Real Space Renormalization Group

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Pyeongchang Summer Institute, July, 2015

Coarse Graining and Field Theory

- How to do RG systematically
- Ising universality class: $H = -J\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$, $(s_i = \pm 1)$
- Landau-Ginzburg-Wilson Hamiltonian (Free energy)

$$\mathcal{H} = \int d^d\mathbf{x}\left[\frac{c}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{r}{2}\phi^2(\mathbf{x}) + u\phi^4(\mathbf{x}) - h\phi(\mathbf{x})\right]$$

$\phi(\mathbf{x})$: local order parpameter / coarse grained field

- Partition function:

$$Z = \int \mathcal{D}\phi(\mathbf{x}) e^{-\mathcal{H}}$$

1. Continuous spin model

$$\mathcal{H} = -\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} J(\mathbf{r}-\mathbf{r}')s(\mathbf{r})s(\mathbf{r}') - h\sum_{\mathbf{r}} s(\mathbf{r}) + \lambda\sum_{\mathbf{r}}(s^2(\mathbf{r}) - 1)^2$$

2. Exact transformation: Hubbard-Stratonovich transformation
3. Landau expansion $\rightarrow$ Coarse grained field theory

Exact Transformation: Ising-like system

Partition function:

$$Z = \sum_{\{s_i=\pm 1\}} \exp\left[\frac{1}{2}\sum_{i,j} K_{ij} s_i s_j + \sum_i \beta h_i s_i\right],$$

where $K_{ij} = \beta J$ if (i,j) is n.n. and $K_{ij} = 0$ otherwise, and h_i is local field. One can write

$$e^{\frac{1}{2}\sum_{i,j} K_{ij} s_i s_j} = (const.) \int \prod_i d\psi_i \; e^{-\frac{1}{2}\sum_{i,j} \psi_i (K^{-1})_{ij} \psi_j + \sum_i s_i \psi_i},$$

where $\sum_j K^{-1}_{ij} K_{jk} = \delta_{i,k}$ and $(const.) = (2\pi)^{-N/2} (\det K)^{1/2}$. Spin terms are now local and the trace can now be done as

$$\sum_{\{s_i=\pm 1\}} e^{\sum_i s_i (\psi_i + \beta h_i)} = \prod_i 2\cosh(\psi_i + \beta h_i) \equiv e^{-\sum_i A(\psi_i + \beta h_i)},$$

where $A(x) = -\ln(2\cosh x) \simeq -\ln 2 - x^2/2 + x^4/12 + O(x^6)$.

Partition function now becomes a functional integral over a field ψ_i

$$Z = (const.)\int \prod_i d\psi_i \; e^{-\frac{1}{2}\sum_{i,j}\psi_i (K^{-1})_{ij}\psi_j - \sum_i A(\psi_i+\beta h_i)} \tag{1}$$

$$= (const.)\int \prod_i d\psi_i \; e^{-\frac{1}{2}\sum_{i,j}(\psi_i-\beta h_i)(K^{-1})_{ij}(\psi_j-\beta h_j) - \sum_i A(\psi_i)} \tag{2}$$

Local Magnetization

$$\langle s_i\rangle = \frac{\partial \ln Z}{\partial(\beta h_i)} = \left\langle \sum_j (K^{-1})_{ij}(\psi_j - \beta h_j)\right\rangle_{(2)} = \left\langle \sum_j (K^{-1})_{ij}\psi_j \right\rangle_{(1)}$$

Define $\phi_i \equiv \sum_j (K^{-1})_{ij}\psi_j$, then $\langle s_i\rangle = \langle \phi_i\rangle$, where the last average is w.r.t.

$$\int \prod_i d\phi_i \exp[-S[\phi]],$$

where

$$\boxed{S[\phi] = \frac{1}{2}\sum_{i,j}\phi_i K_{ij}\phi_j + \sum_i A\left(\sum_j K_{ij}\phi_j + \beta h_i\right)}$$

- Evaluate the functional integral using saddle point approximation
  Saddle point equation

$$\phi_i = -A'\left(\sum_j K_{ij}\phi_j + \beta h_i\right) = \tanh\left(\sum_j K_{ij}\phi_j + \beta h_i\right)$$

  $\Rightarrow$ Mean field theory
- Could do a systematic loop expansion;
  Expansion parameter $\sim 1/$ (number of identical indep. spins on a site)

