String Theory Compactifications

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Abstract

In this series of six-hour lectures, we give an introduction to supersymmetric compactifications of string theory. For the first two hours, the idea of Kaluza-Klein reduction is reviewed and some rudiments of complex and Kähler geometries, as well as the notion of holonomy, are introduced. For the second two hours, Calabi-Yau property is defined (and an example provided, depending on how fast we proceed). The moduli space is also explored for Calabi-Yau geometries. Finally, for the last two hours, we compactify type II supergravities on a Calabi-Yau three-fold and study the resulting effective theories. If time permitted, before closing, we shall either survey construction methods for a Calabi-Yau three-fold or discuss a couple of famous flux compactification scenarios of type II string theory.

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Prerequisites

Basic notions of

- differential geometry/topology on a real manifold (Riemannian metric, covariant derivative and differential forms)
- representations of a rotation (spinorial representation in particular)

Refs

- Geometry:

- M.Nakahara, "Geometry, Topology and Physics"
- P.Candelas, "Lectures on Complex Manifolds"

- Useful reviews:

- A. Font and S. Theisen, "Introduction to String Compactification," Lect.Notes Phys.**668** (2005) 101-181 (main source)
- M. Douglas and S. Kachru, "Flux Compactification," Rev.Mod.Phys. **79** (2007) 733-796
- M. Grana, "Flux compactifications in string theory: A Comprehensive review," Phys.Rept. **423** (2006) 91-158

- Moduli spaces:

- P. Candelas and X. de La Ossa, Moduli Space of Calabi-Yau Manifolds, Nucl.Phys. B355 (1991) 455-481

- Textbook:

- T.Hubsch, Calabi-Yau Manifolds - A Bestiary for Physicists, World Scientific

1 Introduction

1.1 Restrictions on the string theory background

- Polyakov action
$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN}(X) \ , \tag{1.1}$$
where $X : \Sigma \hookrightarrow \mathcal{M}$ with $M = 0, \cdots, D-1$ (c.f. $\alpha = 0, 1$); $\alpha' = l_s^2$.

- 2d NLSM with target space $\mathcal{M}$ (free if $G_{MN}(X) = \eta_{MN}$)
- S_P is Weyl invariant (local rescaling, $h_{\alpha\beta} \to e^{\omega(\sigma)}h_{\alpha\beta}$); also demanded after quantisation
- Anomalies produce, $\beta^{(G)}_{MN} = \alpha' R_{MN} + \mathcal{O}(\alpha'^2)$ and $\beta = 0$ gives Einstein equations.

- Compactification

- Only those backgrounds that solve the $\beta = 0$ are viable (pert.) string backgrounds
- Other beta functions for, B_{MN} and ϕ, vanish for $\phi = const.$ and $B = 0$ only if $D = 10$ (taking into account of susy partners of X and h).
- Thus, $\mathcal{M}_{10}$ is a Ricci-flat manifold with Lorentzian signature (NB. $\mathcal{O}(\alpha'^2)$ corrections ignored).
- A way to make the space-time look four-dimensional and Lorentz inv.:
$$\mathcal{M}_{10} = \mathcal{M}_4 \times K_6 \ , \tag{1.2}$$
where $\mathcal{M}_4$ is either AdS, Mink, or dS and K_6 is "small" (than length scales already probed).
- Eff. theory on $\mathcal{M}_4$ determined by the geometry of K_6 (physics det. by geometry).
- E.g. [CHSW, '85] with $K_6 = \mathrm{CY}_3$, the 4d theory has a minimal susy and $\chi(K_6)$ counts matters.

1.2 The aim of the lectures

- Introduction to string compactifications on a Calabi-Yau manifold (c.f. there are other compactifications, too!)

2 KK reduction

- KK unified gravity and EM in 4d by deriving the both from pure 5d gravity. Can this be generalised for string theory compactifications from 10d to 4d?
- In this lecture, KK reduction will be applied to the field theory limit of string theory, which neglects the tower of massive excitation modes and reproduces the scattering amplitudes.

2.1 Dimensional reduction

- D to d dimensions. A toy example: $D = 5$ to $d = 4$ for a real massless scalar ϕ,
$$S = -\frac{1}{2}\int \mathrm{d}^5x\partial_M\phi\partial^M\phi \ , \tag{2.3}$$
where the spacetime has the product form $\mathcal{M}_5 = \mathrm{Mink}_4 \times S^1_R$ with $\eta = (-, +, +, +, +)$.

- Coordinates: $x_M = (x_\mu, y)$, with $y \in [0, 2\pi R]$.
- E.o.m.: $\Box\phi = 0 \Rightarrow \partial_\mu\partial^\mu\phi + \partial_y^2\phi = 0$
- Fourier expansion: $\phi(x,y) = \frac{1}{\sqrt{2\pi R}}\sum \phi_n(x)e^{iny/R} \Rightarrow \partial_\mu\partial^\mu\phi_n - n^2/R^2\phi_n = 0$
- i.e. in 4d, there's a massless scalar ϕ_0 and an infinite tower of massive scalars.
- In the limit $R \to 0$, only the zero mode ϕ_0 remains light and the other very massive modes are discarded (*dimensional reduction*)
- Remark: the notion of zero modes generalises to curved internal compact spaces too, but zero modes do not have to be indep. of the internal coordinates in general (consistency of the discarding the heavy modes in the sense that the lower-dim eom also solves the full-dim one has to be questioned in general).

• Not just scalar: branching of D-dim Lorentz reps, $SO(1, d-1) \times SO(D-d) \subset SO(1, D-1)$.

