

# Real Space Renormalization Group

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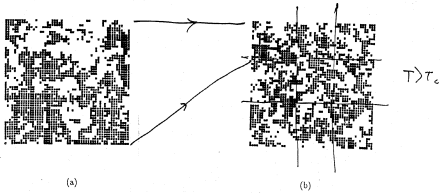
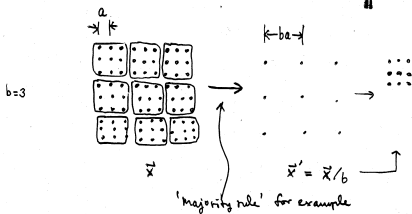
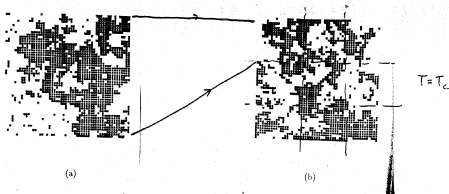
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Pyeongchang Summer Institute, July, 2015

# Renormalization Group Transformation

## Block Spin transformation



$T = 1.22T_c$

ORIGINAL LATTICE



FIRST-STAGE BLOCK SPINS



SECOND-STAGE BLOCK SPINS



THIRD-STAGE  
BLOCK SPINS



FOURTH-STAGE  
BLOCK SPINS



$T = T_c$

ORIGINAL

LATTICE



FIRST-STAGE

BLOCK SPINS



SECOND-STAGE BLOCK SPINS



THIRD-STAGE  
BLOCK SPINS



FOURTH-STAGE  
BLOCK SPINS



$T = 99T_c$

ORIGINAL LATTICE



FIRST-STAGE BLOCK SPINS



SECOND-STAGE BLOCK SPINS



THIRD-STAGE  
BLOCK SPINS



FOURTH-  
STAGE  
BLOCK  
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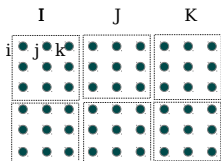
## Idea of RG

- Integrate out short distance degrees of freedom:

$$-\mathcal{H} \equiv -\beta H = NK_0 + K_1 \sum_i \sigma_i + K_2 \sum_{i,j} \sigma_i \sigma_j + K_3 \sum_{i,j,k} \sigma_i \sigma_j \sigma_k + \dots$$

is described by  $\{K\}$

- Block Spins



$$b=3$$

$$s_I = \frac{1}{b^d} \sum_{i \in I} \sigma_i, \quad \text{or} \quad s_I = \text{sign} \left( \sum_{i \in I} \sigma_i \right)$$

- Projection:

$$P(s_I; \{\sigma_i\}) = \delta \left( s_I - \text{sign} \left( \sum_{i \in I} \sigma_i \right) \right)$$

Note that

$$\sum_{\{s_I\}} \prod_I P(s_I; \{\sigma_i\}) = 1$$

$$\begin{aligned}
 Z_N[K] &= \sum_{\{\sigma_i\}} e^{-\mathcal{H}[\sigma]} = \sum_{\{\sigma_i\}} \sum_{\{s_l\}} \prod_l P(s_l; \{\sigma_i\}) e^{-\mathcal{H}[\sigma]} \\
 &\equiv \sum_{\{s_l\}} e^{-\mathcal{H}'[s_l]} = Z_{N'}[K'],
 \end{aligned}$$

where  $N' = N/b^d$  and

$$e^{-\mathcal{H}'[s_l]} \equiv \sum_{\{\sigma_i\}} \prod_l P(s_l; \{\sigma_i\}) e^{-\mathcal{H}[\sigma]}$$

RG transformation

$$\mathcal{R}_b : \{K\} \rightarrow \{K'\}$$

In practice, this is difficult to do!

## RG / 1d Ising model

Recall that

$$\mathcal{H} = -K \sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i,$$

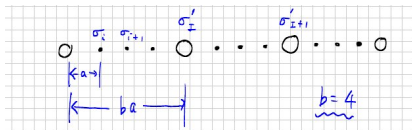
with  $K \equiv \beta J$  and  $B \equiv \beta h$

$$Z = \sum_{\{\sigma_i = \pm 1\}} T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} \cdots T_{\sigma_N \sigma_1} = \text{Tr } \mathbb{T}^N,$$

➤ Decimation: Trace over  $b - 1$  spins and leave a spin at every  $b$ -th site.

$$Z_N(K, B) = \text{Tr } \mathbb{T}^N = \text{Tr } (\mathbb{T}^b)^{N'} = Z_{N'}(K', B'),$$

where  $N' = N/b$ .