- Expansion and Truncation: Note that

$$\frac{1}{2}\sum_{i,j}\phi_i K_{ij}\phi_j = \frac{1}{2}\sum_{\mathbf{q}}\phi_{-\mathbf{q}}K_{\mathbf{q}}\phi_{\mathbf{q}},$$

$$-\frac{1}{2}\sum_{i}\left(\sum_{j}K_{ij}\phi_j\right)^2 = -\frac{1}{2}\sum_{\mathbf{q}}K_{-\mathbf{q}}\phi_{-\mathbf{q}}K_{\mathbf{q}}\phi_{\mathbf{q}}$$

  and

$$K_{\mathbf{q}} = z\beta J - cq^2 + O(q^4), \quad zJ = T_c, \quad K_{-\mathbf{q}} = K_{\mathbf{q}},$$

Adding two terms,

$$\frac{1}{2}\sum_{\mathbf{q}}K_{\mathbf{q}}(1-K_{\mathbf{q}})|\phi_{\mathbf{q}}|^2 \simeq \frac{1}{2}\sum_{\mathbf{q}}(r+cq^2)|\phi_{\mathbf{q}}|^2 + O(q^4),$$

where $r \sim (T-T_c)/T_c$ and we assumed $|T-T_c| \ll T_c$

Taking the continuum limit and neglecting the higher order, higher derivative terms, we obtain the Ginzburg-Landau-Wilson Hamiltonian

$$S[\phi] = \int d^d\mathbf{x} \left[\frac{c}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{r}{2}\phi^2(\mathbf{x}) + u\phi^4(\mathbf{x}) - h\phi(\mathbf{x}) \right]$$

Neglected terms are irrelevant near the fixed points we are interested in.

Landau Theory

- Define an order parameter ϕ: zero in disordered phase and nonzero in ordered phase
- Construct the Landau free energy $F_L[\phi, T]$
  - F_L has all the symmetries of the system (e.g. translation, rotational, internal etc)
  - F_L is assumed to be analytic (polynomial) function of ϕ

$$f_L = \frac{F_L}{V} = \sum_n a_n[T]\phi^n$$

  - For an inhomogeneous system, order parameter $\to \phi(\mathbf{r})$; F_L includes the derivatives of $\phi(\mathbf{r})$
  - Temperature dependence of coefficients: Typically a_2 changes sign at T_c:

$$a_2[T] = a_2^0(T - T_c) + \ldots$$

- Minimize F_L with respect to ϕ to determine the nature of phase transition

Meaning of Landau Free Energy

Coarse graining: order parameter

$$\phi(\mathbf{r}) = \frac{1}{N_\Lambda} \sum_{i \in \Lambda} S_i \equiv \langle S_i \rangle_\Lambda$$

Field theory:

$$\begin{aligned}
\sum_{\{S\}} e^{-\beta H[S]} &= \sum_{\{S\}} \int \mathcal{D}\phi \prod_{\Lambda} \delta(\phi(\mathbf{r}) - \langle S_i \rangle_\Lambda) e^{-\beta H[S]} \\
&= \int \mathcal{D}\phi \sum_{\{S\}} \prod_{\Lambda} \delta(\phi(\mathbf{r}) - \langle S_i \rangle_\Lambda) e^{-\beta H[S]} \\
&\simeq \int \mathcal{D}\phi \; g(\phi) e^{-\beta H[\phi]} = \int \mathcal{D}\phi \; e^{-\beta F_L[\phi]}
\end{aligned}$$

Nature of Landau Free Energy

- By evaluating the functional integral using the saddle point approximation, we obtain the Landau free energy.
- Landau theory is a Mean Field Theory which neglects the fluctuation of the order parameter.
- To take into account of fluctuation effects, we have to do the functional integral $\rightarrow$ Renormalization Group approach
- Landau free energy can be derived from a microscopic theory in some cases (superconductors, magnets etc.). But in most cases, this is difficult to do.
- Landau theory is essentially phenomenological in nature (to be motivated by physical reasoning, i.e. symmetry of the system and the structure of order parameter etc.)

