Topology in Condensed Matter Physics: Quantum Hall Effect, Chern Number, Topological Insulators, and All That Jazz

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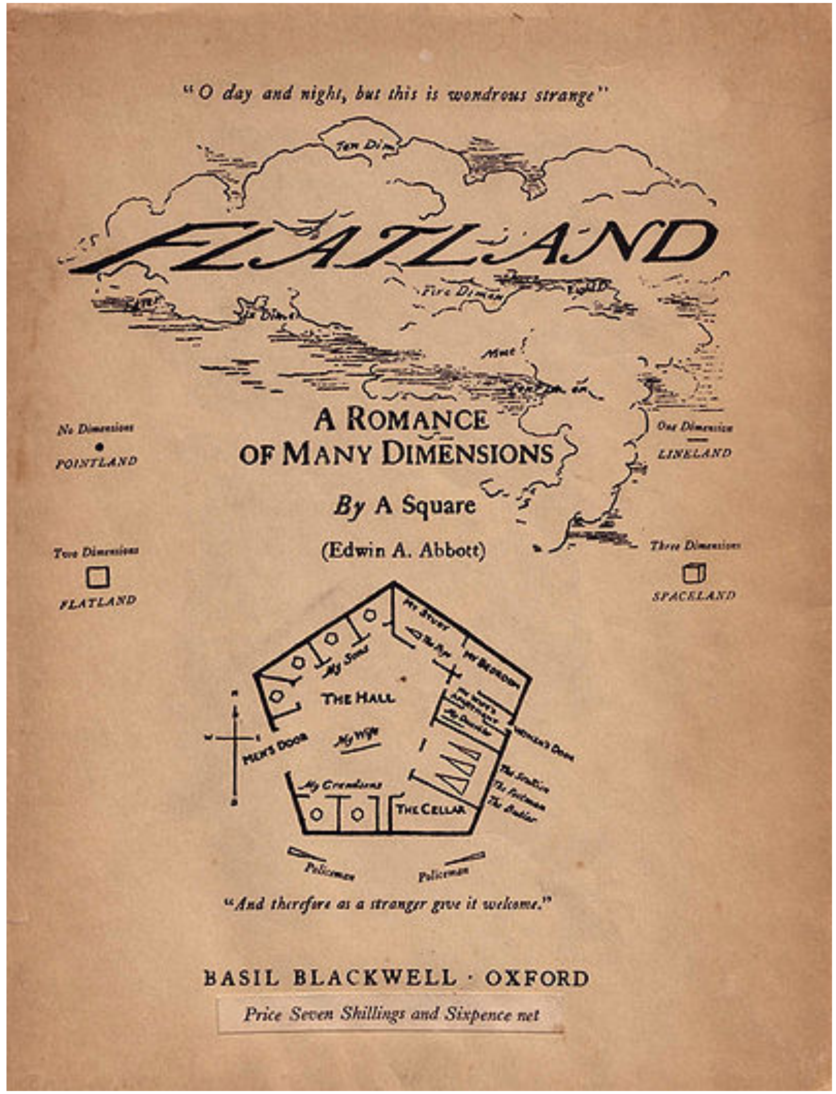
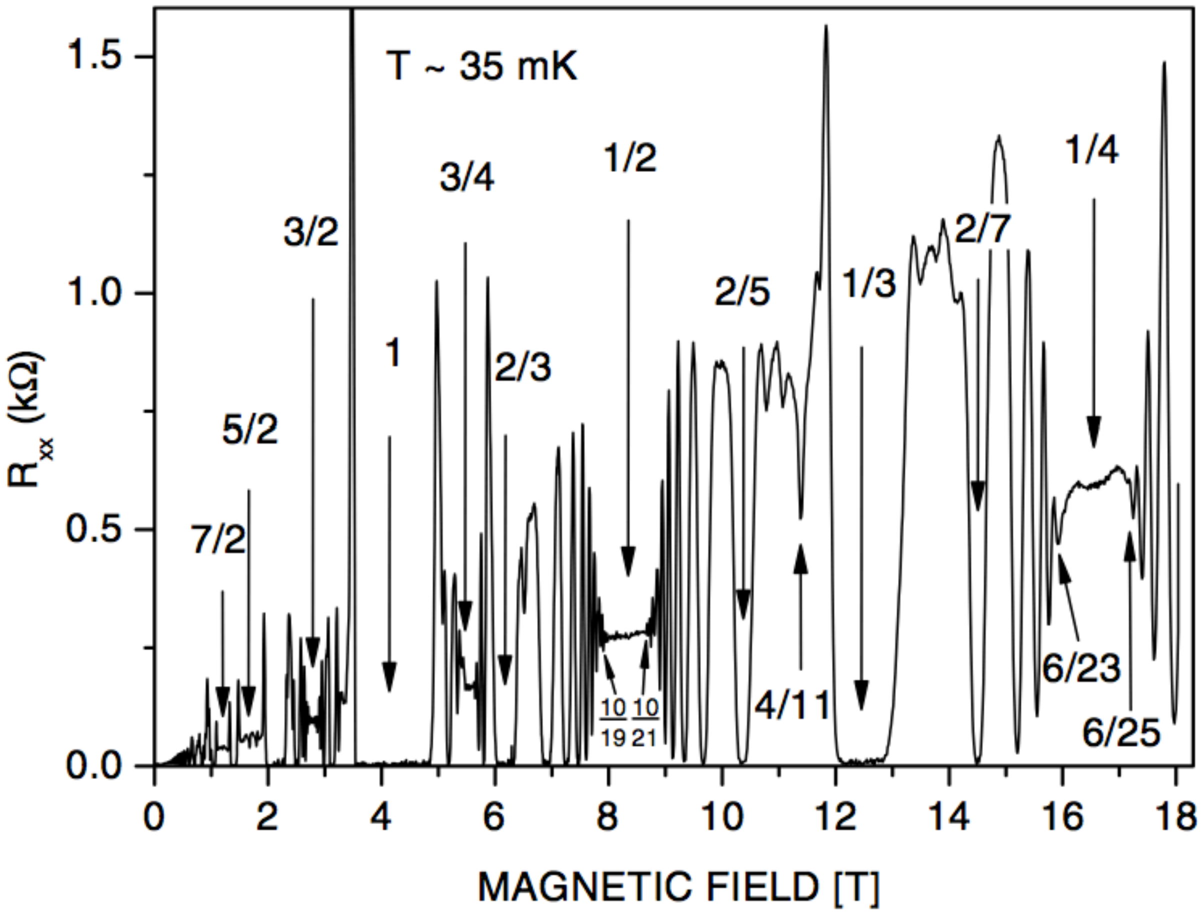
The quantum Hall effect (QHE) is one of the most fascinating phenomena in physics, induced by a topological quantum state of matter.



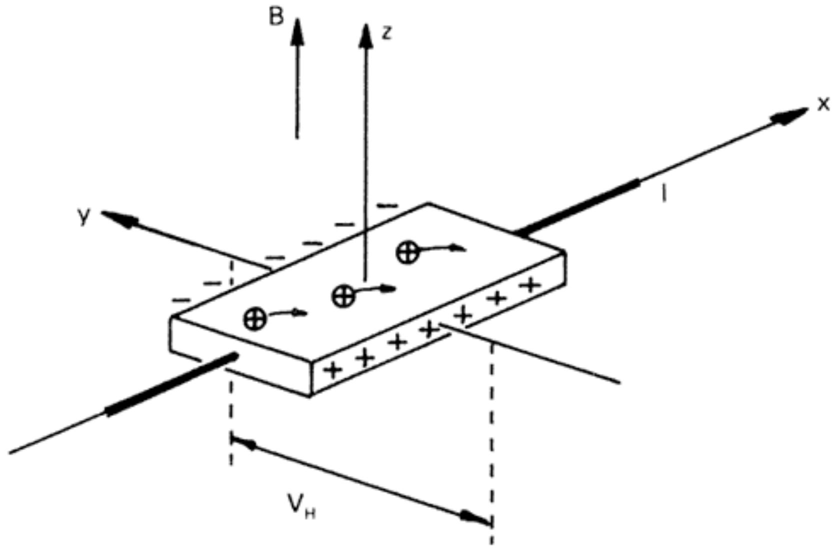
Written by Edwin A. Abbott in 1884

"Electrons in Flatland" under a high magnetic field: QHE



Pan *et al.,* PRL **88**, 176802 (02)

Classical Hall effect



Lorentz force due to magnetic field

$$\mathbf{F} = \frac{e}{c}\mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\rho c} \times \mathbf{B}$$

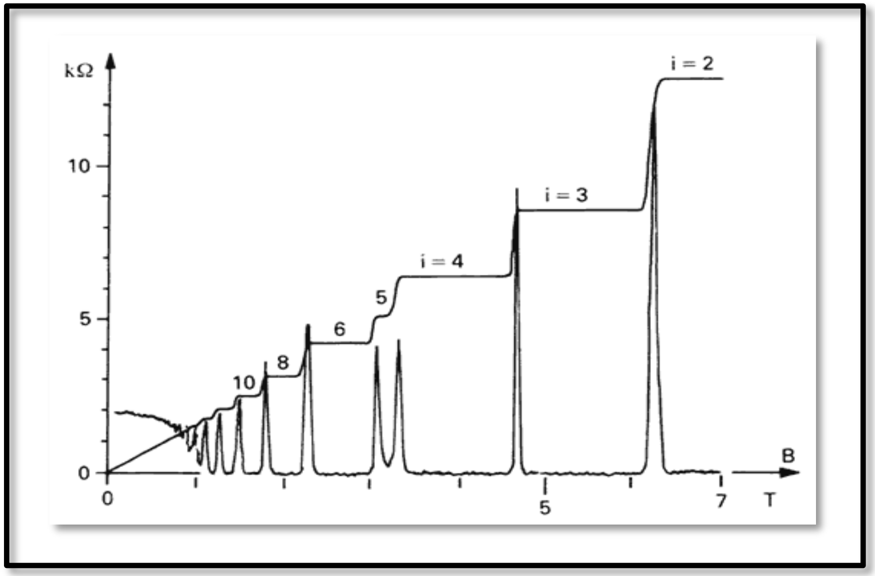
Electric force due to charge accumulation

$$\mathbf{F} = e\mathbf{E}$$

The Hall resistance is given by the steady-state condition balancing the two forces.

$$R_{xy} = \frac{E_y}{j_x} = \frac{B}{\rho e c}$$

Integer quantum Hall effect (IQHE)



$$R_{xy} = \frac{h}{ne^2}$$

Von Klitzing, Dorda, Pepper (1980)

Figure: Nobel prize press release (1988)

- With help of the disorder-induced Anderson localization, the incompressibility of completely filled Landau levels at integer filling factors can explain the IQHE.

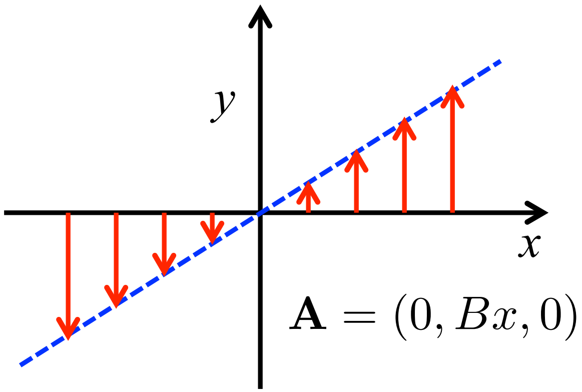
Magnetic algebra

- There is a close similarity between the Hamiltonian of a particle moving in 2D under a uniform magnetic field and that of an 1D harmonic oscillator.

$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 = \frac{1}{2m}(\pi_x^2 + \pi_y^2)$	$H = \frac{1}{2m}\left[p^2 + (m\omega)^2 x^2\right]$
$[\pi_x, \pi_y] = \frac{i\hbar e}{c}\nabla \times \mathbf{A} = \frac{i\hbar^2}{l_B^2}$ $\quad l_B^2 = \frac{\hbar c}{eB}$	$[x, p] = i\hbar$
$a = \frac{l_B/\hbar}{\sqrt{2}}(\pi_x + i\pi_y)$ $a^\dagger = \frac{l_B/\hbar}{\sqrt{2}}(\pi_x - i\pi_y)$	$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)$ $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right)$
$H = \hbar\omega_C\left(a^\dagger a + \frac{1}{2}\right)$ $\quad \omega_C = \frac{eB}{mc}$	$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$

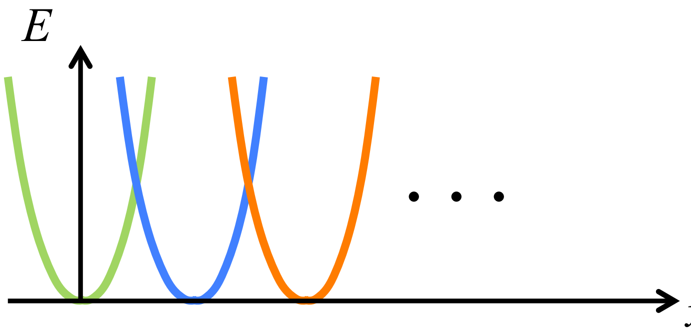
Landau levels in the Landau gauge

- In the Landau gauge, the Hamiltonian reduces to that of an 1D harmonic oscillator with its center location proportional to the perpendicular momentum, $k_y l_B^2$.



