global SUSY

$m_{3/2}^2 |\phi_i|^2$: soft mass terms for sfermions , $\quad m_{3/2} \left\{ \phi_i \frac{\partial W}{\partial \phi_i} + (b_i^* (a_i + b_i) - 3) W_0 + \text{h.c.} \right\}$ ↪ "A-terms"

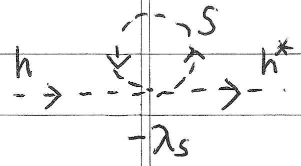
↓ positive, ~~E~~W radiatively ∵ y_t is large enough.

MSSM



$-\mathcal{L}_Y \supset \lambda_f h \psi \psi^c + h.c.$

$\Delta m_h^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{UV}^2 + \cdots$ indep. of m_h^2



$-\mathcal{L}_{pot} \supset \lambda_s |h|^2 |S|^2$

$\Delta m_h^2 = \frac{\lambda_s}{16\pi^2}\left[\Lambda_{UV}^2 - 2m_S^2 \ln\frac{\Lambda_{UV}}{m_S} + \cdots\right]$

} Hard to get a small scalar mass ($\ll \Lambda_{UV}$)

Require $(\psi\ \psi^c) \longleftrightarrow (S\ S^c)$ i.e. 2 complex scalars corresponding to $\{\psi\ \psi^c\}$

$|\lambda_f|^2 = \lambda_s$

Satisfied In SUSY! : $W \supset \lambda_t H_u \Phi_t \Phi_{t^c} \rightarrow -\mathcal{L} \supset \lambda_t h_u \psi_t \psi_{t^c} + |\lambda_t|^2 |h_u|^2 (|\phi_t|^2 + |\phi_{t^c}|^2)$

Basically $\exists$ Chiral sym, for fermions in SM which can protect small fermion masses

No useful sym. for scalars (or Higgs)

In SUSY, $m_B = m_F$ and so small scalar masses can also be protected against quantum corrections by the chiral sym. of fermions and supersym.

In the SM.

$Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix} = (3, 2, 1/6)$ Ⓕ $\quad u_{Ri} = (3, 1, 2/3)$ Ⓕ + Gauge Bosons

$d_{Ri} = (3\ 1\ {-1/3})$ Ⓕ

$L_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix} = (1\ 2\ {-1/2})$ Ⓕ $\quad e_{Ri} = (1\ 1\ {-1})$ Ⓕ

$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix} = (1\ 2\ 1/2)$ Ⓑ $\quad [\ \nu_{Ri} = (1\ 1\ 0)\ ]$ Ⓕ

In the MSSM, χ superfields (LHed)

		$\textcircled{R_P}$
Q_i	$(3, 2, 1/6)$	−
U_i^c	$(\bar{3}\ 1\ {-2/3})$	−
D_i^c	$(\bar{3}\ 1\ 1/3)$	−
L_i	$(1\ 2\ {-1/2})$	−
E_i^c	$(1\ 1\ 1)$	−
H_d	$(1\ 2\ {-1/2})$	+
H_u	$(1\ 2\ 1/2)$	+

Needed for { ① u-type quark masses ② Anomaly cancellation

$[N^c = (1\ 1\ 0)]$

$W_R = y_{ij}^d Q_i H_d D_j^c + y_{ij}^u Q_i H_u U_j^c + y_{ij}^e L_i H_d E_j^c + y_{ij}^\nu L_i H_u N^c + M_{ij} N^c N^c + \mu H_u H_d$

$\left[W_{\not R} = \frac{1}{2}\lambda_{[ij]k} L_i L_j E_k^c + \lambda'_{ijk} Q_i L_j D_k^c + \lambda''_{[ij]k} D_i^c D_j^c U^c + \mu'_i L_i H_u \right]$

$\not B$ or $\not L$

$$G_\alpha = -\frac{1}{4}\bar{D}^2 e^{-G^a\frac{\lambda^a}{2}} D_\alpha e^{G^a\frac{\lambda^a}{2}}, \quad W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-W^i\frac{\sigma^i}{2}} D_\alpha e^{W^i\frac{\sigma^i}{2}}, \quad B_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha B$$

$$\mathcal{L}_V = \frac{1}{16k}\,\mathrm{Tr}\Big[\frac{1}{g_3^2}(G^\alpha G_\alpha + \bar{G}_{\dot\alpha}\bar{G}^{\dot\alpha}) + \frac{1}{g_2^2}(W^\alpha W_\alpha + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}) + \frac{1}{g_1^2}(B^\alpha B_\alpha + \bar{B}_{\dot\alpha}\bar{B}^{\dot\alpha}\Big]_{\theta^2}$$