Momentum shell RG procedure

$$\phi(\mathbf{x}) = \int^{\Lambda} \frac{d^d\mathbf{q}}{(2\pi)^d}\; e^{i\mathbf{q}\cdot\mathbf{x}} \phi_{\mathbf{q}}$$

Cutoff=$\Lambda \sim \pi/a$ (1st Brillouin zone; a =lattice spacing)

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3. Rescale fields via $\phi_{\mathbf{q}} = \phi_{\mathbf{q}'/b} = \zeta\phi'_{\mathbf{q}'}$: c.f. block spins $s'(\mathbf{x}') = (1/|m_b|)s_{ave}(\mathbf{x})$, $\mathbf{x}' = \mathbf{x}/b$ with $s_{ave}(\mathbf{x}) = \sum_{\mathbf{y}} s(\mathbf{y})$: The extra factor is needed to make $\langle s'\rangle = \pm 1$

- Decomposition of Fourier components: $\phi_{\mathbf{q}} = \phi_{\mathbf{q}}^{<} + \phi_{\mathbf{q}}^{>}$

$$\phi_{\mathbf{q}}^{<} \equiv \begin{cases} \phi_{\mathbf{q}}, & 0 < q < \frac{\Lambda}{b} \\ 0, & \frac{\Lambda}{b} < q < \Lambda \end{cases}, \qquad \phi_{\mathbf{q}}^{>} \equiv \begin{cases} 0, & 0 < q < \frac{\Lambda}{b} \\ \phi_{\mathbf{q}}, & \frac{\Lambda}{b} < q < \Lambda \end{cases}$$

- $\phi^{<}(\mathbf{x})$ and $\phi^{>}(\mathbf{x})$ are Fourier tranforms of $\phi_{\mathbf{q}}^{<}$ and $\phi_{\mathbf{q}}^{>}$, respectively. Then $\phi(\mathbf{x}) = \phi^{>}(\mathbf{x}) + \phi^{<}(\mathbf{x})$.
- Meaning of ζ: Consider the field term

$$h \int d^d\mathbf{x}\, \phi(\mathbf{x}) = h\phi_{\mathbf{q}=0} \to h\zeta\phi'_{\mathbf{q}'=0} = h'\phi'_{\mathbf{q}'=0}$$

From $h' = b^{y_h} h$ and $y_h = (d + 2 - \eta)/2$,

$$\boxed{\zeta = y_h = \frac{d + 2 - \eta}{2}}$$

Gaussian fixed point: $u = 0$

$$\mathcal{H}_\Lambda = \frac{1}{2}\int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\,(cq^2+r)|\phi_\mathbf{q}|^2$$
$$= \frac{1}{2}\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\,(cq^2+r)|\phi^<_\mathbf{q}|^2 + \frac{1}{2}\int_{\Lambda/b}^{\Lambda} \frac{d^d\mathbf{q}}{(2\pi)^d}\,(cq^2+r)|\phi^>_\mathbf{q}|^2$$

1. Integrate out $\phi^>_\mathbf{q}$

$$e^{-\mathcal{H}_{\Lambda/b}} \equiv e^{-\mathcal{H}^<_\Lambda}\int \mathcal{D}\phi^>_\mathbf{q}\; e^{-\mathcal{H}^>_\Lambda} \equiv e^{-\mathcal{H}^<_\Lambda} Z_0$$

2. Rescale $\mathbf{q}' = b\mathbf{q}$

$$\mathcal{H}_{\Lambda/b} \to \mathcal{H}'_\Lambda = b^{-d}\frac{1}{2}\int_0^\Lambda \frac{d^d\mathbf{q}'}{(2\pi)^d}\,(cb^{-2}q'^2+r)|\phi_{\mathbf{q}'/b}|^2$$

3. Rescale fields; $\phi_{\mathbf{q}'/b} = \zeta\phi'_{\mathbf{q}'}$

$$\mathcal{H}'_\Lambda = \frac{1}{2}\int_0^\Lambda \frac{d^d\mathbf{q}'}{(2\pi)^d}\,(c\zeta^2 b^{-d-2}q'^2 + \zeta^2 b^{-2}r)|\phi'_{\mathbf{q}'}|^2$$

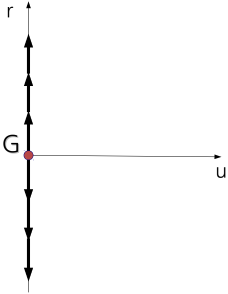
We recover the original form of Hamiltonian if we use

$$c' = \zeta^2 b^{-d-2} c, \quad r' = \zeta^2 b^{-d} r$$

Look for fixed points when $\zeta^2 = b^{d+2}$ or $\eta = 0$. Then $\boxed{r' = b^2 r}$.