- E.g. $A_M = (A_{\mu=0,\ldots,d-1}, A_{m=d,\cdots,D-1})$ (branching of $\mathbf{D} = (\mathbf{d}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{D}-\mathbf{d})$)
- E.g. $B_{MN} = (B_{\mu\nu}, B_{\mu m}, B_{mn})$

• Not just bosonic fields

- Our interest lies in compactification of susy theories with a certain number of conserved spinorial charges, $Q^{I=1,\cdots,\mathcal{N}}$, and hence, branching of spinors is relevent.
- Remark: Dirac spinors and Gamma matrices, $\Gamma^M = (\Gamma^\mu, \Gamma^m)$, are of size $2^{\left[\frac{D}{2}\right]}$ and the lower-dim gamma matrices act on all the components of higher-dim spinor.
- E.g. $D = 11$ spinor to $d = 4$: $\mathbf{32} = (\mathbf{4}, \mathbf{8})$, all three reps are Majorana spinors.

2.2 Supersymmetry and Calabi-Yau-ness

• Sugras in high dimensions

- $D = 11, \mathcal{N} = 1$; $D = 10, \mathcal{N} = (1,1)$ (non-chiral, IIA), $D = 10, \mathcal{N} = (2,0)$ (chiral, IIB), $D = 10, \mathcal{N} = 1$ may couple to SYM with $G = E_8 \times E_8$ or $SO(32)$.
- The sugras above are a low-energy limit of various string theories.

• 4d susy theories are obtained by dimensionally reducing the above (i.e. torus compactification).

- Susy in 4d is attractive but too many are not desirable (non-chiral); beyond torus comp.!
- Remark: Susy is not just phenomenologically desirable but supersymmetric string compactifications are stable (c.f. tachyons or tadpoles of non-susy vacua).

• Let us have the gravity back in. Dynamics (eom) should allow for

$$\mathcal{M}_D = \mathcal{M}_d \times K_{D-d} \ , \tag{2.4}$$

in which case the system is said to admit spontaneous compactification.

- With the vev ansatz, $\langle G_{MN}(x,y)\rangle = \overline{g}_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + \overline{g}_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n$ (which can be generalised to a warped geometry maintaining maximal space-time symmetry in $\mathcal{M}_d$), we demand susy in the effective theory (with the eoms to be checked a posteriori).

- i.e. There must be a spinor field, ϵ, for which $\langle \delta_\epsilon \Phi_b \rangle \sim \langle \Phi_f \rangle = 0$ and $\langle \delta_\epsilon \Phi_f \rangle \sim \langle \Phi_b \rangle = 0$; the former is zero since fermions are zero for maximal symmetry in $\mathcal{M}_d$ (they transform non-trivially under Lorentz, for instance) and we only need to demand the latter.

- In sugra, there's a gravitino, ψ_M that transforms as $\delta_\epsilon \psi_M = \nabla_M \epsilon + \cdots$, where $\cdots$ contains other bosonic fields (dilaton, H and F's) which we turn off.

- Thus, we demand that $\langle \nabla_M \epsilon \rangle \equiv \overline{\nabla}_M \epsilon = 0 \Rightarrow \overline{\nabla}_m \epsilon = 0$ and $\overline{\nabla}_\mu \epsilon = 0$.

• Consequence of the existence of Killing spinors

- (dropping the bars from now) $[\nabla_M, \nabla_N]\,\epsilon = \frac{1}{4} R_{MNPQ} \Gamma^{PQ} \epsilon = 0$, where $\Gamma_{AB} = \frac{1}{2}[\Gamma_A, \Gamma_B]$.

- Thus, $\overline{R}_{MQ} = 0$ ($\overline{R}_{MQ} \overline{\Gamma}^Q \epsilon = 0$ follows from $\Gamma^N \Gamma^{PQ} = \Gamma^{NPQ} + G^{NP}\Gamma^Q - G^{NQ}\Gamma^P$ and the Bianchi, $R_{MNPQ} + R_{MQNP} + R_{MPQN} = 0$).

- Thus, $\mathcal{M}_d$ is Mink and K_{D-d} admits a Ricci-flat metric.

• Detour: Holonomy

- On an m-dim mfld, parallel transport of a vector v along a closed curve gives Uv with $U \in O(m)$; $\mathcal{H} = \{U\}$ is the holonomy group $\subset O(m)$ ($SO(m)$, if orientable).

- Rmk: $\delta V^P = -\frac{12}{\delta} a^{MN} R_{MN}{}^P{}_Q V^Q$ implies that for a simply connected manifold to have a non-trivial holonomy it has to have curvature.

- We restrict to $D = 10$ and $d = 4$ case from now. Cov const spinor, ϵ, is a singlet under $\mathcal{H}$. But it is a $SO(6)$ spinor and has R-/L-chirality piece transforming as $\mathbf{4}$ ($\overline{\mathbf{4}}$) of $SO(6) \simeq SU(4)$.

- Suppose $\mathcal{H} = SU(3)$. Since $\mathbf{4}_{SU(4)} = \mathbf{3}_{SU(3)} + \mathbf{1}_{SU(3)}$, there is one cov const spinor of positive and one of negative chirality, $\epsilon_\pm$, so that the allowed susy parameter takes the form,

$$\epsilon = \epsilon_R \otimes \epsilon_+(y) + \epsilon_L \otimes \epsilon_-(y) \ , \tag{2.5}$$

  with $\epsilon_L = \epsilon_R^*$ and $\epsilon_-(y) = \epsilon_+^*(y)$ in a Majorana basis for ϵ.

• **Schematic def**. a $2n$-dim compact Riemannian manifold with a metric whose holonomy is $SU(n) \subset SO(2n)$ is called a Calabi-Yau manifold.

• A glimpse at Calabi-Yau manifolds (obvious by now)

- Admit a covariantly constant spinor

- Ricci flat.

• Remarks

- No $SU(n)$ holonomy metrics are known except for the $n = 1$ case.