For canonical norm. $V \to 2gV$

$$\begin{aligned}\mathcal{L}_K = \Big[&\Sigma\, Q^\dagger \exp[\,g_3 G^a\lambda^a + g_2 W^i\sigma^i + g_1 B\,]\,Q \\ &+ U^{c\dagger}\exp[\,g_3 G^a\lambda^a - \tfrac{4}{3}g_1 B\,]\,U^c + D^{c\dagger}\exp[\,g_3 G^a\lambda^a + \tfrac{2}{3}g_1 B\,]\,D^c \\ &+ L^\dagger \exp[\,g_2 W^i\sigma^i - g_1 B\,]\,L + E^{c\dagger}\exp[\,2g_1 B\,]\,E^c \\ &+ H_d^\dagger \exp[\quad '' \qquad '' \quad]\,H_d + H_u^\dagger \exp[\,g_2 W^i\sigma^i + g_1 B\,]\,H_u\Big]_{\theta^2\bar\theta^2}\end{aligned}$$

$$\begin{aligned}-\mathcal{L}_{soft} = {}& m_{\tilde q}|\tilde q|^2 + m_{\tilde u^c}^2|\tilde u^c|^2 + m_{\tilde d^c}^2|\tilde d^c|^2 + m_{\tilde l}^2|\tilde l|^2 + m_{\tilde e^c}^2|\tilde e|^2 \\ &+ m_{hu}^2|h_u|^2 + m_{hd}^2|h_d|^2 \\ &+ \tfrac{1}{2}M_3\tilde g\tilde g + \tfrac{1}{2}M_2\tilde W\tilde W + \tfrac{1}{2}M_1\tilde B\tilde B \\ &+ A_u\tilde q h_u\tilde u^c + A_d\tilde q h_d\tilde d^c + A_e\tilde l h_d\tilde e^c \\ &+ B_\mu h_u h_d\end{aligned}$$

can be generated via a spontaneous SUSY breaking mechanism.

the Higgs quartic coupling is given by the SM gauge couplings in the MSSM.

$$V_h = \overset{m_{hd}^2+|\mu|^2}{m_1^2}|h_d|^2 + \overset{m_{hu}^2+|\mu|^2}{m_2^2}|h_u|^2 + (m_3^2 h_u h_d + h.c.) + \frac{g_2^2+g_1^2}{8}(|h_u|^2-|h_d|^2)^2$$

$$\frac{\partial V_h}{\partial h_u} = \frac{\partial V_h}{\partial h_d} = 0$$

* minimize V_h

$$\begin{cases} -2B_\mu = (m_{hd}^2 - m_{hu}^2)\tan 2\beta + M_Z^2 \sin 2\beta \\ \dfrac{M_Z^2}{2} + |\mu|^2 = \dfrac{m_{hd}^2 - m_{hu}^2 t^2\beta}{t^2\beta - 1}\end{cases} \qquad t\beta \equiv \frac{v_u}{v_d}$$

$$M_Z^2 = \frac{1}{4}(g_2^2+g_1^2)\underbrace{(v_d^2+v_u^2)}_{\approx(246\,\mathrm{GeV})^2},$$

* at tree level $\quad m_{H,h}^2 = \frac{1}{2}\Big[m_A^2 + M_Z^2 \pm \sqrt{(m_A^2+M_Z^2)^2 - 4M_Z^2 m_A^2\cos^2 2\beta}\,\Big]$

the Higgs' quartic coupling is too small.

$$m_h^2 \lesssim M_Z^2\cos^2 2\beta \quad \text{for } m_A^2 \gg M_Z^2$$

* radiative Higgs mass $\simeq \frac{3}{4\pi^2}y_t^2 m_t^2 \ln\frac{\tilde m_t^2}{m_t^2}$

* $m_2^2 = m_{hu}^2 + |\mu|^2 = B_\mu\cot\beta + \frac{M_Z^2}{2}\cos 2\beta - \frac{3}{16\pi^2}y_t^2\big[f(m_{\tilde t}^2) - f(m_t^2)\big]$

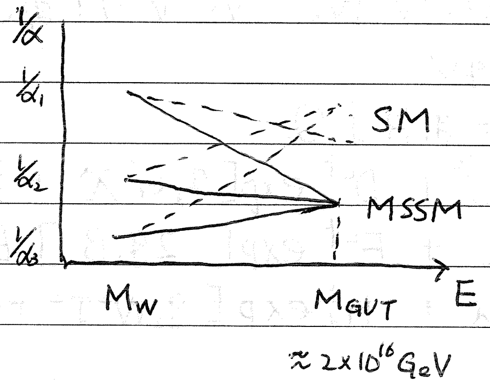
where $f(m^2) \equiv 2m^2\Big(\ln\frac{m^2}{Q} - 1\Big)$

renorm. scale

Radiative EW sym. breaking!

A Fine-tuning needed for an excessively heavy ($\gtrsim$ 1 TeV?) stop mass.

⊛ Gauge coupling unification



⊛ Dark Matter

$\Omega_\Lambda \approx 0.7$

$\Omega_m \approx 0.3$

$\Omega_{DM} \approx 1/4$

$\Omega_b \approx 0.04$

$\Omega_{lum} \approx 0.01$

(weakly interacting massive ptl.)

The MSSM provides a good WIMP DM candidate.

LSP (lightest superparticle) ≡ neutralino

≈ wino + bino ($\tilde{W}^0 + \tilde{B}$)

Z_2 = R-parity $\subset U(1)_R$