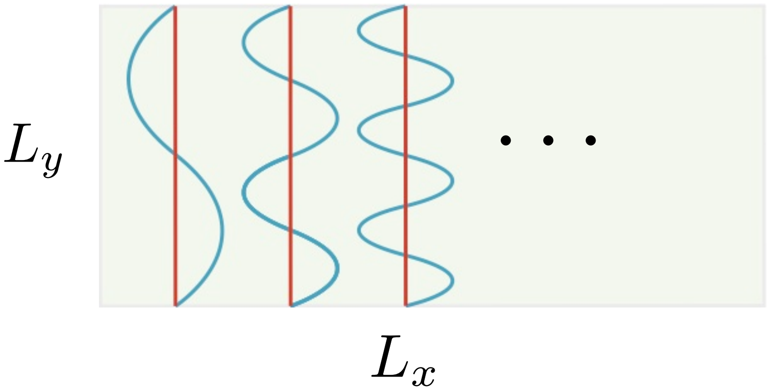
$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y - \frac{eB}{c}x\right)^2\right]$$
$$\Rightarrow \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_C^2(x - k_y l_B^2)^2$$

$$\Psi(x, y) = \psi(x)e^{ik_y y}$$



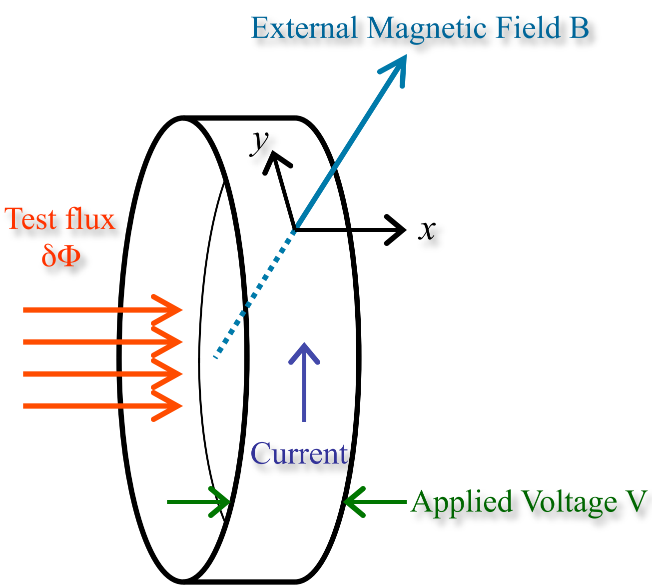
- Each Landau level has a macroscopic degeneracy!

$$k_{y,\max} l_B^2 = L_x$$
$$\frac{2\pi n_{\max}}{L_y} l_B^2 = L_x$$
$$n_{\max} = \frac{L_x L_y}{2\pi l_B^2} = \frac{BA}{2\pi\hbar c/e} = \frac{\Phi}{\phi_0}$$



Laughlin's gauge argument for the quantized Hall resistance

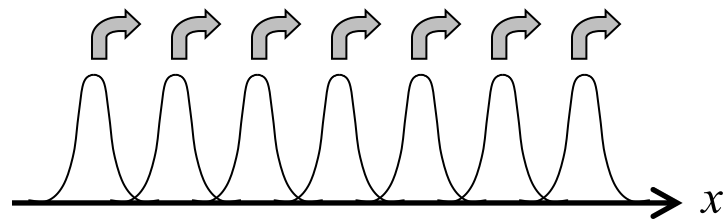
Laughlin, PRB **23**, 5632 (1981)



- Magnetic moment:
  $\mu = IA/c$
- Energy change:
  $\delta E = \mu\,\delta B = \mu\,\delta\Phi/A = I\,\delta\Phi/c$
- Therefore, $I = c\,\delta E/\delta\Phi$

Eigenstate wave function in the Landau gauge:

- Gaussian wave packet in the *x*-direction
- Plane wave in the *y*-direction
- $x_0=k_y l^2$ with l a magnetic length



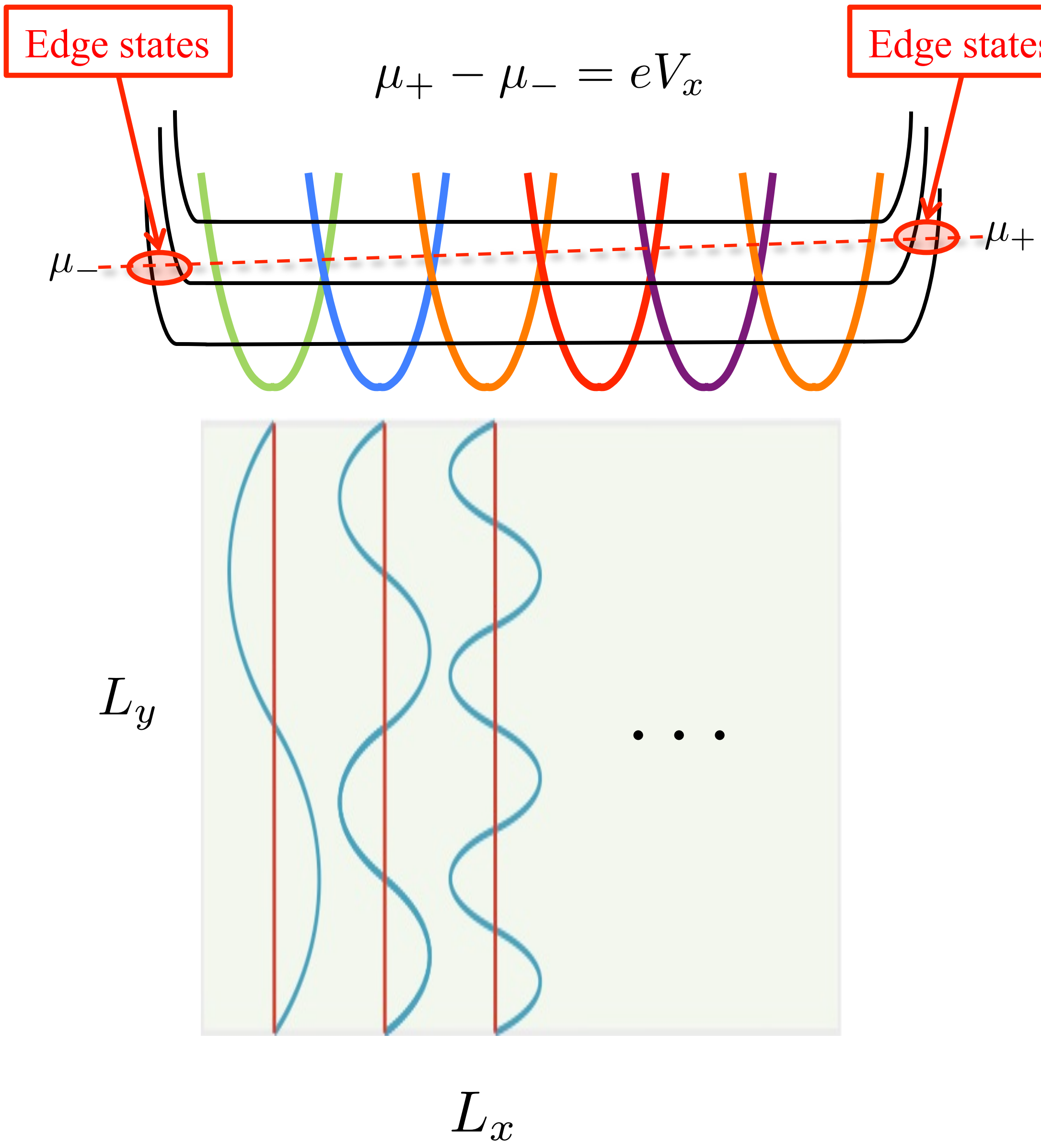
Effect of adding test flux δΦ:

- δΦ increases a vector potential shifting k_y.
- The center of wave packets moves.
- When $\delta\Phi=hc/e$, the physics must be invariant.
- Therefore, the wave packets move by one unit.
- When *n* Landau level is filled, *n* electrons are transferred with the energy gain $\delta E = neV$.

$$R_{xy} = \frac{V}{I} = \frac{h}{ne^2}$$

The Hall resistance is so precisely quantized since only the extended states can respond to the global flux change.

"Landauer" approach for the quantized Hall resistance



$$I_y = e\int \frac{dp_y}{2\pi\hbar}\langle v_y\rangle$$

$$= e\int \frac{dp_y}{2\pi\hbar}\left\langle \frac{\partial H}{\partial p_y}\right\rangle$$

$$= e\sum_n \int \frac{dp_y}{2\pi\hbar}\frac{\partial E_n}{\partial p_y}$$

$$= e\sum_n \int_{\mu_-}^{\mu_+} \frac{dE_n}{2\pi\hbar}$$

$$= \frac{ne}{h}(\mu_+ - \mu_-)$$

$$= \frac{ne^2}{h}V_x$$

$$\vec{v} = \frac{\vec{\pi}}{m} = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)$$

$$\mu_+ - \mu_- = eV_x$$

$$R_H = \frac{V_x}{I_y} = \frac{h}{ne^2}$$

Linear response theory for the quantized Hall resistance

- The fluctuation-dissipation theorem is a general result of statistical mechanics stating that the fluctuation in a given system at equilibrium is related with the response of the system to a small applied perturbation.

Fluctuation/Correlation	Dissipation/Response function
Density-density	Dielectric function
Spin-spin	Spin susceptibility
Current-current along the same direction	Conductivity
Current-current between orthogonal directions	**Hall conductivity**

- The Kubo formula provides the precise mathematical formulation of the fluctuation-dissipation theorem.

$$\sigma_{xy} \propto \int_0^\infty dt \, \langle J_x(t) J_y(0) \rangle$$

Connection between the Chern number and the Hall conductance

- Kubo formula: $$\sigma_{\alpha\beta}(\mathbf{q},\omega) = \frac{i}{\omega}\left[\Pi_{\alpha\beta}(\mathbf{q},\omega) + \frac{n_0 e^2}{m}\delta_{\alpha\beta}\right]$$

$$\Pi_{\alpha\beta}(\mathbf{q},\omega) = -\frac{i}{V}\int_{-\infty}^{\infty} dt e^{i\omega t}\theta(t)\langle\Psi|[j_\alpha^\dagger(\mathbf{q},t), j_\beta(\mathbf{q},0)]|\Psi\rangle$$

- Current operator in the long wavelength limit:

$$j_\alpha(\mathbf{q}\to 0) = e v_\alpha = e\sum_{\mathbf{k}} \hat{v}_\alpha(\mathbf{k}) = e\sum_{\mathbf{k}} \mathbf{c}_{\mathbf{k}}^\dagger \frac{\partial H_{\mathbf{k}}}{\partial k_\alpha}\mathbf{c}_{\mathbf{k}}$$

- Hall conductivity in the long wavelength limit:

$$\sigma_{xy}(\omega+i\eta) = \frac{-ie^2}{V(\omega+i\eta)}\sum_{n,m} f(\epsilon_n)\left[\frac{\langle\epsilon_n|v_x|\epsilon_m\rangle\langle\epsilon_m|v_y|\epsilon_n\rangle}{\omega+i\eta+\epsilon_n-\epsilon_m} + \frac{\langle\epsilon_n|v_y|\epsilon_m\rangle\langle\epsilon_m|v_x|\epsilon_n\rangle}{-(\omega+i\eta)+\epsilon_n-\epsilon_m}\right]$$

Chern number as a topological order parameter: TKNN formula

$$\sigma_{xy} \propto \int_{\mathbf{k}\in\mathrm{BZ}} d^2\mathbf{k} \sum_{\epsilon_\mu(\mathbf{k})<\epsilon_\mathrm{F}<\epsilon_\nu(\mathbf{k})} \frac{\langle u_\mu(\mathbf{k})|\nabla_\mathbf{k} H(\mathbf{k})|u_\nu(\mathbf{k})\rangle \times \langle u_\nu(\mathbf{k})|\nabla_\mathbf{k} H(\mathbf{k})|u_\mu(\mathbf{k})\rangle}{[\epsilon_\mu(\mathbf{k})-\epsilon_\nu(\mathbf{k})]^2} \cdot \hat{z}$$

$$= \int_{\mathbf{k}\in\mathrm{BZ}} d^2\mathbf{k} \sum_{\epsilon_\mu(\mathbf{k})<\epsilon_\mathrm{F}} \langle \nabla_\mathbf{k} u_\mu(\mathbf{k})| \times |\nabla_\mathbf{k} u_\mu(\mathbf{k})\rangle \cdot \hat{z}$$

where $u_{\mu\mathbf{k}}(\mathbf{r})$ is the periodic part of a Bloch wave function:

$$\left[\frac{1}{2m}\left(-i\hbar\nabla + \hbar\mathbf{k} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 + U(\mathbf{r})\right] u_{\mu\mathbf{k}}(\mathbf{r}) = \epsilon_{\mu\mathbf{k}} u_{\mu\mathbf{k}}(\mathbf{r})$$