- Manifolds with a special holonomy appear a lot in string theory. Berger's classification for a simply connected manifold (refs):

$$U(n), SU(n), Sp(n/2), Sp(n/2)\cdot Sp(1), G_2(7d), Spin(7)(8d) \tag{2.6}$$

2.3 Zero modes, harmonic forms and cohomologies

- Generalities of KK reduction on a curved internal space: $\Phi_{\mu\nu\cdots mn\cdots}(x,y)$

- Expand around a vev, $\Phi_{\mu\cdots m\cdots}(x,y) = \langle\Phi_{\mu\cdots m\cdots}(x,y)\rangle + \phi_{\mu\cdots m\cdots}(x,y)$

- D-dim eom under the split metric ansatz:

$$\mathcal{O}_d\phi + \mathcal{O}_{\text{int}}\phi = 0 \ , \tag{2.7}$$

  where $\mathcal{O}_d$ and $\mathcal{O}_{\text{int}}$ are diff operators of order $p = 1$ or 2.

- Expand $\phi_{\mu\cdots m\cdots}$ in terms of eigenfunctions $Y^a_{m\cdots}(y)$ of $\mathcal{O}_{\text{int}}$ in K_{D-d} (eig. val λ_a):

$$\phi_{\mu\cdots m\cdots}(x,y) = \sum \phi^a_{\mu\cdots}(x) Y^a_{m\cdots}(y) \ . \tag{2.8}$$

- Remark: $\lambda_a \sim 1/R^p$ det. mass of $\phi^a_{\mu\cdots}(x)$ and zero modes of $\mathcal{O}_{\text{int}}$ corresp. to massless fields.

- Caution in tunctation of heavy modes

- In general not consistent to simply set the massive fields to zero (they might induce interactions of ϕ_0 not suppressed by heavy mass, e.g. $\phi_0\phi_0\phi_h$).

- Even when the heavy fields can't be naively discarded, the eff action might be consistently det.

- Examples

- Scalar: $\mathcal{O}_{\text{int}} = \Delta$ has a unique scalar zero mode, leading to a single massless scalar in d dim.

- Dirac: $\mathcal{O}_{\text{int}} = \nabla\!\!\!/ (= \Gamma^m \cdot \nabla_m)$ has its zero mode count dep. on topology of K_{D-d} (index theorem).

- p-form: the action is

$$S_p = -\frac{1}{2(p+1)!}\int_{\mathcal{M}_D} F_{p+1} \wedge \star F_{p+1} \ , \tag{2.9}$$

  leading to the eom $\Delta_D A_p = 0$, for $\Delta_D = dd^* + d^*d$, upon fixing the gauge freedom by $\mathrm{d}^* A_p = 0$. Since $\mathcal{O}_{\text{int}} = \Delta_{\text{int}}$, the zero mode counting becomes a cohomology problem. p-form in D-dim gives "b_n" massless fields, $n = 0, \cdots, p$, that correspond to $(p-n)$-forms in d-dim.

- Detour: (Co)homologies (Betti numbers) in relation with zero modes and harmonic forms

- to come shortly after the following quick remarks.

- Remark: Compactification of string theory v.s. that of its field theoretic limit

- At length scales near/below $l_s = \sqrt{\alpha'}$, classical geomtry has to be modified to 'stringy geometry'.

- Different geometries may correspond to the same physics, e.g. T-duality, mirror symmetry, etc.

- In this lecture, we shall mainly consider sugras, as the field theory limit of string theory.

- Back to the detour: (Co)homologies and Betti numbers

1. Homology

- On a smooth, connected (real) manifold, M, the p-th homology group of M is,

$$H_p = Z_p/B_p \ , \tag{2.10}$$

  where $B_p \subset Z_p \subset C_p$ are $C_p = \{a_p = \sum_i c_i N_i \mid N_i \text{ are p-dim oriented submflds}\}$ (chains), $Z_p = \{a_p \mid \partial a_p = \varnothing\}$ (cycles), and $B_p = \{\partial a_{p+1}\}$ (boundaries).

- Eliments of H_p are equiv classes $[z_p]$ of p-cycles, $z_p \simeq z_p + \partial a_{p+1}$ (homology class).
- Poincare duality: $H_p \simeq H_{m-p}$, where $m = \text{dim}M$.
- p-th Betti number, $b_p = \text{dim}H_p(M, \mathbb{R})$.

2. Cohomology

- p-th de Rham cohomology group of M is,

$$H^p = Z^p/B^p \ , \tag{2.11}$$

where $B^p \subset Z^p \subset A^p$ are exact forms, closed forms, and differential p-forms, respectively.

- Eliments of H^p are equiv classes $[\omega_p]$ of closed p-forms, $\omega_p \simeq \omega_p + \mathrm{d}\alpha_{p-1}$ (cohomology class).

3. Relationship

- Inner product (period of ω_p over z_p),

$$\pi(z_p, \omega_p) = \int_{z_p} \omega_p \ , \tag{2.12}$$

is bilinear and non-degenerate, thus giving an isom (de Rham), $H_p \simeq H^p$.

- The notion of cohomology will naturally be refined for the manifolds with additional structures, which we will turn to tomorrow!

3 Kähler geometry

Mflds of $SU(3)$ hol, important for susy, happen to be a complex mfld in particular. Let us start by recalling some basics of complex manifolds.