- $(c^*, r^*) = (c, 0)$: Gaussian fixed point $\to$ physical situation for $d > 4$
- $(c^*, r^*) = (c, +\infty)$: Far above T_c or $T = \infty$
- $(c^*, r^*) = (c, -\infty)$: Far below T_c or $T = 0$

[The case where $\zeta^2 = b^d$ or $\eta = 2$ gives either $T = 0$ or $T = \infty$.]



At the Gaussian f.p.

- r is relevant: $y_r = 2 = y_t = 1/\nu$.
- Recall that $y_h = (d+2)/2$ or $\eta = 0$
- Using scaling relations, $\gamma = 1$, $\alpha = \frac{4-d}{2}$, $\delta = \frac{d+2}{d-2}$, $\beta = \frac{d-2}{4}$
- MF values: $\nu = 1/2$, $\eta = 0$, $\gamma = 1$, $\alpha = 0$, $\delta = 3$, $\beta = 1/2$

What about u?

$u \int d^d\vec{x}\ \phi^4(\vec{x})$

$= u \int \frac{d^d\vec{q}_1}{(2\pi)^d} \int \frac{d^d\vec{q}_2}{(2\pi)^d} \int \frac{d^d\vec{q}_3}{(2\pi)^d}\ \phi_{\vec{q}_1}\ \phi_{\vec{q}_2}\ \phi_{\vec{q}_3}\ \phi_{-\vec{q}_1-\vec{q}_2-\vec{q}_3}$

Consider the following near Gaussian f.p.

$u \int \frac{d^d\vec{q}_1}{(2\pi)^d} \int \frac{d^d\vec{q}_2}{(2\pi)^d} \int \frac{d^d\vec{q}_3}{(2\pi)^d}\ \phi^<_{\vec{q}_1}\ \phi^<_{\vec{q}_2}\ \phi^<_{\vec{q}_3}\ \phi^<_{-\vec{q}_1-\vec{q}_2-\vec{q}_3}$

$\downarrow$

$u \zeta^4 b^{-3d} \int \frac{d^d\vec{q}_1}{(2\pi)^d} \int \frac{d^d\vec{q}_2}{(2\pi)^d} \int \frac{d^d\vec{q}_3}{(2\pi)^d}\ \phi'_{\vec{q}_1'}\ \phi'_{\vec{q}_2'}\ \phi'_{\vec{q}_3'}\ \phi'_{-\vec{q}_1'-\vec{q}_2'-\vec{q}_3'}$

$$u' = \zeta^4 b^{-3d} u = b^{2(2+d)-3d} u$$

$$= b^{4-d} u$$

$$\boxed{y_u = 4 - d}$$

- For $d > 4$, u is irrelevant near G.
- For $d < 4$, u is relevant near G.
- For $d > 4$, the scaling exponents at G.f.p. are OK? $\rightarrow$ No!
- u is "dangerously irrelevant".

dangerously irrelevant variables

Consider once again the flow of singular part of free energy

$$f_s(t, h, u) = b^{-d} f_s(b^{y_t} t, b^{y_h} h, b^{y_u} u)$$

Stop at $b^{n y_t} t = 1$.

$$f_s(t, h, u) = |t|^{d/y_t} \Phi\left(\frac{h}{|t|^{y_h/y_t}}, \frac{u}{|t|^{y_u/y_t}}\right)$$

Consider the magnetization

$$m = \left.\frac{\partial f}{\partial h}\right|_{h=0} \sim (-t)^{(d-y_h)/y_t} F_m\left(\frac{u}{(-t)^{y_u/y_t}}\right)$$

Scaling law works only if F_m is nonsingular as $u \to 0$. But we know that for $d > 4$

$$F_m(x) \sim \frac{1}{\sqrt{x}}$$

N-vector model in $4-\epsilon$ dimensions

Generalize to a model with $O(N)$ symmetry: $\phi_i(\mathbf{x})$, $i=1,2,\cdots,N$.
($N=1$ Ising)

$$\mathcal{H}=\int d^d\mathbf{x}\,\left\{\sum_{i=1}^{N}\left[\frac{1}{2}(\nabla\phi_i(\mathbf{x}))^2+\frac{r}{2}\phi_i^2\right]+u\left(\sum_i\phi_i^2(\mathbf{x})\right)^2\right\}\equiv\mathcal{H}_0+\mathcal{H}_4$$