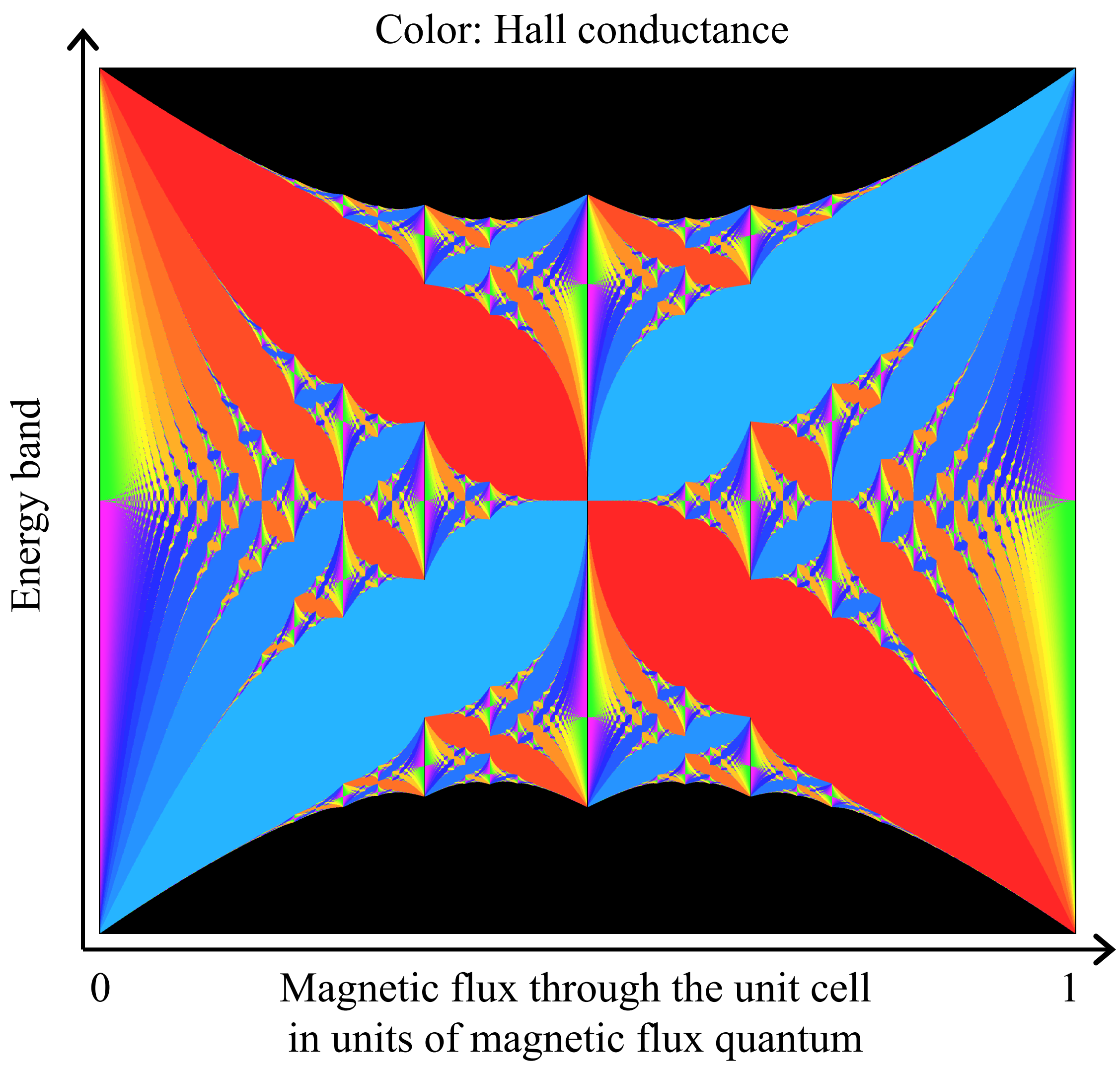
- This is the famous Thouless-Kohmoto-Nightingale-Nijs (TKNN) formula relating the the topologically invariant "order parameter" called the Chern number with the Hall conductivity via $\sigma_{xy}=ne^2/h$.

$$n = \sum_{\epsilon_\mu(\mathbf{k})<\epsilon_\mathrm{F}} \mathcal{C}_\mu$$

$$\boxed{\mathcal{C}_\mu = \frac{i}{2\pi} \int_{\mathbf{k}\in\mathrm{BZ}} d^2\mathbf{k} \langle \nabla_\mathbf{k} u_\mu(\mathbf{k})| \times |\nabla_\mathbf{k} u_\mu(\mathbf{k})\rangle \cdot \hat{z}}$$

Berry curvature flux piercing through the Brillouin zone (BZ) for the μ-th energy band

QHE in the lattice: Hofstadter’s Butterfly



Topology plays an intriguing role in quantum physics under the name of the Berry phase.

Adiabatic evolution and geometrical phase

- Adiabatic theorem (originally by Born and Fock, 1928):

A physical system remains in its *instantaneous* energy eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the spectrum.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

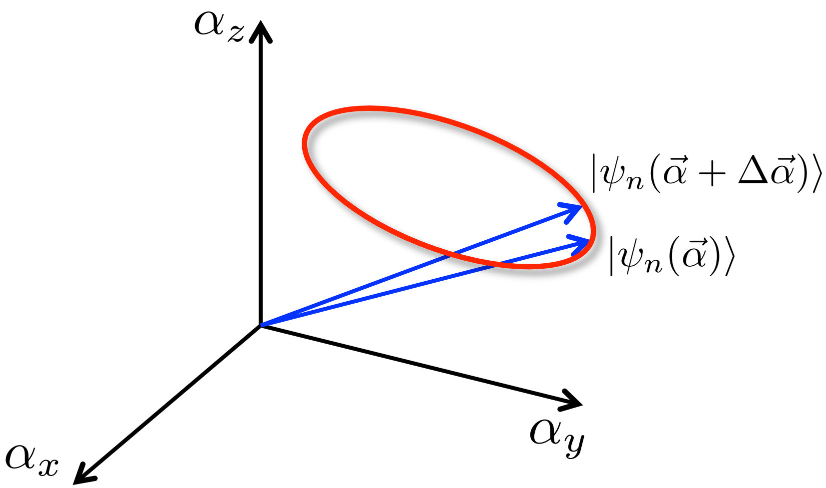
$$H(t)|\psi_{n*}(t)\rangle = E_{n*}(t)|\psi_{n*}(t)\rangle$$

$$|\psi_n(t)\rangle \simeq |\psi_{n*}(t)\rangle e^{-\frac{i}{\hbar}\int_0^t dt' E_{n*}(t')} e^{i\Gamma_n(t)}$$

Berry realized that there was an additional geometrical phase that can have a physical effect.

Berry, "Quantal Phase Factors Accompanying Adiabatic Changes," Proc. R. Soc. A **392**, 45 (1984)

Formal theory of the Berry phase



$$\langle\psi_n(\vec{\alpha})|\psi_n(\vec{\alpha}+\Delta\vec{\alpha})\rangle$$
$$= 1 + \Delta\vec{\alpha}\,\langle\psi_n(\vec{\alpha})|\,\nabla_{\vec{\alpha}}\,|\psi_n(\vec{\alpha})\rangle$$
$$= e^{-i\Delta\vec{\alpha}\cdot\vec{\mathcal{A}}_n(\vec{\alpha})}$$

- Berry connection: *vector potential*

$$\vec{\mathcal{A}}_n(\vec{\alpha}) = i\,\langle\psi_n(\vec{\alpha})|\,\nabla_{\vec{\alpha}}\,|\psi_n(\vec{\alpha})\rangle$$

- Berry curvature: *magnetic field*

$$\vec{\mathcal{B}}_n(\vec{\alpha}) = \nabla_{\vec{\alpha}} \times \vec{\mathcal{A}}_n(\vec{\alpha}) = i\,\langle\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})| \times |\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})\rangle$$

- Berry phase: *Aharonov-Bohm phase*

$$\Gamma_n = \oint_C d\vec{\alpha}\cdot\vec{\mathcal{A}}_n(\vec{\alpha}) = \int_A d\vec{S}\cdot\vec{\mathcal{B}}_n(\vec{\alpha}) = i\int_A d\vec{S}\cdot\langle\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})| \times |\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})\rangle$$

Formal theory of the Berry phase

$$\Gamma_n = i\int_A d\vec{S}\cdot\langle\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})|\times|\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})\rangle$$
$$= i\int_A d\vec{S}\cdot\sum_{m\neq n}\langle\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})|\psi_m(\vec{\alpha})\rangle\times\langle\psi_m(\vec{\alpha})|\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})\rangle$$
$$= i\int_A d\vec{S}\cdot\sum_{m\neq n}\frac{\langle\psi_n(\vec{\alpha})|\nabla_{\vec{\alpha}}H(\vec{\alpha})|\psi_m(\vec{\alpha})\rangle\times\langle\psi_m(\vec{\alpha})|\nabla_{\vec{\alpha}}H(\vec{\alpha})|\psi_n(\vec{\alpha})\rangle}{[E_n(\vec{\alpha})-E_m(\vec{\alpha})]^2}$$

Note

$$H(\vec{\alpha})|\psi_n(\vec{\alpha})\rangle = E_n(\vec{\alpha})|\psi_n(\vec{\alpha})\rangle$$

↓ Multiplying both sides by $\langle\psi_m(\vec{\alpha})|\nabla_{\vec{\alpha}}$

$$\langle\psi_m(\vec{\alpha})|\nabla_{\vec{\alpha}}\psi_n(\vec{\alpha})\rangle = \frac{\langle\psi_m(\vec{\alpha})|\nabla_{\vec{\alpha}}H(\vec{\alpha})|\psi_n(\vec{\alpha})\rangle}{E_n(\vec{\alpha})-E_m(\vec{\alpha})}$$

TKNN formula revisited

- The TKNN formula tells us that the Hall conductivity is proportional to the Berry phase of a closed path encompassing the entire Brillouin zone.

$$\Gamma_n = i \int_A d\vec{S} \cdot \langle \nabla_{\vec{\alpha}} \psi_n(\vec{\alpha}) | \times | \nabla_{\vec{\alpha}} \psi_n(\vec{\alpha}) \rangle$$

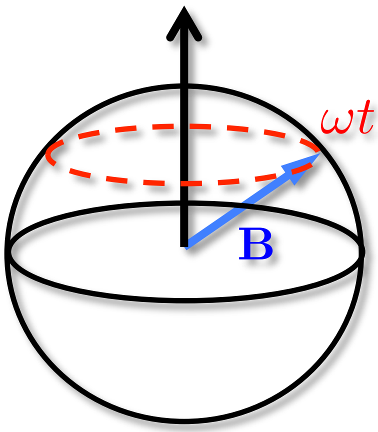
$$\vec{\alpha} \leftrightarrow \mathbf{k}$$

$$\sigma_{xy} \propto \int_{\mathbf{k}\in \mathrm{BZ}} d^2\mathbf{k} \sum_{\epsilon_\mu(\mathbf{k})<\epsilon_\mathrm{F}} \langle \nabla_{\mathbf{k}} u_\mu(\mathbf{k}) | \times | \nabla_{\mathbf{k}} u_\mu(\mathbf{k}) \rangle \cdot \hat{z}$$

When does the topology become non-trivial?

An answer to this question reveals that there is a profound connection between the Rabi oscillation and topological insulators.