□  $h = 0$  Case:

Recall that

$$\begin{aligned} T_{\sigma\sigma'} &= \cosh K(1 + \sigma\sigma' \tanh K) \\ (T^b)_{\sigma\sigma'} &= 2^{b-1} \cosh^b K(1 + \sigma\sigma' \tanh^b K) \\ &= \left( \frac{2^{b-1} \cosh^b K}{\cosh K'} \right) \cosh K'(1 + \sigma\sigma' \tanh K') \end{aligned}$$

Note that constant term is generated. We may as well start from

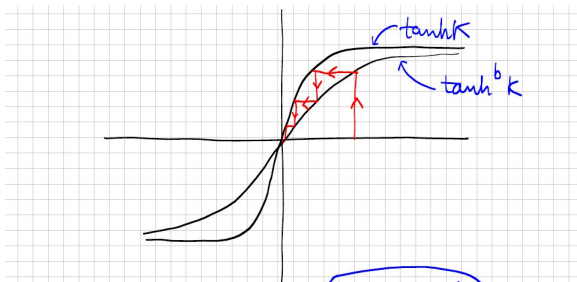
$$-\mathcal{H} = Ng + K \sum_i \sigma_i \sigma_{i+1} \rightarrow -\mathcal{H}' = N'g' + K' \sum_l \sigma_l \sigma_{l+1}$$

$$\begin{aligned} T_{\sigma\sigma'} &= e^g \cosh K(1 + \sigma\sigma' \tanh K) \\ (T^b)_{\sigma\sigma'} &= e^{bg} 2^{b-1} \cosh^b K(1 + \sigma\sigma' \tanh^b K) \\ &= e^{g'} \cosh K'(1 + \sigma\sigma' \tanh K') \end{aligned}$$

We therefore have for  $h = 0$  the RG flow equation  $(K', g') = \mathcal{R}_b(K, g)$  as

$$g' = bg + \ln \left( \frac{2^{b-1} \cosh^b K}{\cosh K'} \right)$$

$$K' = \tanh^{-1} \left( \tanh^b K \right)$$



RG flow diagram:

fixed points

stable

unstable

$$K^* = 0$$

$$K^* = \infty$$

$$T = \infty$$

$$T = 0$$

disordered

ordered

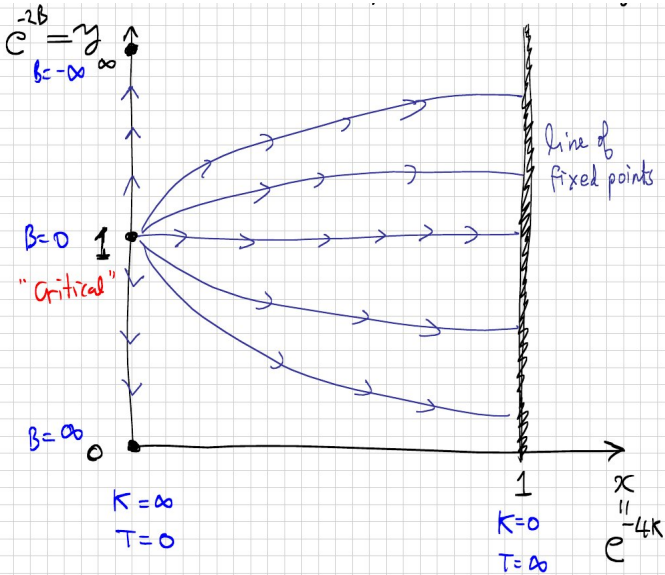
□  $h \neq 0$  Case: Take  $b = 2$  for simplicity

$$\begin{aligned}\mathbb{T}^2 &= e^{2g} \begin{pmatrix} e^{2K+2B} + e^{-2K} & e^B + e^{-B} \\ e^B + e^{-B} & e^{2K-2B} + e^{-2K} \end{pmatrix} \\ &\equiv e^{g'} \begin{pmatrix} e^{K'+B'} & e^{-K'} \\ e^{-K'} & e^{K'-B'} \end{pmatrix}\end{aligned}$$

$$y' \equiv e^{-2B'} = x \frac{(1+y)^2}{(x+y)(1+xy)}$$

$$x' \equiv e^{-4K'} = y \frac{x+y}{1+xy},$$

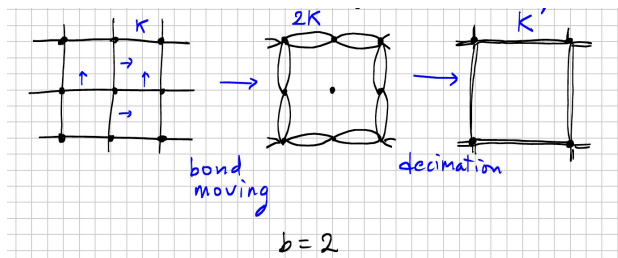
where  $x \equiv e^{-4K}$  and  $y \equiv e^{-2B}$ . (Derive these.)