$d<4$: u is a relevant perturbation to Gaussian fixed point
Expect new fixed point $u^*\sim O(\epsilon)$ with $\epsilon=4-d\Rightarrow O(\epsilon)$ calculation
Follow 3-step procedure:

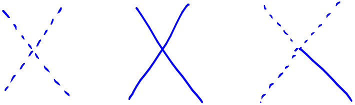
1. Integrate over $\phi_i^>(\mathbf{x})$: Need to evaluate (cumulant expansion)

$$\langle e^{-\mathcal{H}_4}\rangle_>,$$

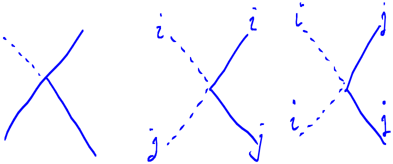
where

$$\langle\cdots\rangle_>=\frac{1}{Z_0^>}\int\mathcal{D}\phi^>e^{-\mathcal{H}_0^>}(\cdots)$$

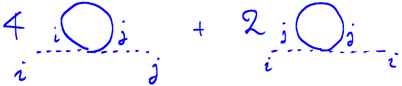
$$\vec{\phi}^4 = \vec{\phi}_<^4 + \vec{\phi}_>^4 + 4\vec{\phi}_<^2(\vec{\phi}_< \cdot \vec{\phi}_>)$$



$$+ 4\vec{\phi}_>^2(\vec{\phi}_< \cdot \vec{\phi}_>) + 4(\vec{\phi}_< \cdot \vec{\phi}_>)^2 + 2\vec{\phi}_>^2\vec{\phi}_<^2$$



Diagrams for r'

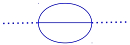


$$= u\left[4 + 2N\right] \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{r+q^2} \int^{\Lambda/b} \frac{d^d \vec{k}}{(2\pi)^d} \left|\vec{\phi}^{<}_{\vec{k}}\right|^2$$

$$\xrightarrow{\text{step (2),(3)}} \underbrace{u\left[4+2N\right] b^{-d}\zeta^2 K_d \int_{\Lambda/b}^{\Lambda} dq \, \frac{q^{d-1}}{r+q^2}}_{= \frac{1}{2}\delta r} \int_0^{\Lambda} \frac{d^d \vec{k}'}{(2\pi)^d} \left|\vec{\phi}'_{\vec{k}'}\right|^2$$

$$K_d \equiv \frac{\Omega_d}{(2\pi)^d}$$

No contribution to c' to $O(u)$ order



$O(u^2)$ contribution: $\rightarrow \eta = O(\epsilon^2)$

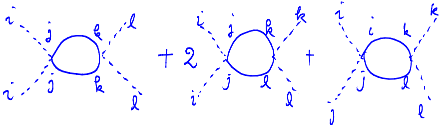
$$\begin{aligned} r' =& b^{2-\eta}\left[r + u\, 2(4+2N)K_d \int_{\Lambda/b}^{\Lambda} dq\, \frac{q^{d-1}}{r+q^2} \right] \\ =& (1+\delta\ell)^2 \left[r + (\delta\ell) u\, 2(4+2N)K_d \frac{\Lambda^{d-1}}{r+\Lambda^2} \right], \end{aligned}$$

where $b = 1 + \delta\ell$ ($b = e^{\ell}$)

$$\boxed{\frac{dr}{d\ell} = 2r + u\, 4(2+N)K_d \frac{\Lambda^d}{r+\Lambda^2}}$$

Diagrams for u'

Let's take external momentum=0 for the moment.



$$\left[N(2u)^2 + 2\cdot(2u)(4u) + (4u)^2\right] K_d \int_{\Lambda/b}^{\Lambda} dq\, \frac{q^{d-1}}{(r+q^2)^2}$$

$$\times \int^{\Lambda/b} \prod_{\ell}^{4} \frac{d^d\vec{k}_\ell}{(2\pi)^d}\, \phi^{i<}_{\vec{k}_1}\, \phi^{i<}_{\vec{k}_2}\, \phi^{j<}_{\vec{k}_3}\, \phi^{j<}_{\vec{k}_4}\, (2\pi)^d \delta^{(d)}(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

$$u' = b^{-3d}\zeta^4\left[u - 4(N+8)u^2 K_d \int_{\Lambda/b}^{\Lambda} dq\, \frac{q^{d-1}}{(r+q^2)^2}\right]$$

Following the same procedure as before,

$$\boxed{\frac{du}{d\ell} = \epsilon u - u^2\, 4(8+N)K_d \frac{\Lambda^d}{(r+\Lambda^2)^2}}$$

➢ Irrelevant terms:

① $u_4(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = u + Aq^2 + \cdots$; higer order terms are irrelevant

② $u_6(\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_5)$ etc.