Rabi oscillation



$$H(t) = \frac{\hbar\Omega}{2}\hat{\sigma}_z + \gamma(\cos\omega t\hat{\sigma}_x + \sin\omega t\hat{\sigma}_y)$$

$$\Omega = |g|B_0 \qquad \gamma = \hbar|g|B_1/2$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle = \begin{pmatrix} \hbar\Omega/2 & \gamma e^{-i\omega t} \\ \gamma e^{i\omega t} & -\hbar\Omega/2 \end{pmatrix}|\psi(t)\rangle$$

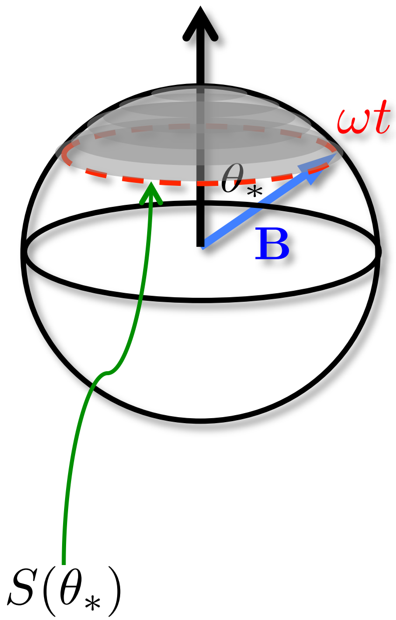
- Exact solution:

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\frac{\omega t}{2}} & 0 \\ 0 & e^{i\frac{\omega t}{2}} \end{pmatrix}\begin{pmatrix} \cos\tilde{\Omega}t - i\cos\theta\sin\tilde{\Omega}t & -i\sin\theta\sin\tilde{\Omega}t \\ -i\sin\theta\sin\tilde{\Omega}t & \cos\tilde{\Omega}t + i\cos\theta\sin\tilde{\Omega}t \end{pmatrix}|\psi(0)\rangle$$

$$\hbar\tilde{\Omega} = \sqrt{\hbar^2(\Omega-\omega)^2/4 + \gamma^2} \qquad \sin\theta = \frac{\gamma}{\hbar\tilde{\Omega}} \qquad \cos\theta = \frac{\hbar(\Omega-\omega)/2}{\hbar\tilde{\Omega}}$$

Geometrical meaning of the Berry phase

- In the adiabatic limit, i.e., $\omega << \Omega$, the exact solution says:



$$E_{\pm *}(t) = \pm\sqrt{\hbar^2\Omega^2/4 + \gamma^2} = \pm\hbar\Omega_*/2$$

$$|\psi_+(t)\rangle \simeq \begin{pmatrix} \cos\frac{\theta_*}{2} e^{-i\frac{\omega t}{2}} \\ \sin\frac{\theta_*}{2} e^{i\frac{\omega t}{2}} \end{pmatrix} e^{-i\frac{\tilde{\Omega} t}{2}}$$

$$\simeq \begin{pmatrix} \cos\frac{\theta_*}{2} e^{-i\frac{\omega t}{2}} \\ \sin\frac{\theta_*}{2} e^{i\frac{\omega t}{2}} \end{pmatrix} e^{-i\frac{\Omega^* t}{2}} e^{\frac{i}{2}\frac{\Omega}{\Omega_*}\omega t}$$

$$\Gamma_+(t) = \frac{1}{2}\frac{\Omega}{\Omega_*}\omega t = \frac{1}{2}\cos\theta_*\omega t$$

$$\langle\psi_+(t = 2\pi/\omega)|\psi_+(t = 0)\rangle = e^{-i\frac{\pi\Omega_*}{\omega}} e^{-\frac{i}{2}S(\theta_*)}$$

Solid angle enclosed by a circular path

Magnetic monopole in the Rabi oscillation

$$H(\vec{B}) = |g|\vec{S}\cdot\vec{B} \longrightarrow H(\vec{\alpha}) = \vec{\sigma}\cdot\vec{\alpha}$$

$$\vec{\mathcal{B}}_+(\vec{\alpha}) = i\frac{\langle\psi_+(\vec{\alpha})|\nabla_{\vec{\alpha}}H(\vec{\alpha})|\psi_-(\vec{\alpha})\rangle \times \langle\psi_-(\vec{\alpha})|\nabla_{\vec{\alpha}}H(\vec{\alpha})|\psi_+(\vec{\alpha})\rangle}{[E_+(\vec{\alpha}) - E_-(\vec{\alpha})]^2}$$

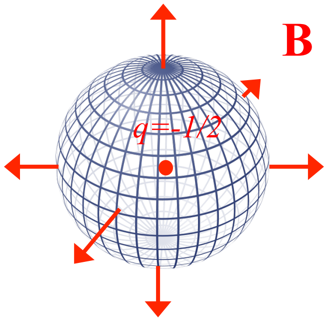
Note

$$\nabla_{\vec{\alpha}}H(\vec{\alpha}) = \vec{\sigma} \qquad |\psi_+(\vec{\alpha})\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}$$

$$E_\pm(\vec{\alpha}) = \pm|\vec{\alpha}| = \pm\alpha \qquad |\psi_-(\vec{\alpha})\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}$$

$$\vec{\mathcal{B}}_+(\vec{\alpha}) = -\frac{\hat{\alpha}}{2\alpha^2}$$

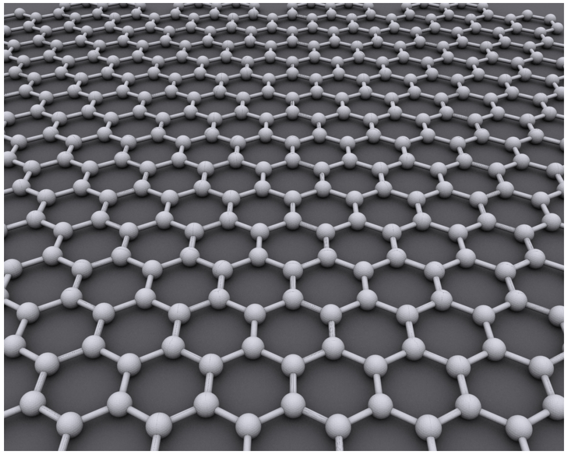
"Magnetic field" exerted by a Dirac monopole at the center with the strength -1/2

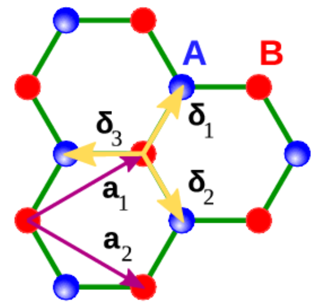


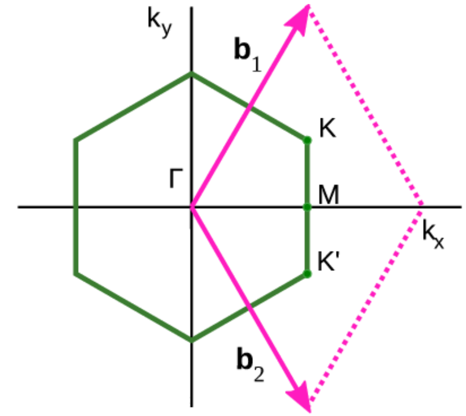
$$\Gamma_+ = \int_A d\vec{S}\cdot\vec{\mathcal{B}}_+(\vec{\alpha}) = -\frac{1}{2}(\text{Solid angle})$$

Topological insulators (TIs) are generated by a magnetic monopole in the pseudospin space.

Graphene







$$\mathbf{K}, \mathbf{K}' = \left(\frac{2\pi}{3a}, \pm\frac{2\pi}{3\sqrt{3}a}\right)$$

- Graphene Hamiltonian:

$$H = -t \sum_{\langle i,j \rangle} \left(|\mathbf{r}_i\rangle\langle\mathbf{r}_j| + |\mathbf{r}_j\rangle\langle\mathbf{r}_i| \right)$$

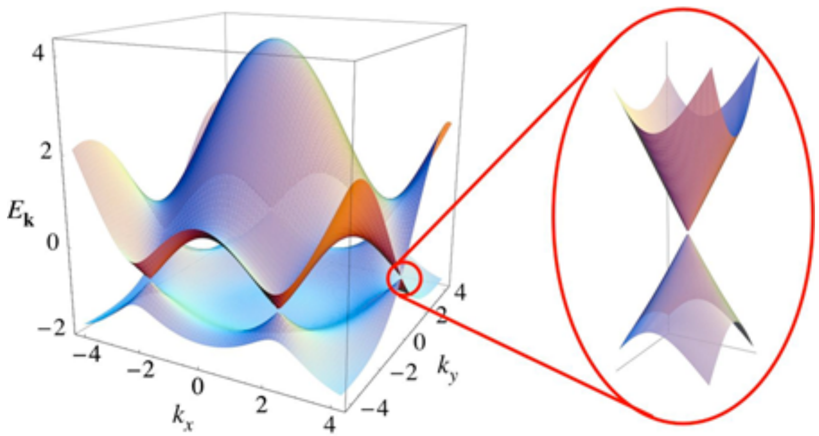
$$H = -t \int_{\mathrm{BZ}} d^2\mathbf{k} \left[(e^{i\mathbf{k}\cdot\boldsymbol{\delta}_1} + e^{i\mathbf{k}\cdot\boldsymbol{\delta}_2} + e^{i\mathbf{k}\cdot\boldsymbol{\delta}_3}) |\mathbf{k}, \mathrm{A}\rangle\langle\mathbf{k}, \mathrm{B}| + \mathrm{H.c.} \right]$$

$$= \int_{\mathrm{BZ}} d^2\mathbf{k} \begin{pmatrix} |\mathbf{k}, \mathrm{A}\rangle & |\mathbf{k}, \mathrm{B}\rangle \end{pmatrix} \begin{pmatrix} 0 & f_{\mathbf{k}} \\ f_{\mathbf{k}}^* & 0 \end{pmatrix} \begin{pmatrix} \langle\mathbf{k}, \mathrm{A}| \\ \langle\mathbf{k}, \mathrm{B}| \end{pmatrix}$$

Graphene

$$H_{\mathbf{k}} = \begin{pmatrix} 0 & f_{\mathbf{k}} \\ f_{\mathbf{k}}^* & 0 \end{pmatrix} \qquad f_{\mathbf{k}} = -t(e^{i\mathbf{k}\cdot\boldsymbol{\delta}_1} + e^{i\mathbf{k}\cdot\boldsymbol{\delta}_2} + e^{i\mathbf{k}\cdot\boldsymbol{\delta}_3})$$

- Energy dispersion: $E_{\mathbf{k}} = \pm|f_{\mathbf{k}}|$



$$H_{\mathbf{k}=\mathbf{K}+\mathbf{q}} \simeq \hbar v_{\mathrm{F}} \mathbf{q} \cdot \boldsymbol{\sigma}^*$$

$$H_{\mathbf{k}=\mathbf{K}'+\mathbf{q}} \simeq \hbar v_{\mathrm{F}} \mathbf{q} \cdot \boldsymbol{\sigma}$$

$$\hbar v_{\mathrm{F}} = 3at/2$$

Massless Dirac Hamiltonian, or the Hamiltonian for "Zeeman coupling" with **B** replaced by **q**

Haldane model

VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

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(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.

$$\mathbf{H}(\mathbf{k}) = 2t_2\cos\phi\left(\sum_i \cos(\mathbf{k}\cdot\mathbf{b}_i)\right)\mathbf{I} + t_1\left(\sum_i [\cos(\mathbf{k}\cdot\mathbf{a}_i)\sigma^1 + \sin(\mathbf{k}\cdot\mathbf{a}_i)\sigma^2]\right) + \left[M - 2t_2\sin\phi\left(\sum_i \sin(\mathbf{k}\cdot\mathbf{b}_i)\right)\right]\sigma^3$$

$\mathbf{a}_i$ / $\mathbf{b}_i$: vectors connecting between nearest/next nearest neighbors

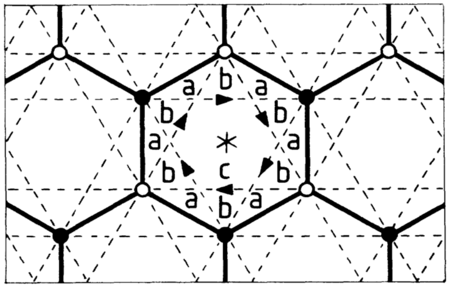


FIG. 1. The honeycomb-net model ("2D graphite") showing nearest-neighbor bonds (solid lines) and second-neighbor bonds (dashed lines). Open and solid points, respectively, mark the *A* and *B* sublattice sites. The Wigner-Seitz unit cell is conveniently centered on the point of sixfold rotation symmetry (marked "*") and is then bounded by the hexagon of nearest-neighbor bonds. Arrows on second-neighbor bonds mark the directions of positive phase hopping in the state with broken time-reversal invariance.

$$H(\mathbf{k}) = \begin{pmatrix} g_{\mathbf{k}} & f_{\mathbf{k}} \\ f^*_{\mathbf{k}} & -g_{\mathbf{k}} \end{pmatrix} = \mathbf{d}_{\mathbf{k}}\cdot\boldsymbol{\sigma}$$

$$\mathbf{d}_{\mathbf{k}} = (\mathrm{Re} f_{\mathbf{k}}, -\mathrm{Im} f_{\mathbf{k}}, g_{\mathbf{k}})$$

$$f_{\mathbf{k}} = t_1 \sum_i e^{i\mathbf{k}\cdot\mathbf{a}_i}$$

$$g_{\mathbf{k}} = M + 2t_2 \sum_i \cos(\mathbf{k}\cdot\mathbf{b}_i + \phi)$$

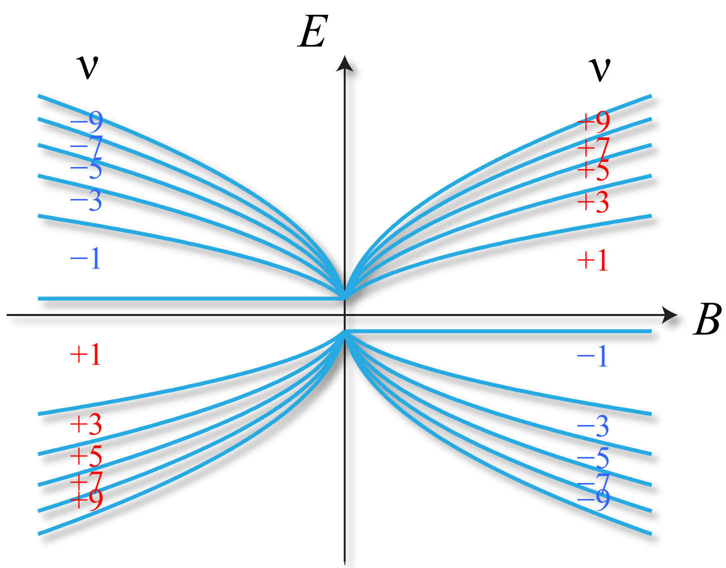
Chern number from the Landau level structure