3.1 Complex manifolds

- Def: a complex manifold M is a diff manifold that admits an open cover $\{U_a\}_{a\in A}$ and coordinate maps, $z_a : U_a \to \mathbb{C}^n$ s.t. $z_a \circ z_b^{-1}$ is hol. on $z_b(U_{ab}) \subset \mathbb{C}^n$ for all $a, b \in A$.
- i.e. z_a give local hol coords, $(z_a^1, \cdots, z_a^n)$, and on U_{ab}, $z_a^i = f_{ab}^i(z_b^j)$ are hol (no dep on $\bar{z}_b^j$).
- An atlas $\{(U_a, z_a)\}$ defines a complex structure on M and $n =: \text{dim}_{\mathbb{C}}M$.
- Not all real manifolds admit a complex str. (e.g. S^2 is complex, but S^{2n} for $n > 1$ are not).
- An obvious example: $\mathbb{C}^m$ with a single coord patch, with coords, $z^{j=1,\cdots,m}$. By writing $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, $j = 1, \cdots, m$, it can be thought of as a real mfld with coords, $x^{j=1,\cdots,m}$ and $x^{m+j} \equiv y^j$ for $j = 1, \cdots, m$. Note that

$$\partial_j \equiv \frac{\partial}{\partial z^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}) \ , \quad \bar{\partial}_j \equiv \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}) \ , \tag{3.13}$$

and that

$$\mathrm{d}z^j = \mathrm{d}x^j + i\mathrm{d}y^j \ , \quad \mathrm{d}\bar{z}^j = \mathrm{d}x^j - i\mathrm{d}y^j \ . \tag{3.14}$$

As a real mfld, complex mflds can be seen orientable always.

- Tangent bundle $T(M)$ complexified to $T_{\mathbb{C}}(M) = T(M)\otimes\mathbb{C}$ so that a tangent vector, v, can have complex coefficients. Given a complex structure, v decomposes as the holo and the antiholo parts, and the bundle splits as

$$T_{\mathbb{C}}(M) = T^{1,0}(M) \oplus T^{0,1}(M) \ , \tag{3.15}$$

where $T^{1,0}$ and $T^{0,1}$ are spanned, resp., by $\{\partial_i\}$ and $\{\bar{\partial}_i\}$.

- A (holo?) section of $T^{1,0}(M)$ is called a holo v.f. (its component fns are holo).

- Cotangent bundle

$$T_{\mathbb{C}}(M)^* = T^{1,0}(M)^* \oplus T^{0,1}(M)^* \ , \tag{3.16}$$

where $T^{1,0}(M)^*$ and $T^{0,1}(M)^*$ are spanned, resp., by $\{\mathrm{d}z^i\}$ and $\{\mathrm{d}\bar{z}^i\}$.

- (p,q)-forms as a section of $\wedge^p T^{1,0}(M)^* \wedge^q T^{0,1}(M)^*$. The space of (p,q)-forms is denoted by $A^{p,q}$ and that of r-forms by A^r (sections of $\wedge^r T_{\mathbb{C}}(M)^*$).

- obviously, $\overline{A^{p,q}} = A^{q,p}$; $A^r = \bigoplus\limits_{p+q=r} A^{p,q}$.

- $d = \partial + \bar{\partial}$, with $\partial : A^{p,q} \to A^{p+1,q}$ and $\bar{\partial} : A^{p,q} \to A^{p,q+1}$, satisfying

$$\partial^2 = 0; \quad \bar{\partial}^2 = 0; \quad \text{and } \partial\bar{\partial} + \bar{\partial}\partial = 0 \ . \tag{3.17}$$

- $\omega \in A^p$ is a hol p-form if it is of type $(p,0)$ and $\bar{\partial}\omega = 0$ and is an anti-hol p-form if it is of type $(0,p)$ and $\partial\omega = 0$.

- The space of hol. p-forms is denoted by $\Omega^p(M)$.

3.2 Kähler manifolds

- Hermitian metric

- a hermitian metric is a covariant tensor field of the form, $\sum g_{i\bar{j}}(z,\bar{z})\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j$ s.t. $\overline{g_{i\bar{j}}(z,\bar{z})} = g_{j\bar{i}}(z,\bar{z})$. and $g_{i\bar{j}}$ is positive definite (i.e. $\forall v^i \in \mathbb{C}^n, v^i g_{i\bar{j}}\bar{v}^{\bar{j}} \geq 0$ with equality only if $v = 0$).

- To a hermitian metric is associated a two-form field $\omega = i\sum g_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$ (the fund form); ω is a real (1,1)-form and $\omega^n \sim g(z,\bar{z})\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n}$ is a volume form.

- Inverse of $g_{i\bar{j}}$ is $g^{i\bar{j}}$ s.t. $g^{j\bar{i}}g_{j\bar{k}} = \delta^{\bar{i}}_{\bar{k}}$ and $g_{i\bar{j}}g^{k\bar{j}} = \delta^k_i$.

- Kähler metric

- a hermitian metric g whose associated fund form ω is closed is a Kähler metric; a complex mfld endowed with a Kähler metric is a Kähler manifold (and ω is the Kähler form).

- $d\omega = 0 \Rightarrow \partial\omega = \bar{\partial}\omega = 0 \Rightarrow$

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}, \quad \bar{\partial}_i g_{j\bar{k}} = \bar{\partial}_k g_{j\bar{i}} \ , \tag{3.18}$$

implying the only non-vanishing components of the Riemannian connection are $\Gamma^k_{ij} = g^{k\bar{l}}\partial_i g_{j\bar{l}}$ and $\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = g^{l\bar{k}}\bar{\partial}_{\bar{i}} g_{l\bar{j}} \Rightarrow$ holo and anti-holo tangent spaces do not mix under parallel transport.

- Remark: a complex manifold admits a hermitian metric, but may not admit a Kähler metric.

- Remark: a complex submanifold of a Kähler manifold is again Kähler (induced metric).