Fixed Points to $O(\epsilon)$

$$\left.\frac{dr}{d\ell}\right|_{r^*,u^*} = \left.\frac{du}{d\ell}\right|_{r^*,u^*} = 0$$

$$2r^* + 4u^*(2+N)\frac{K_d\Lambda^d}{r^* + \Lambda^2} = 0$$

$$\epsilon u^* - 4u^{*2}(8+N)\frac{K_d\Lambda^d}{(r^* + \Lambda^2)^2} = 0$$

① $r^* = u^* = 0$: Gaussian fixed point

② Wilson-Fisher fixed point

$$u^* = \frac{\epsilon}{4(N+8)K_d\Lambda^{d-4}} + O(\epsilon^2), \qquad r^* = -\frac{1}{2}\left(\frac{N+2}{N+8}\right)\Lambda^2\epsilon + O(\epsilon^2)$$

Linearized RG

Let $\delta r \equiv r - r^*$ and $\delta u \equiv u - u^*$.

$$\begin{pmatrix} \frac{d}{d\ell}\delta r \\ \frac{d}{d\ell}\delta u \end{pmatrix} = \begin{pmatrix} 2 - u^* 4(N+2)\frac{K_d \Lambda^d}{(r^*+\Lambda^2)^2} & 4(N+2)\frac{K_d\Lambda^d}{r^*+\Lambda^2} \\ O(\epsilon^2) & \epsilon - 8u^*(N+8)\frac{K_d\Lambda^d}{(r^*+\Lambda^2)^2} \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}$$

Eigenvalues of the matrix: $\rightarrow y_t$ and y_u [Recall $K' = b^{y_K} K$, etc]

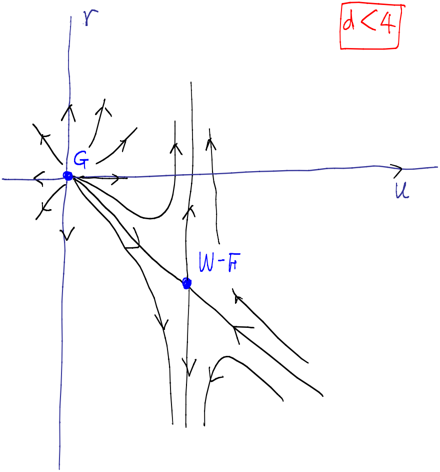
➢ Gaussian fixed point: $y_t = 2 = \nu^{-1}$ and $y_u = \epsilon > 0$

➢ Wilson-Fisher fixed point:

$$\begin{pmatrix} 2 - \epsilon\left(\frac{N+2}{N+8}\right) & 4(N+2)\frac{K_d\Lambda^d}{r^*+\Lambda^2} \\ O(\epsilon^2) & -\epsilon + O(\epsilon^2) \end{pmatrix}$$

$$y_t = 2 - \epsilon\left(\frac{N+2}{N+8}\right) = \nu^{-1}, \qquad y_u = -\epsilon$$

RG flow diagram for $d = 4 - \epsilon$



$\boxed{d<4}$

$$\nu = \frac{1}{2} + \frac{1}{4}\left(\frac{N+2}{N+8}\right)\epsilon,$$

$$\eta = 0,$$

$$\alpha = \frac{4-N}{2(N+8)}\epsilon,$$

$$\beta = \frac{1}{2} - \frac{3}{2(N+8)}\epsilon,$$

$$\gamma = 1 + \frac{N+2}{2(N+8)}\epsilon,$$

$$\delta = 3 + \epsilon$$

Summary

- Renormalization Group Transformation
  - Coarse graining + Change in length scale
  - Useful tool to study problems where fluctuations at many length (and time) scales are important
  - Great success in treating problems near the critical point of a continuous phase transition
  - Applications to other fluctuation-dominated physical problems
- Real-space RG
  - Conceptually easy to understand
  - Practically not a systematic calculational method
  - Field theory and momentum shell RG