- Energy of the Landau levels:

$$E_n = \mathrm{sgn}(n)\sqrt{(m_\alpha v_\mathrm{F}^2)^2 + 2e\hbar v_\mathrm{F}^2 |nB|} \quad (n \neq 0)$$

$$E_0 = \alpha m_\alpha v_\mathrm{F}^2 e \,\mathrm{sgn}(B)$$

$$m_\alpha v_\mathrm{F}^2 = M - 3\sqrt{3}\alpha t_2 \sin\phi$$



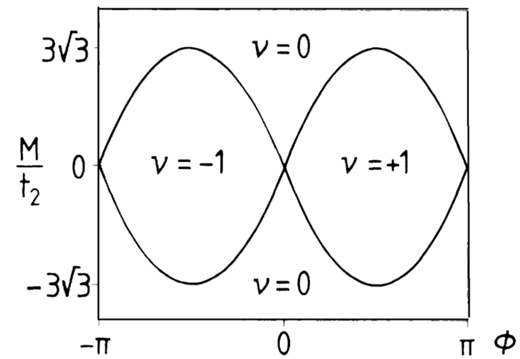
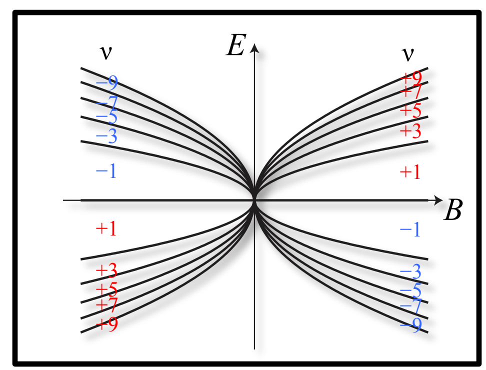


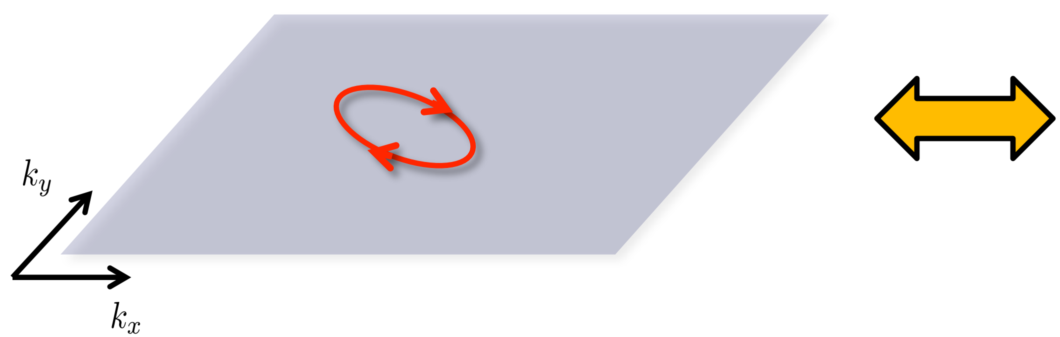
FIG. 2. Phase diagram of the spinless electron model with $|t_2/t_1| < \frac{1}{3}$. Zero-field quantum Hall effect phases ($\nu = \pm 1$, where $\sigma^{xy} = \nu e^2/h$) occur if $|M/t_2| < 3\sqrt{3}|\sin\phi|$. This figure assumes that t_2 is positive; if it is negative, ν changes sign. At the phase boundaries separating the anomalous and normal ($\nu = 0$) semiconductor phases, the low-energy excitations of the model simulate undoubled massless chiral relativistic fermions.

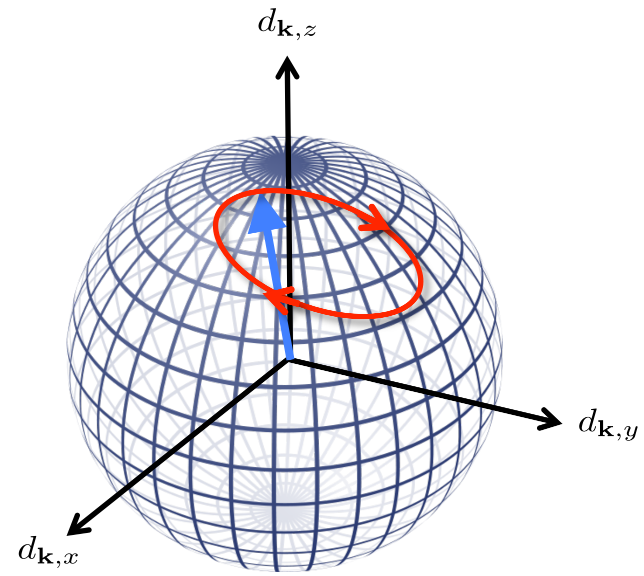


Graphene Landau levels

Chern number from the Berry phase:
Connection between the Rabi oscillation and TI

$$H(\mathbf{k}) = \begin{pmatrix} g_{\mathbf{k}} & f_{\mathbf{k}} \\ f^*_{\mathbf{k}} & -g_{\mathbf{k}} \end{pmatrix} = \mathbf{d}_{\mathbf{k}} \cdot \boldsymbol{\sigma}$$





Momentum space

Bloch sphere

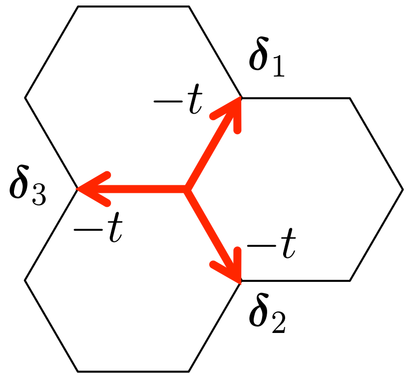
Chern number, or the winding number:

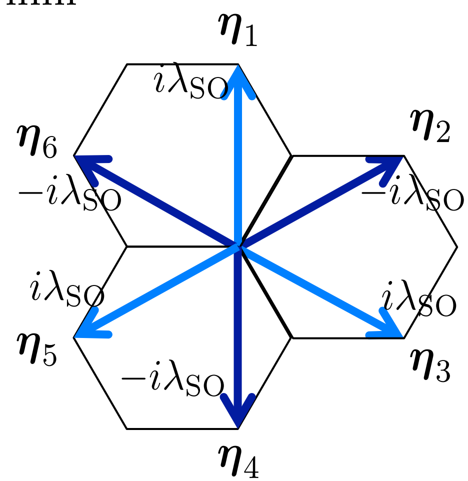
$$\mathcal{C} = \frac{1}{4\pi}\int d^2\mathbf{k}\ \ \hat{\mathbf{d}}_{\mathbf{k}} \cdot (\partial_{k_x}\hat{\mathbf{d}}_{\mathbf{k}} \times \partial_{k_y}\hat{\mathbf{d}}_{\mathbf{k}})$$

Kane-Mele model: spin-orbit-coupled graphene

Kane, Mele, PRL **95**, 226801 (2005)

$$H_{\mathrm{KM}}^{(\sigma)} = H_{\mathrm{nn}} + H_{\mathrm{nnn}}^{(\sigma)}$$





$$H_{\mathrm{nn}} = -t \sum_{\langle i|j \rangle} c_i^\dagger c_j$$

$$H_{\mathrm{nnn}}^{(\uparrow)} = \lambda_{\mathrm{SO}} \sum_{\langle\langle i|j \rangle\rangle} e^{i\phi_{ij}} c_i^\dagger c_j$$

$$H_{\mathrm{KM}}^{(\uparrow)} = \int_{\mathrm{BZ}} d^2\mathbf{k} \begin{pmatrix} c_{\mathbf{k},\mathrm{A}}^\dagger & c_{\mathbf{k},\mathrm{B}}^\dagger \end{pmatrix} \begin{pmatrix} g_{\mathbf{k}} & f_{\mathbf{k}} \\ f_{\mathbf{k}}^* & -g_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k},\mathrm{A}} \\ c_{\mathbf{k},\mathrm{B}} \end{pmatrix}$$

$$f_{\mathbf{k}} = -t \sum_{i=1}^{3} e^{i\mathbf{k}\cdot\boldsymbol{\delta}_m} \qquad g_{\mathbf{k}} = -i\lambda_{\mathrm{SO}} \sum_{i=1}^{6} (-1)^m e^{i\mathbf{k}\cdot\boldsymbol{\eta}_m}$$

Low-energy effective Hamiltonian for the Kane-Mele model

$$H^{(\uparrow)}_{\text{KM}}(\mathbf{k}) = \begin{pmatrix} g_{\mathbf{k}} & f_{\mathbf{k}} \\ f^*_{\mathbf{k}} & -g_{\mathbf{k}} \end{pmatrix} = \mathbf{d}_{\mathbf{k}} \cdot \boldsymbol{\sigma} \qquad \mathbf{d}_{\mathbf{k}} = (\mathrm{Re} f_{\mathbf{k}}, -\mathrm{Im} f_{\mathbf{k}}, g_{\mathbf{k}})$$

$$H^{(\downarrow)}(\mathbf{k}) = H^{(\uparrow)*}(-\mathbf{k})$$

Time-reversal symmetry

---

(1) The energy spectrum is gapped: $E_{\pm,\mathbf{k}} = \pm|\mathbf{d}_{\mathbf{k}}| = \pm\sqrt{|f_{\mathbf{k}}|^2 + g^2_{\mathbf{k}}}$

---

(2) The low-energy effective Hamiltonian:

- Around $\mathbf{k} = \mathbf{K} + \mathbf{q}$,

$$f_{\mathbf{q}} = \hbar v_{\mathrm{F}}(q_x + iq_y) \qquad g_{\mathbf{q}} = -3\sqrt{3}\lambda_{\mathrm{SO}} + \frac{9\sqrt{3}}{4}\lambda_{\mathrm{SO}} a^2(q_x^2 + q_y^2)$$

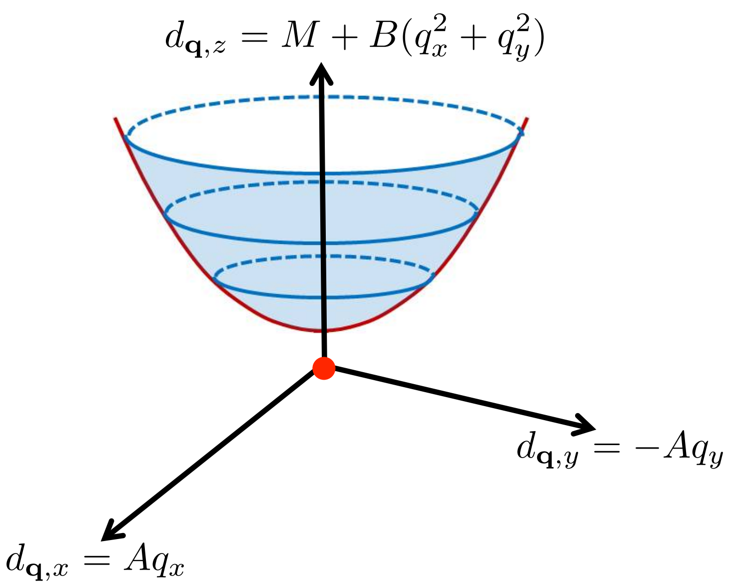
- Around $\mathbf{k} = \mathbf{K}' + \mathbf{q}$,

$$f_{\mathbf{q}} = \hbar v_{\mathrm{F}}(q_x - iq_y) \qquad g_{\mathbf{q}} = 3\sqrt{3}\lambda_{\mathrm{SO}} - \frac{9\sqrt{3}}{4}\lambda_{\mathrm{SO}} a^2(q_x^2 + q_y^2)$$

Condition for topological non-triviality

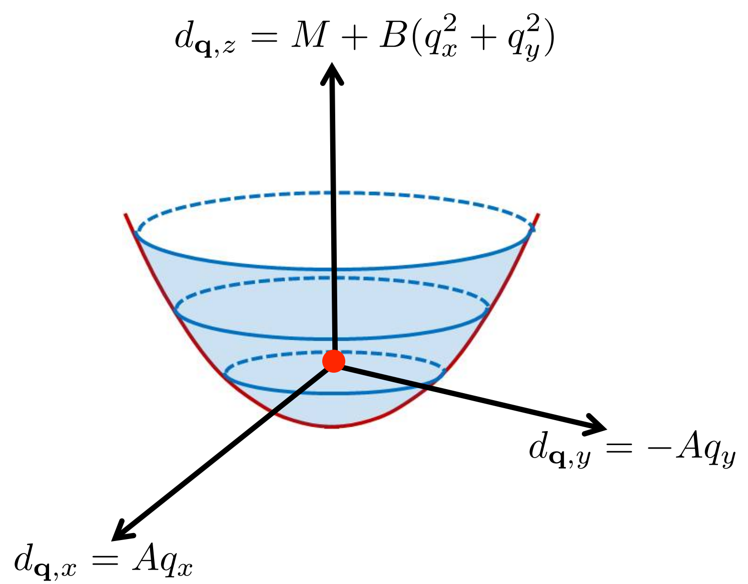
$$H^{(\uparrow)}_{\mathrm{KM}}(\mathbf{k}) = \mathbf{d}_{\mathbf{k}} \cdot \boldsymbol{\sigma} \qquad H^{(\downarrow)}(\mathbf{k}) = H^{(\uparrow)*}(-\mathbf{k})$$

$$\mathbf{d}_{\mathbf{k}=\mathbf{K}+\mathbf{q}} \simeq (Aq_x, -Aq_y, M + B(q_x^2 + q_y^2))$$