- Kähler potential
- (3.18) says that locally there exists a real Kähler potential K s.t.

$$g_{i\bar{j}} = \partial_i \bar{\partial}_j K \ (\omega = i\partial\bar{\partial}K) \ . \tag{3.19}$$

- K is not unique: e.g. $K(z,\bar{z})$ and $K(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$ define the same metric.
- Riemann curvature tensor, Ricci tensor, and Ricci-form:
- Exercise: $R_{i\bar{j}k\bar{l}} = -\partial_i\bar{\partial}_{\bar{j}} g_{k\bar{l}} + g^{m\bar{n}}(\partial_i g_{k\bar{n}})(\bar{\partial}_{\bar{j}} g_{m\bar{l}})$ (sign convenction: $[\nabla_i, \nabla_{\bar{j}}] V_k = -R_{i\bar{j}k}{}^l V_l$) are the only non-vanishing components.
- Exercise: $R_{i\bar{j}} = -\partial_i \bar{\partial}_{\bar{j}}(\log\,(\det g))$.
- Ricci-form (of type (1,1)) is

$$\mathcal{R} = iR_{j\bar{k}} dz^j \wedge d\bar{z}^k = -i\partial\bar{\partial}(\log\,(\det g)); \quad d\mathcal{R} = 0 \text{ (and } \mathcal{R} = \bar{\mathcal{R}}). \tag{3.20}$$

- $\mathcal{R}$ only depends on the volume form of the Kähler metric and on the complex str.
- Under the change $g \to g'$,

$$\mathcal{R}(g') = \mathcal{R}(g) - i\partial\bar{\partial}\log\left(\frac{\det(g'_{k\bar{l}})}{\det(g_{k\bar{l}})}\right) \ , \tag{3.21}$$

where the ratio in log is globally well-defined non-vanishing fn on M and hence, its cohomology class is metric independent.

- Example: $\mathbb{P}^n = \{[z^0 : \cdots : z^n] \mid z^a \in \mathbb{C}, \lambda \in \mathbb{C}^*\}/(z \sim \lambda z)$
- Exercise: Complex - $U_a = \{z^a = 1\} \simeq \mathbb{C}^n$ gives an atlas and a complex str (transition is holo).
- Exercise: Kähler - on U_a, the locally defined Kähler potential is $K = \log(1+\sum_{b\neq a}|z^b|^2)$, leading to the Fubini-Study metric.

3.3 Holonomy group of Kähler manifolds

- Holonomy of a Kähler manifold
- (3.18) implied that parallel transport does not mix elts of $T^{1,0}(M)$ and $T^{0,1}(M)$.
- Since the length of a vector does not change under parallel transport, the holonomy is $U(n)$, where $n = \dim_{\mathbb{C}}(M)$.
- elts of $T^{1,0}(M)$ ($T^{0,1}(M)$) transform as $\mathbf{n}$ ($\bar{\mathbf{n}}$).
- Holonomy of a Ricci-flat Kähler manifold
- Parallel transport:

$$\delta V^P = -\frac{1}{2}\delta a^{MN} R_{MN}{}^P{}_Q V^Q \Rightarrow \delta V^i = -\delta a^{k\bar{l}} R_{k\bar{l}}{}^i{}_j V^j \ , \tag{3.22}$$

and hence, $-\delta a^{k\bar{l}} R_{k\bar{l}}{}^i{}_j$ is an elt of $u(n)$.

- Its trace, proportional to Ricci tensor, generates the $u(1)$ part in $u(n) \simeq su(n) \oplus u(1)$. Thus, the holonomy group of a Ricci-flat Kähler manifold is a subgroup of $SU(n)$.
- Converse is also true:
- $2n$-dim manifold with $U(n)$ ($SU(n)$) holonomy admits a (Ricci-flat) Kähler metric.

3.4 Cohomology of Complex and Kähler manifolds

- Cohomology theory on real manifolds also applies to complex and Kähler manifolds. But one can use the complex structure to define Dolbeault cohomology ($\bar{\partial}$-cohomology):

$$H^{p,q}_{\bar{\partial}} = \frac{Z^{p,q}_{\bar{\partial}}}{\bar{\partial}A^{p,q-1}} \ , \tag{3.23}$$

where $\bar{\partial}A^{p,q-1} \subset Z^{p,q}_{\bar{\partial}} \subset A^{p,q}$ are $\bar{\partial}$-exact, $\bar{\partial}$ closed (p,q) forms.

- Remark: $\bar{\partial}$-Poincare lemma (Dolbeault): Dolbeault cohomology groups are locally trivial.

- Hodge numbers, $h^{p,q}(M) = \dim_{\mathbb{C}}(H^{p,q}_{\bar{\partial}}(M))$

- for a Kähler manifold, often arranged in the Hodge diamond (draw, say, for 3-fold!).

- Inner product between two (p,q)-forms:

$$\begin{aligned} \varphi &= \frac{1}{p!q!}\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(z)dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q} \ , & (3.24)\\ \psi &= \frac{1}{p!q!}\psi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(z)dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q} \ , & (3.25)\end{aligned}$$

define

$$(\varphi,\psi)(z) = \frac{1}{p!q!}\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(z)\bar{\psi}^{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(z) \ , \tag{3.26}$$

where $\bar{\psi}^{i_1\cdots\bar{j}_1\cdots}(z) = g^{i_1\bar{k}_1}\cdots g^{i_p\bar{k}_p}g^{l_1\bar{j}_1}\cdots g^{l_q\bar{j}_q}\overline{\psi_{k_1\cdots k_p\bar{l}_1\cdots\bar{l}_q}(z)}$. Then the i.p. is:

$$(\varphi,\psi) = \int_M (\varphi,\psi)(z)\frac{\omega^n}{n!} \ . \tag{3.27}$$

- Hodge-$*$: $A^{p,q}\to A^{n-q,n-p}$ is defined so that

$$(\varphi,\psi)(z)\frac{\omega^n}{n!} = \varphi(z)\wedge *\bar{\psi}(z) \ , \tag{3.28}$$

where $\bar{\psi} = \frac{1}{p!q!}\overline{\psi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}dz^{i_1}\wedge\cdots\wedge d\bar{z}^{j_q}} = \frac{1}{p!q!}\bar{\psi}_{j_1\cdots j_q\bar{i}_1\cdots\bar{i}_p}dz^{j_1}\wedge\cdots\wedge dz^{j_q}\wedge d\bar{z}^{i_1}\wedge\cdots\wedge d\bar{z}^{i_p}$

- Exercise: $*\bar{\psi} = \overline{*\psi}$

- Exercise: $**\psi = (-1)^{p+q}\psi$, where ψ is of type (p,q).