$M/B > 0$

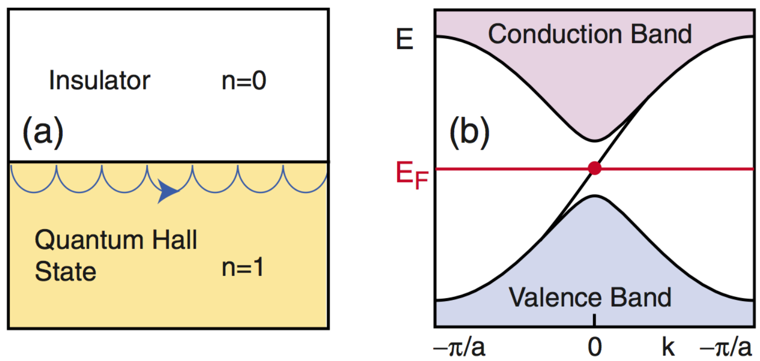
Topologically trivial



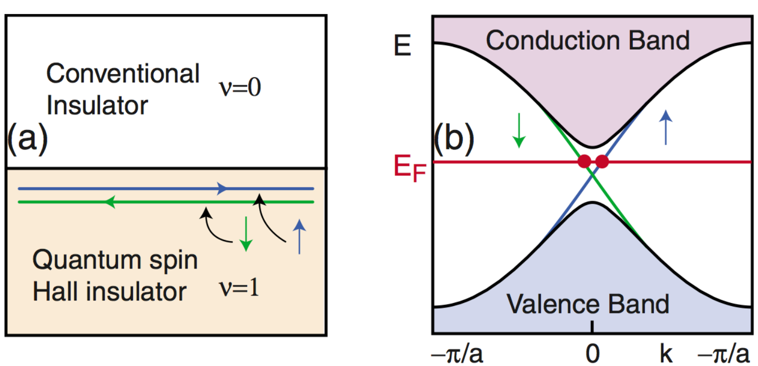
$M/B < 0$

Topologically non-trivial

How to measure: quantum spin Hall effect (QSHE)

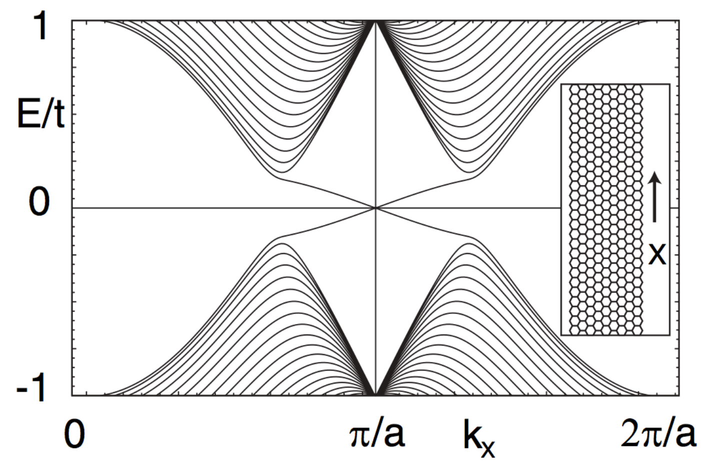


Interface between a quantum Hall insulator (in the Haldane model) and an ordinary insulator



Interface between a quantum spin Hall insulator (in the Kane-Mele model) and an ordinary insulator

Hasan, Kane, RMP **82**, 3045 (2010)



1D energy bands for a strip of spin-orbit coupled graphene as described by the Kane-Mele model

Kane, Mele, PRL **95**, 226801 (2005)

Bernevig-Hughes-Zhang (BHZ) model: HgTe quantum well

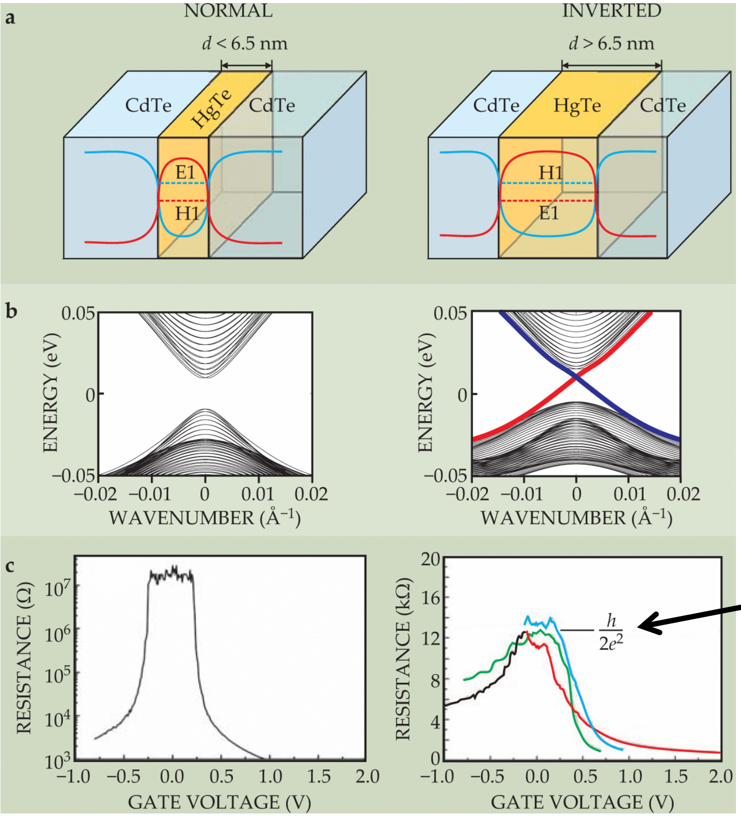
•**<u>Schematically speaking, a *half* of the Kane-Mele model</u>**

Bernevig, Hughes, Zhang, Science **314**, 1757 (2006)

$$H(\mathbf{k}) = \epsilon_{\mathbf{k}}\mathbb{I} + \begin{pmatrix} M + B(k_x^2+k_y^2) & A(k_x+ik_y) & 0 & 0 \\ A(k_x-ik_y) & -[M+B(k_x^2+k_y^2)] & 0 & 0 \\ 0 & 0 & M+B(k_x^2+k_y^2) & -A(k_x-ik_y) \\ 0 & 0 & -A(k_x+ik_y) & -[M+B(k_x^2+k_y^2)] \end{pmatrix}$$

Spin-up electrons in the *s*-like E1 conduction and the p-like H1 valence bands

Spin-down electrons in the *s*-like E1 conduction and the p-like H1 valence bands



$M/B > 0$

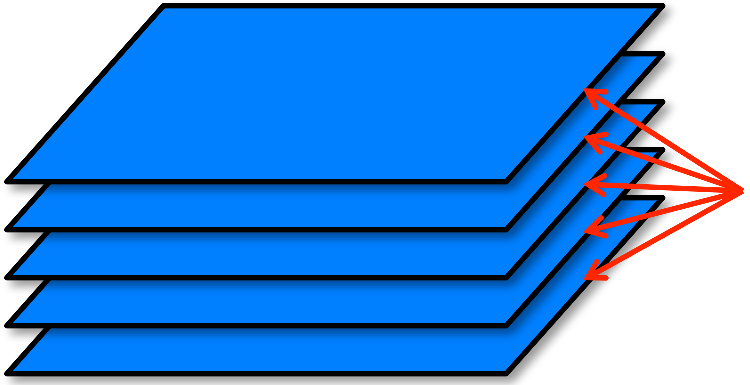
$M/B < 0$

<u>Two-terminal charge conductance, not the spin-filtered Hall conductance!</u>

König *et al.*, Science **318**, 766 (2007), adapted by Qi, Zhang, Phys. Today **63**, 33 (2010)

How to promote TI from 2D to 3D?

3D TI as a system of stacked 2D TI layers: weak TI



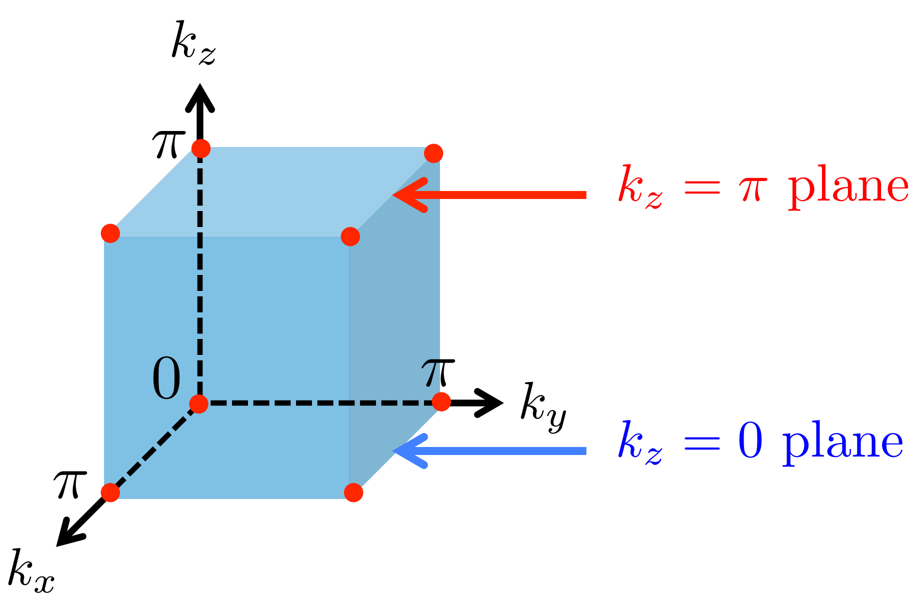
• Unfortunately, unlike the 2D helical edge states, the time-reversal symmetry does not protect the surface states in a ***weak*** TI. Here, the surface states may be localized in the presence of disorder.

Strong 3D TI: BiSe-family materials

$$H(\mathbf{k}) = \epsilon_{\mathbf{k}}\mathbb{I} + \begin{pmatrix} M + B_1 k_\perp^2 + \boxed{B_2 k_z^2} & A_1(k_x + ik_y) & 0 & \boxed{A_2 k_z} \\ A_1(k_x - ik_y) & -(M + B_1 k_\perp^2 + \boxed{B_2 k_z^2)} & \boxed{A_2 k_z} & 0 \\ 0 & \boxed{A_2 k_z} & M + B_1 k_\perp^2 + \boxed{B_2 k_z^2} & -A_1(k_x - ik_y) \\ \boxed{A_2 k_z} & 0 & -A_1(k_x + ik_y) & -(M + B_1 k_\perp^2 + \boxed{B_2 k_z^2)} \end{pmatrix}$$

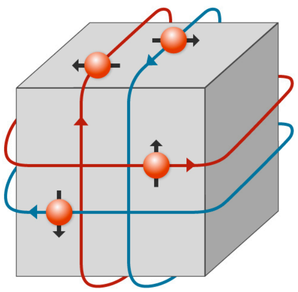
- Lattice regularization: $k_\alpha \to \sin(k_\alpha a)/a \qquad k_\alpha^2 \to 2(1 - \cos(k_\alpha a))/a^2$

- Strong TI:

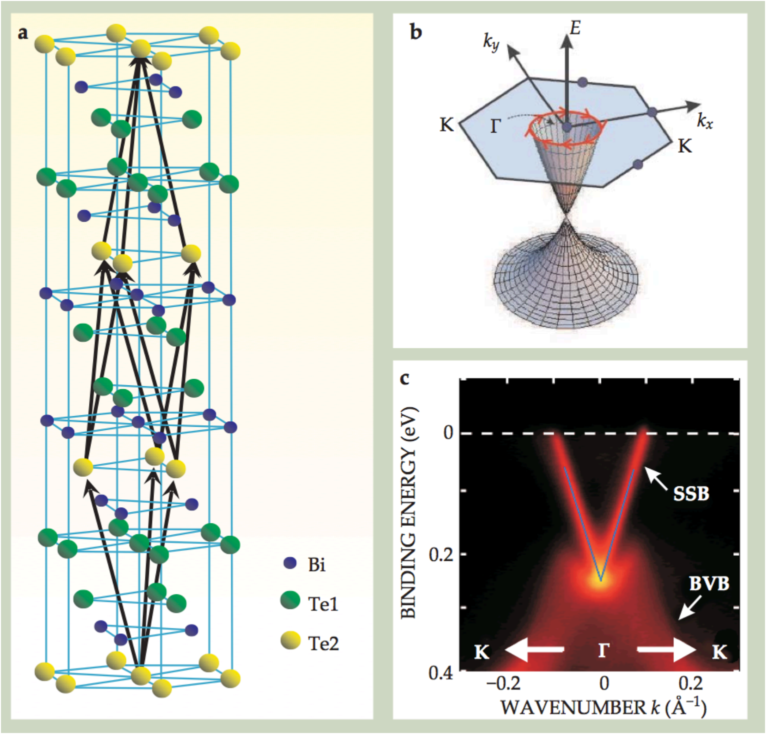


A 3D TI becomes a ***strong*** TI if band topology is opposite between two 2D subsystems in the **k**-space containing one set of time-reversal invariant momenta (TRIM) and the other.