- Exercise: On a 3-fold, $\Omega\in A^{3,0}$, $\alpha\in A^{2,1}$,

$$*\Omega = -i\Omega \ , \qquad *\alpha = i\alpha \ . \tag{3.29}$$

- $\bar{\partial}^*$ defined via the i.p. $(\bar{\partial}^*,\varphi) = (\psi,\bar{\partial}\varphi)$, $\psi\in A^{p,q}$ and $\varphi\in A^{p,q-1}$.

- Exercise: $\bar{\partial}^* = -*\partial*$

- $\bar{\partial}$-Laplacian: $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$: $Ap,q \to A^{p,q}$

- $\mathcal{H}^{p,q} := \{\psi\in A^{p,q}\mid \Delta_{\bar{\partial}}\psi = 0\}$

- (Exercise:) On a compact complex manifold, ψ is harmonic if and only if $\bar{\partial}\psi = \bar{\partial}^*\psi = 0$.

- Hodge Theorem: $A^{p,q}$ has a unique orthogonal decomposition

$$A^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\partial} A^{p,q-1} \oplus \bar{\partial}^* A^{p,q+1} \Rightarrow Z^{p,q}_{\bar{\partial}} = \mathcal{H}^{p,q} \oplus \bar{\partial} A^{p,q-1} \Rightarrow H^{p,q}_{\bar{\partial}} \simeq \mathcal{H}^{p,q} \ , \tag{3.30}$$

i.e., every $\bar{\partial}$-cohomology class has a unique harmonic rep and vice versa.

- Similarly, ∂-Laplacian: $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and the Laplacian $\Delta = dd^* + d^*d$. Then, very importantly, (Exercise)

$$\Delta_{\bar{\partial}} = \Delta_\partial = \frac{1}{2}\Delta_d \ , \tag{3.31}$$

from which lots of interesting consequences follow:

- The three harmonicities agree.
- Δ_d does not change the index type.
- On a Kähler manifold, every holo p-form is harmonic (if $\alpha \in \Omega^p \subset A^{p,0}$, $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$), and every harmonic $(p,0)$ form is holo ($\Delta\alpha = 0$ implies $\bar{\partial}\alpha = 0$, which, for $\alpha \in \mathcal{H}^{p,0}$ means $\alpha \in \Omega^p$).
- $\sum\limits_{p+q=r} h^{p,q} = b_r$ ($U(n)$ invariant decomposition, $\mu = (i, \bar{\imath})$).

- Constraints on the Hodge numbers (draw Hodge diagram!)

- $h^{p,q} = h^{q,p}$ (complex conjugate)
- $h^{p,q} = h^{n-q,n-p}$ ($[\Delta_d, *] = 0$) $= h^{n-p,n-q}$.
- $h^{p,p} > 0$ (ω^p is closed but not exact).
- $h^{0,0} = 1$ for a connected manifold.
- Summary: all these only leave five independent Hodge numbers $(h^{1,0}, h^{2,0}, h^{1,1}, h^{2,1}, h^{3,0})$

- 1st Chern class

- It has been shown Ricci-form leads to a metric-indep cohomology class and we define

$$c_1(M) = -\left[\frac{1}{2\pi}\mathcal{R}\right] \in H^{1,1}(M, \mathbb{C}) \cap H^{2i}(M, \mathbb{R}) \ , \tag{3.32}$$

4 Calabi-Yau geometry

4.1 Calabi-Yau manifolds

- Def. A CY manifold is a compact Kähler manifold with vanishing first Chern class.

- Ricci-flat Kähler manifold has vanishing first Chern class, but the converse is not trivial.
- Calabi's conjecture: Every rep of $c_1(M)$ is the Ricci-form of a Kähler metric (proved that if there is one, it must be unique).
- Yau's proof: such a metric always exists (his statement: let M be a compact Kähler manifold, ω its Kähler form, $c_1(M)$ its first Chern class. Any closed real two-form of type $(1,1)$ belonging to $2\pi c_1(M)$ is the Ricci form of one and only one Kähler metric in the class of ω).

- For vanishing c_1, a Kähler manifold with $c_1 = 0$ admits a unique Ricci-flat Kähler form in each Kähler class.

- $c_1 = 0$ is necessary to have a Ricci-flat metric. The other way is the hard bit (Yau's proof is for existence and does not give the metric (no CY metric has been constructed)).

- Sum: Compact Kähler manifolds with a vanishing c_1 are precisely those admitting a Kähler metric with zero Ricci tensor (equivalently, with restricted holonomy group contained in $SU(n)$).

- We shall assume that CY manifolds have precisely $SU(n)$ as their holonomy group.

- Reduced holonomy group leads to more susy in 4d and hence is bad.

- Alternative definition: Calabi-Yau manifolds are compact Kähler manifolds with trivial canonical bundle.

- Canonical l.b.: $K = \wedge^n T^{1,0}(M)^*$, whose sections are forms of type $(n, 0)$.

- Exercise: $\left[\nabla_i, \nabla_{\bar{j}}\right] \omega_{i_1 \cdots i_n} = -R_{i\bar{j}} \omega_{i_1 \cdots i_n}$, which means that $c_1(K) = -c_1(M)$ and hence,

$$c_1(M) = 0 \Leftrightarrow c_1(K) = 0 \ . \tag{4.33}$$

- Thus, $K \simeq \mathcal{O}_M$ and there must exist a (unique) globally defined nowhere vanishing holo n-form Ω on M

- Ω is covariantly constant (!!!), meaning in particular that the holonomy is in $SU(n)$.

- Ω can locally be written as $\Omega_{i_1 \cdots i_n} = f(z) \epsilon_{i_1 \cdots i_n}$: It is holo since $\bar{\partial}_{\bar{i}} \Omega_{j_1 \cdots j_n} = \nabla_{\bar{i}} \Omega_{j_1 \cdots j_n} = 0$.

- Ω can explicitly be constructed using the covariantly constant spinor: $\Omega_{ijk} = \epsilon^T \gamma_{ijk} \epsilon$.

- Exercise: show that such an Ω satisfies all the properties above...

- Ω is unique (if Ω' has the same property, as it's a top form, must have $\Omega' = f\Omega$. Since $\bar{\partial}\Omega' = 0$, f must be holo and hence, is constant.

- Existence of Ω implies $c_1 = 0$; as $\mathcal{R} = i\partial\bar{\partial} \log(\det(g_{k\bar{l}})) = i\partial\bar{\partial} \log(\Omega_{i_1 \cdots i_n} \bar{\Omega}_{\bar{j}_1 \cdots \bar{j}_n} g^{i_1 \bar{j}_1} \cdots g^{i_n \bar{j}_n})$, implying $c_1 = 0$.

4.2 Cohomology of Calabi-Yau manifolds

Hodge diagram (draw!)

4.3 Moduli space of Calabi-Yau geometries (three-folds)

- Given a CY background, the parameters deforming it naturally correspond to massless fields in 4d, as they corresponds to parameters that take one vacuum state into a nearby equivalent one.

- Local structure of CY geometries:

- Q: Space of Ricci-flat metrics for M? $g_{\mu\nu}$ and $g_{\mu\nu} + \delta g_{\mu\nu}$ both satisfy $R_{\mu\nu} = 0$. What are the possible fluctuations?

- A: (Some metric deformations only describe coordinate changes and can be eliminated by fixing the gauge by $\nabla^\nu \delta g_{\mu\nu} = 0$) Then possible fluctuations should satisfy (Lichnerowics) (Exercise)

$$\nabla^\rho \nabla_\rho \delta g_{\mu\nu} + 2R_\mu{}^\rho{}_\nu{}^\sigma \delta g_{\rho\sigma} = 0 \ . \tag{4.34}$$

- (Kählerity) Its solutions split to mixed type, $\delta g_{i\bar{j}}$, and pure type, δg_{ij} and $\delta g_{\bar{i}\bar{j}}$, and hence, we can analyse them separately.

1. For a mixed type: we can associate the real (1,1) form $\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ and is harmonic iff the variation $\delta g_{i\bar{j}}$ satisfies Lichnerowicz

* iterpretation: $\Rightarrow h^{1,1}$ real parameters. Correspond to a cohom non-trivial Kähler form change, $\delta\omega$. Yau says that for each class $[\omega + \delta\omega]$ there is a unique Ricci flat Kähler metric. Thus, it should count Ricci-flat metrics.

* For NS two-form B_2, there are internal $(1,1)$-form zero modes $B_{i\bar{j}}$ which are additional (massless) scalars in 4d. These combine with the Kähler form ω to have it complexified.

2. Similarly, for a pure type, one may associate the complex $(2,1)$-form,

$$\Omega_{jk}{}^{\bar{l}} \delta g_{\bar{l}\bar{m}} dz^j \wedge dz^k \wedge d\bar{z}^{\bar{l}} \ , \tag{4.35}$$

which is harmonic iff Lichnerwowics holds.

* Interpretation: $\Rightarrow h^{2,1}$ complex parameters t^α. They generate pure index components for the metric. Thus, should correspond to complex structure deformations.

* Derivatives of $\Omega = \frac{1}{3!}\Omega_{ijk} dz^i \wedge dz^j \wedge dz^k$ are

$$\partial_\alpha \Omega = \frac{1}{3!}\frac{\partial \Omega_{ijk}}{\partial t^\alpha} dz^i \wedge dz^j \wedge dz^k + \frac{1}{2}\Omega_{ijk} dz^i \wedge dz^j \wedge \frac{\partial (dz^k)}{\partial t^\alpha} \ , \tag{4.36}$$

defines an element in $H^{3,0} \oplus H^{2,1}$ (d commutes with ∂_α). Claim: the second term is (4.35), i.e.,

$$\chi_\alpha = \frac{1}{2}(\chi_\alpha)_{ij\bar{k}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{k}} \ , \qquad \text{with } (\chi_\alpha)_{ij\bar{k}} = -\frac{1}{2}\Omega_{ij}{}^{\bar{l}} \frac{\partial g_{\bar{k}\bar{l}}}{\partial t^\alpha} \ . \tag{4.37}$$

Thus, Ω encodes the complex structure dependence.

- The moduli space metric

- Most general metric that can be written in terms of the background quantities:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{V}\int_M g^{\mu\nu} g^{\rho\sigma} (\delta g_{\mu\rho}\delta g_{\nu\sigma} + \delta B_{\mu\rho}\delta B_{\nu\sigma})\sqrt{g}\mathrm{d}^6 x & (4.38) \\ &= \frac{1}{2V}\int_M g^{i\bar{j}} g^{k\bar{l}} \left[\delta g_{ik}\delta g_{\bar{j}\bar{l}} + (\delta g_{i\bar{l}}\delta g_{k\bar{j}} + \delta B_{i\bar{l}}\delta B_{k\bar{j}})\right]\sqrt{g}\mathrm{d}^6 x \ . & (4.39) \end{aligned}$$

- Simple yet important observation: block-diagonal, separating the variation of complex structure and that of Kähler class!