Helical surface states: spin-momentum locking

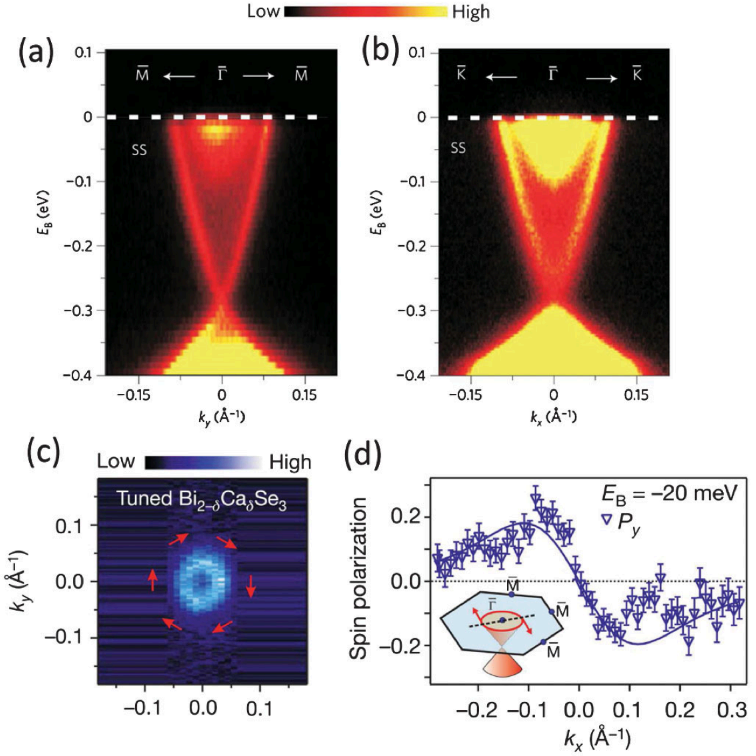


Bi_2Te_3



Chen *et al.*, Science **325**, 178 (2009), adapted by Qi, Zhang, Phys. Today **63**, 33 (2010)

Bi_2Se_3



Xia *et al.*, Nature Phys. **5**, 398 (2009), Hsieh *et al.*, Nature **460**, 1101 (2009), adapted by Qi, Zhang, RMP **83**, 1057 (2011)

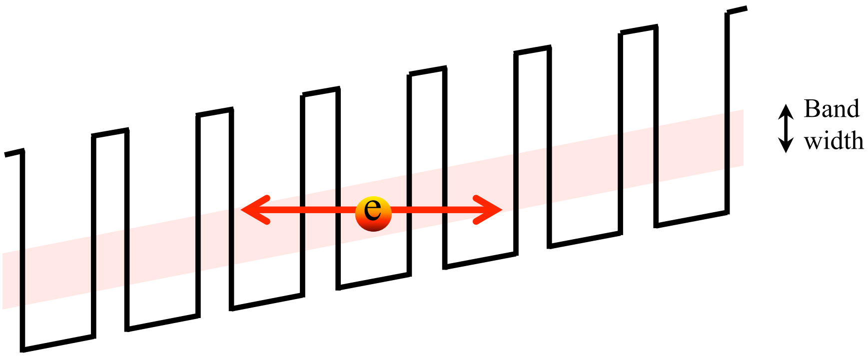
How to measure topology directly in the bulk without reference to boundaries?

A quotation from Lord Kelvin, “To measure is to know.”

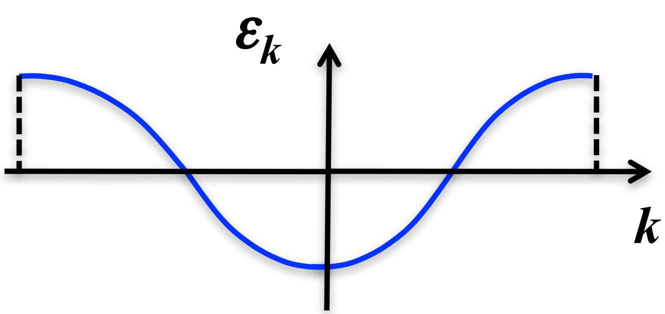
W.-R. Lee & KP, arXiv:1503.01870

Bloch oscillation

Experiments
Waschke *et al.*, PRL (1993): Semiconductor superlattice
Dahan *et al.*, PRL (1996): Optical lattice

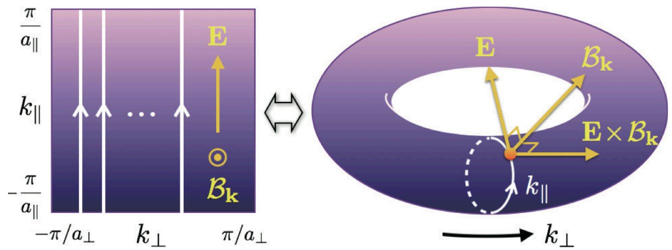


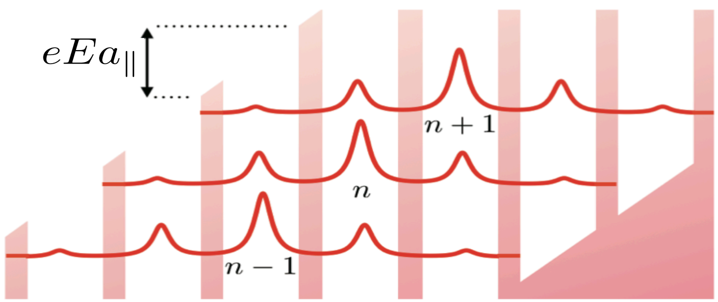
- The electron motion in the lattice is bounded and oscillatory due to the fact that no states are available outside the energy band.



- Another way of viewing this is that the group velocity becomes negative once the crystal momentum crosses the zone boundary.

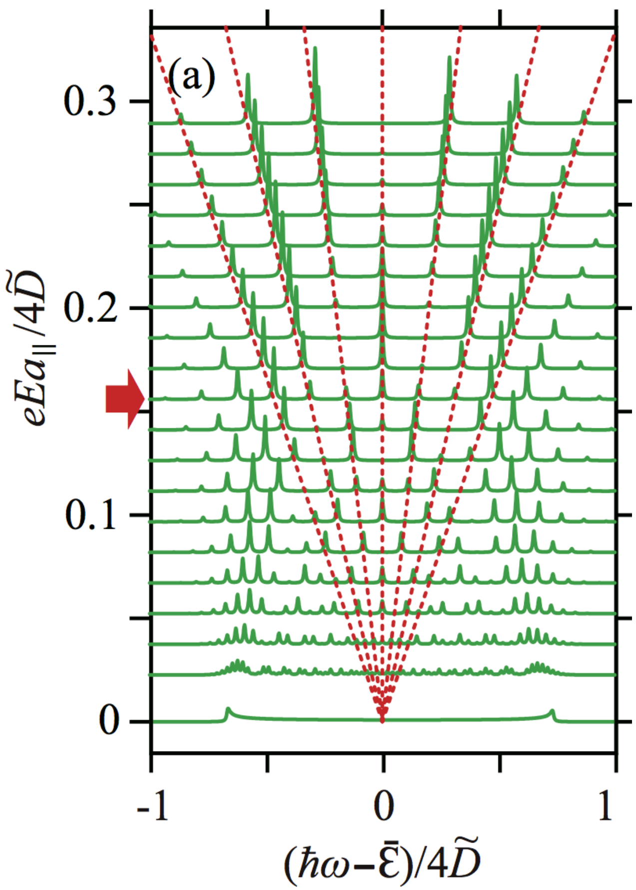
Quantized Bloch oscillation: Wannier-Stark ladder (WSL)



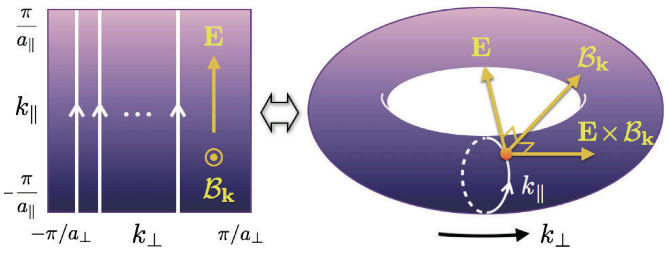


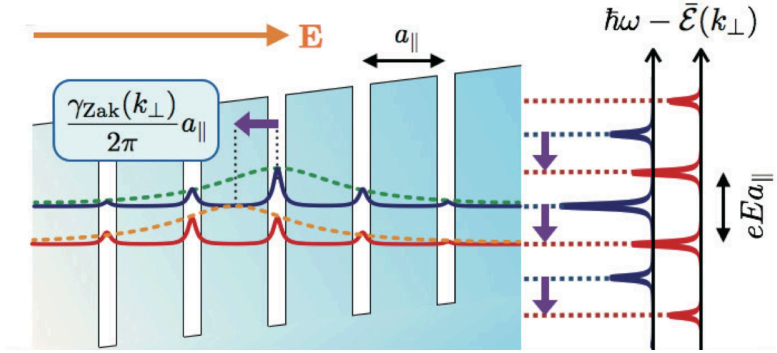
$$\mathcal{E}_n(k_\perp) = \bar{\mathcal{E}}(k_\perp) + neEa_\parallel$$

W.-R. Lee & KP, arXiv:1503.01870



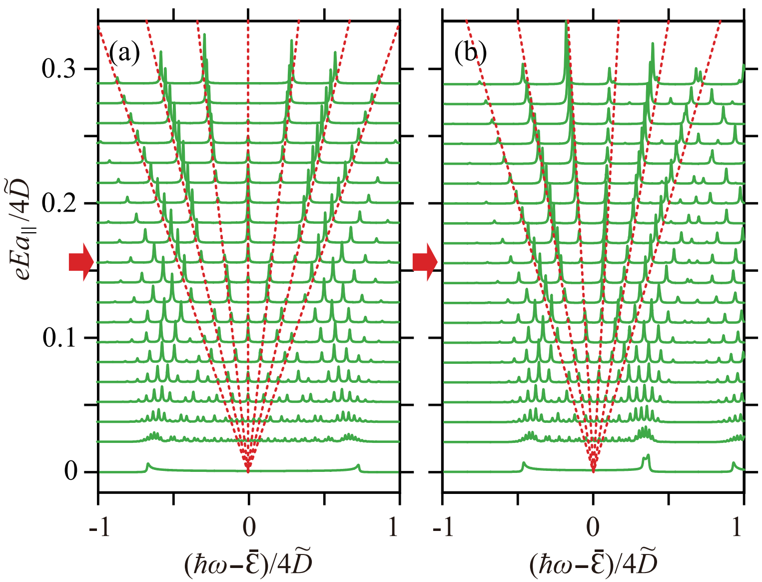
Zak phase: the WSL revisited

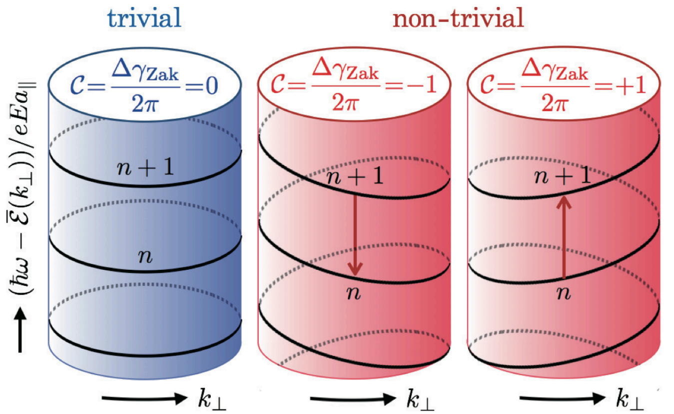




$$\mathcal{E}_n(k_\perp) = \bar{\mathcal{E}}(k_\perp) + \left(n + \frac{\gamma_{\mathrm{Zak}}(k_\perp)}{2\pi}\right) eEa_\parallel$$