- The (2,1) forms

- Have the inverse relation to (4.35):

$$\delta g_{\bar{i}\bar{j}} = -\frac{1}{||\Omega||^2}\bar{\Omega}_{\bar{i}}{}^{kl}(\chi_\alpha)_{kl\bar{j}}\delta t^\alpha \ \text{, with } ||\Omega||^2 = \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk} \text{ constant???} \ . \tag{4.40}$$

- the metric, $2G_{\alpha\bar{\beta}} \mathrm{d}t^\alpha \otimes \mathrm{d}\bar{t}^{\bar{\beta}}$:= the complex str part leads to

$$2G_{\alpha\bar{\beta}} = -\frac{\int_M \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{\int_M \Omega \wedge \bar{\Omega}} \tag{4.41}$$

- Since $\frac{\partial\Omega}{\partial t^\alpha} = k_\alpha\Omega + \chi_\alpha$, we have

$$G_{\alpha\bar{\beta}} = -\frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial \bar{t}^{\bar{\beta}}}\log i\int_M \Omega\wedge\bar{\Omega} \ , \tag{4.42}$$

and hence,

$$K^{2,1} = -\log\left(i\int_M \Omega\wedge\bar{\Omega}\right) \ . \tag{4.43}$$

- The (1,1) forms

- The i.p. on $H^{1,1}$ corresponding to the (4.39) is

$$G(\rho,\sigma) = \frac{1}{2V}\int_M \rho_{i\bar{j}}\sigma_{k\bar{l}}g^{i\bar{l}}g^{k\bar{j}}\sqrt{g}d^6x = \frac{1}{2V}\int_M \rho\wedge *\sigma \ , \tag{4.44}$$

for real (1,1) forms.

- With a real basis of harmonic (1,1) forms, e_a, the metric on moduli space can be written as

$$G_{a\bar{b}} = \frac{1}{2}G(e_a,e_b) = \frac{\partial}{\partial w^a}\frac{\partial}{\partial \bar{w}^{\bar{b}}}K^{1,1} \ , \tag{4.45}$$

where $e^{-K^{1,1}} = \frac{4}{3}\int_M \omega^3$.

5 Type II compactifications on a Calabi-Yau manifold

- Having explored the geometric meaning of $h^{1,1}$ and $h^{1,1}$, could study their relevance to dofs in eff. theory in type II supergravities. Here we summarise the result of those analysis.
- CY3 compactification - 10d origin of the 4d massless fields

1. IIA sugra (non-chiral)

- 10d Field contents, $\{G_{MN}, B_{MN}, \phi, C_M, C_{MNP}(,\psi_M^{(\pm)},\chi^\pm)\}$
- 4d $\mathcal{N}=2$. gravity, hyper, vector (irreps with spin ≤ 2).
- (1) $G_{\mu\nu=0,1,2,3}, \psi_\mu^+, \psi_\mu^-, C_\mu \Rightarrow$ gavtity mult.
- Thanks to supersymmetry, may just work out bosonic states:
- (2) $C_{\mu i\bar{j}} \Rightarrow A_\mu^a$ $(h^{1,1})$
- (3) $G_{i\bar{j}}$ and $B_{i\bar{j}} \Rightarrow w^a$ $(h^{1,1})$.
- (4) $G_{ij} \Rightarrow t^\alpha$ $(h^{2,1})$.
- (5) $C_{ij\bar{k}} \Rightarrow C^\alpha$ $(h^{2,1})$
- (6) ϕ and $B_{\mu\nu} \Rightarrow S$ (1)
- (7) $C_{ijk} \Rightarrow C$ (1)
- grouping: !!! (in the end, 1 gravity mult, $h^{1,1}$ vec mult. and $h^{2,1}+1$ hypermult(incl. universal one from S and C) (or, $2h^{1,1}+4h^{2,1}+4$ scalar fields, $h^{1,1}+1$ vector fields)

2. IIB sugra

- 10d Field contents, $\{G_{MN}, B_{MN}, \phi, C, C_{MN}, \tilde{C}_{MNPQ}(, \psi_M^{(+)}, \tilde{\psi}_M^{(+)}, \chi^{(-)}, \tilde{\chi}^{(-)})\}$

- grouping: !!! (in the end, 1 gravity, $h^{2,1}$ vec mult. and $h^{1,1}+1$ hypermult(incl. universal one from S and C) (or, $2h^{2,1}+4h^{1,1}+4$ scalar fields, $h^{2,1}+1$ vector fields)

- Note that IIA and IIB result in the same spectrum upon exchanging the Hodge numbers, indicating "mirror" symmetry.

5.1 Flux: omitted!

6 Constructing a Calabi-Yau manifold: algebro-geometrical techniques

- Quintic

- Why CY? (see, G.Tian, "Canonical Metrics in Kähler Geometry," Birkhäuser, 200.: given the Fubini-study metric induced to CY, computing the volume form, and read off the first Chern explicitly, resulting in,

$$c_1(M) \sim (n+1-d)\,[\omega] \ . \tag{6.46}$$

- $h^{1,1}=1$ (ambient induced, $h^{1,1}=1$).

- $h^{2,1}=101$ (appearing as the coefficient counting: 126-24-1)(or, via computing $\chi=-200$ first).

- Chern class, Intersection numbers, Kähler cones, etc.

6.1 CICY

- E.g.

$$\left[\begin{array}{c\|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^4 & 1 & 4 \end{array}\right] , \tag{6.47}$$

- What does it mean?

- Why CY?

- Why smooth? Bertini's theorem (c.f. base point freeness...)

- $h^{1,1}=2$ (induced from the ambient? need to be careful - Lefschetz hyperplane)

- $h^{2,1}=86$(or, $\chi=-168$)

- This way, CICYs have been "classified", leading to about 8000 compact CYs

6.2 gCICY

- E.g.

$$\left[\begin{array}{c\|cc} \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^4 & 2 & 3 \end{array}\right] , \tag{6.48}$$

- What does it mean?

- Why CY?

- Why smooth? No reason here!
- $h^{1,1} = 2$ (induced from the ambient? need to be careful, no Lefschetz, need to do bundle-valued cohomology, $h^1(TX^*) = h^2(TX)$)
- $h^{2,1} = 46$(again, no counting, need to do bundle-valued cohomology, $h^1(TX)$, or, $\chi = -88$)