W.-R. Lee & KP, arXiv:1503.01870



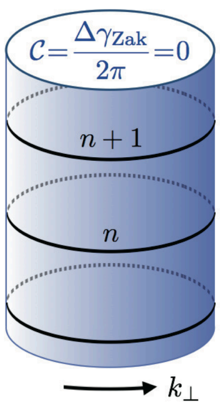


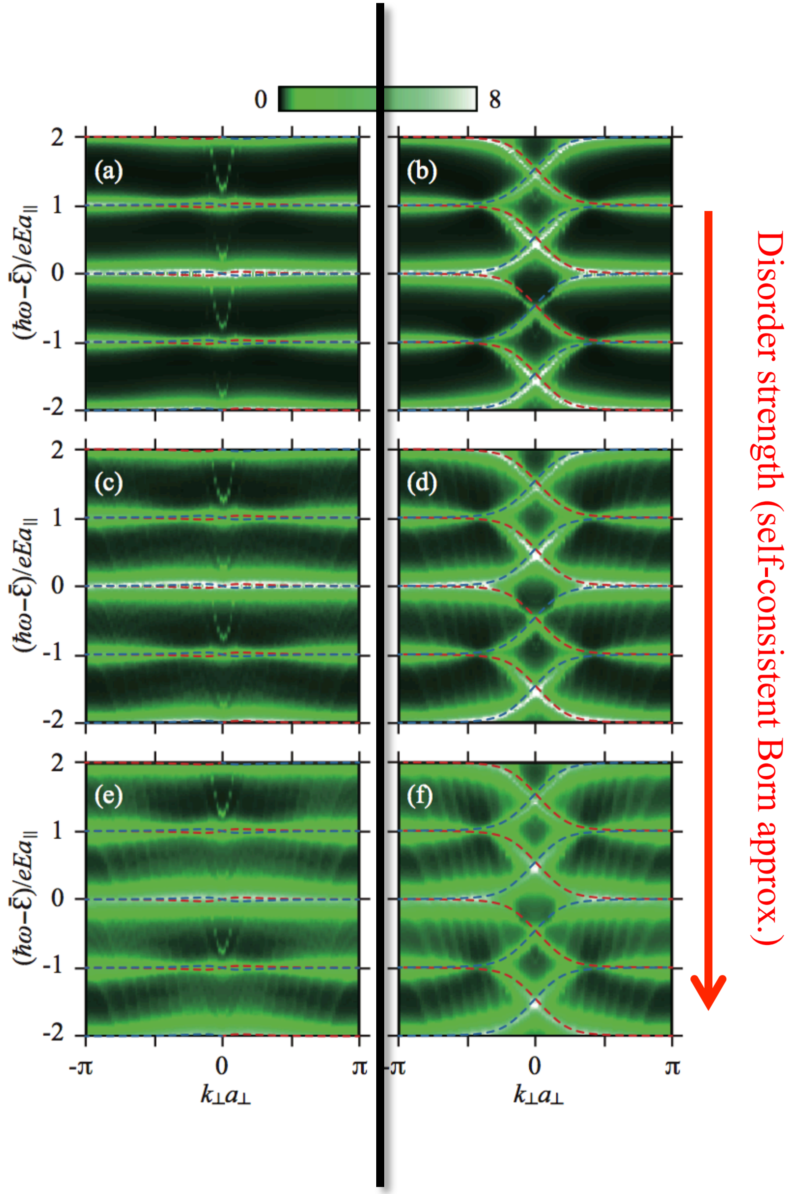
Winding number of the WSL: BHZ model

W.-R. Lee & KP, arXiv:1503.01870

Trivial topology

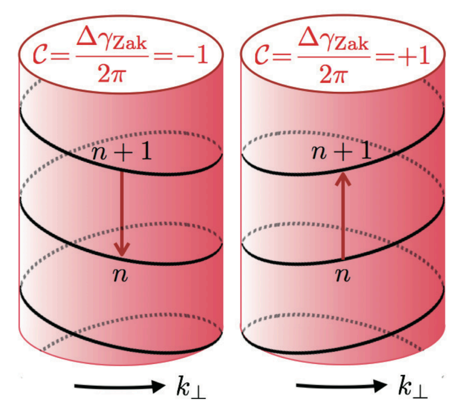
$$M/B > 0$$





Non-trivial topology

$$M/B < 0$$



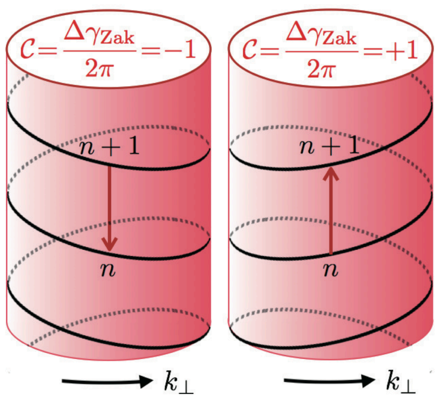
Winding number of the WSL: Kane-Mele model

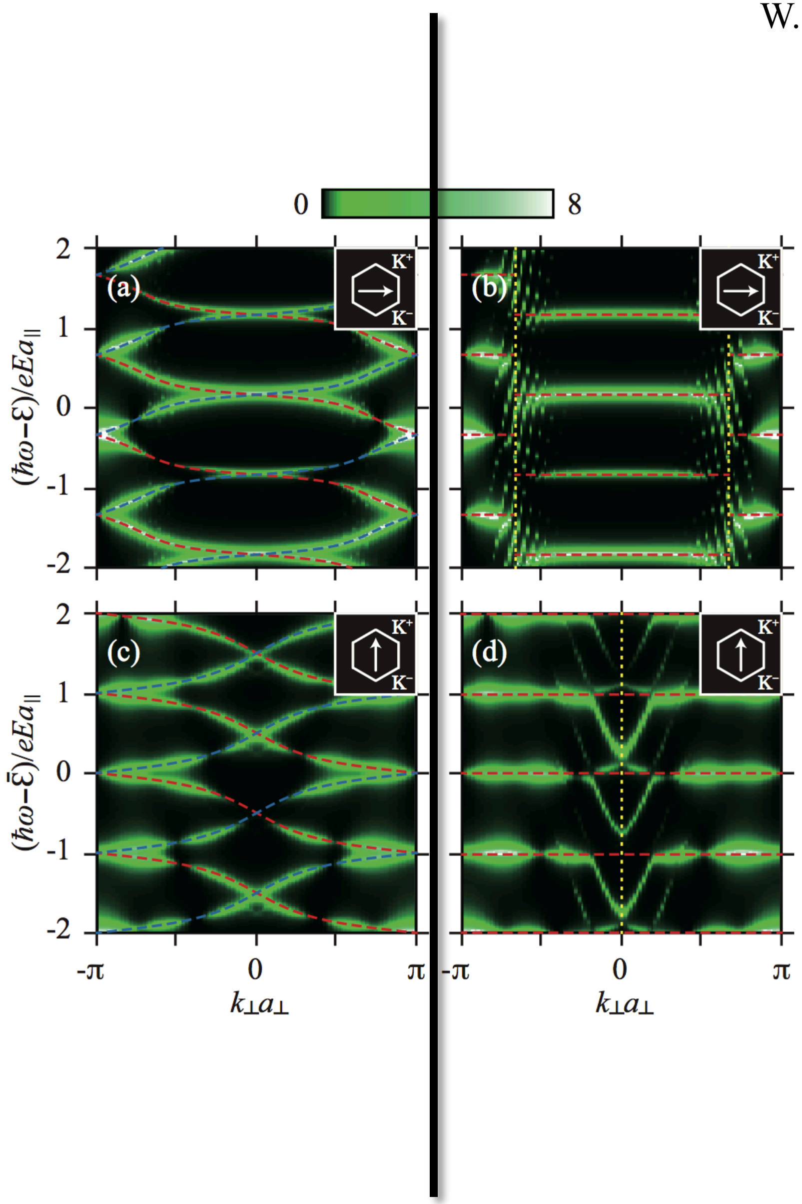
W.-R. Lee & KP, arXiv:1503.01870

Non-trivial topology

$\lambda_{SO} \neq 0$

$M/B < 0$





Critical topology:

Usual graphene

$\lambda_{SO} = 0$

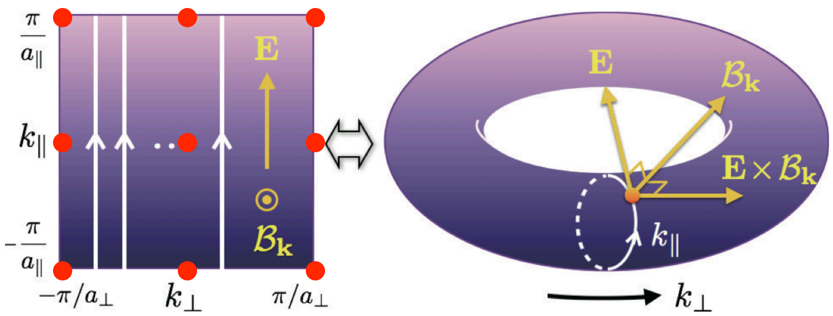
$M = B = 0$

Winding becomes discontinuous!

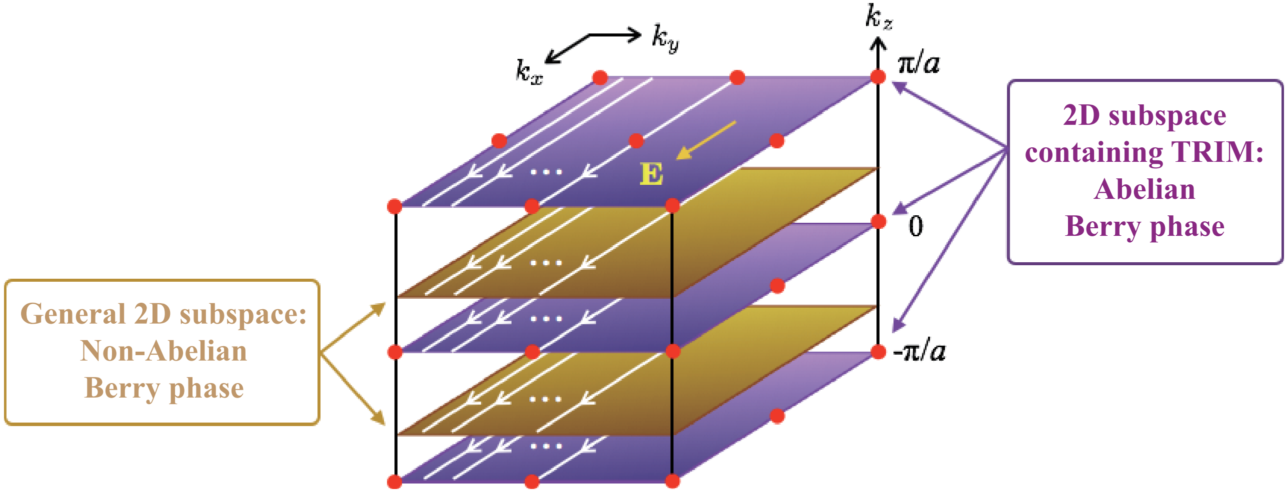
Winding number of the WSL: Strong 3D TI model

W.-R. Lee & KP, arXiv:1503.01870

- **2D TI**



- **3D TI**

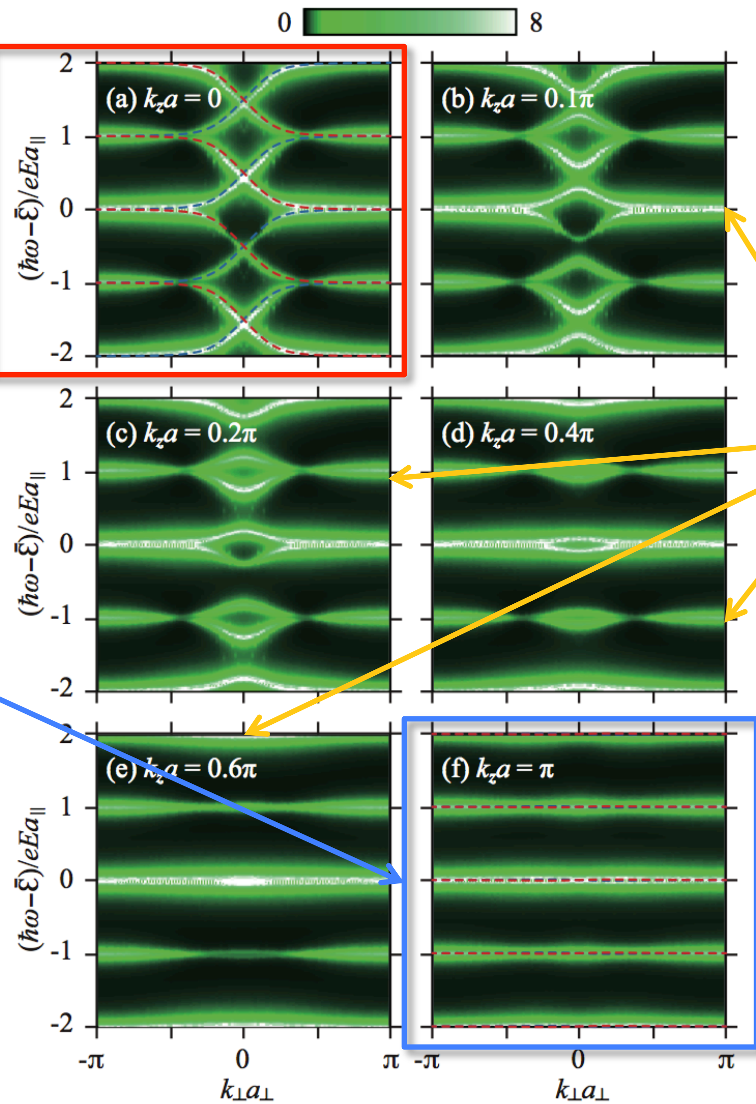


Winding number of the WSL: Strong 3D TI model

W.-R. Lee & KP, arXiv:1503.01870

Non-trivial topology

2D subspace containing TRIM: **Abelian Berry connection**



General 2D subspace: **Non-Abelian Berry connection**

Trivial topology

Topological insulators provide one of the most dramatic physical examples accentuating an intriguing role of the geometrical phase in quantum physics.