# Migdal-Kadanoff Approximation

$d = 2$



Let us recall the recursion relation for decimation in 1d ( $b = 2$ )

$$K' = \tanh^{-1}(\tanh^2 K) = \frac{1}{2} \ln \cosh(2K)$$

From the above figure, we have in this case

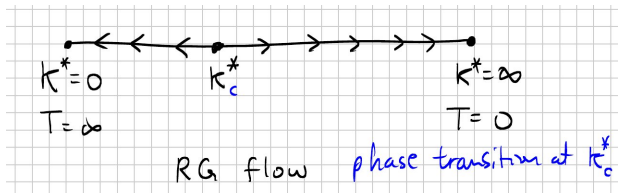
$$K' = \frac{1}{2} \ln \cosh(2 \cdot 2K)$$

## Fixed points

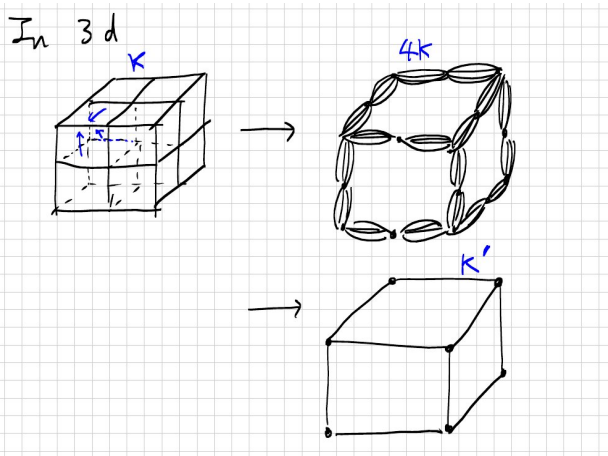
- $K^* = 0$  For  $K \ll 1$ ,  $K' \simeq \frac{1}{2} \ln(1 + 8K^2) \simeq 4K^2 \rightarrow$  Stable
- $K^* = \infty$  For  $K \gg 1$ ,  $K' \simeq \frac{1}{2} \ln(e^{4K}/2) \simeq 2K \rightarrow$  Stable
- Critical fixed point

$$e^{2K_c^*} = \frac{1}{2}(e^{4K_c^*} + e^{-4K_c^*})$$

$$K_c^* \simeq 0.305 \quad (\text{c.f. } K_c^{\text{exact}} \simeq 0.441)$$



$$d = 3$$



$$K' = \frac{1}{2} \ln \cosh(2 \cdot 4K)$$

$$K_c^* \simeq 0.065 \quad (\text{c.f. } K_c^{\text{known}} \simeq 0.222)$$



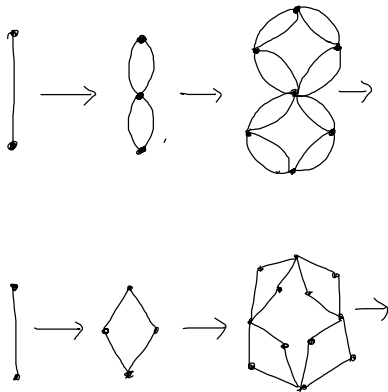
In  $d$  dimensions,

$$K' = \frac{1}{2} \ln \cosh(2 \cdot 2^{d-1} K)$$

MK approximation becomes worse as  $d \rightarrow \infty$ .

It becomes exact on hierarchical lattices

Hierarchical Lattices



For an arbitrary  $b$ ,

$$\tanh K' = [\tanh(b^{d-1}K)]^b$$

Generalize to continuous  $b = 1 + \delta\ell = e^{\delta\ell}$ ;  $K = K(\ell)$

$$\begin{aligned}\tanh K' &= [\tanh((1 + \delta\ell)^{d-1}K)]^{1+\delta\ell} \\ &\simeq [\tanh((1 + (d-1)\delta\ell)K)]^{1+\delta\ell} \\ &\simeq \tanh((1 + (d-1)\delta\ell)K)[1 + \delta\ell \ln \tanh K] \\ &\simeq \tanh((1 + (d-1)\delta\ell)K) + (\delta\ell) \frac{\sinh K \cosh K}{\cosh^2 K} \ln \tanh K \\ &\simeq \tanh \left[ K + (d-1)\delta\ell K + (\delta\ell) \frac{1}{2} \sinh(2K) \ln \tanh K \right]\end{aligned}$$

$$\frac{dK}{d\ell} = (d-1)K + \frac{1}{2} \sinh(2K) \ln \tanh K$$

## Ising model in $1 + \varepsilon$ dimensions

Look at large  $K$  (or small  $T$ ) region.

$$\frac{1}{2} \sinh(2K) \ln \tanh K \simeq \frac{1}{2} \frac{e^{2K}}{2} \ln(1 - 2e^{-2K}) \simeq -\frac{1}{2}$$

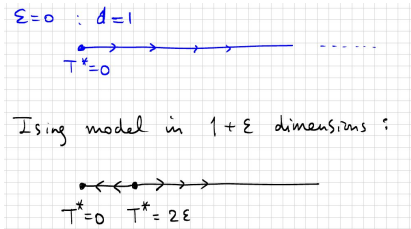
For  $d = 1 + \varepsilon$ ,

$$\frac{dK}{dl} \simeq \varepsilon K - \frac{1}{2}$$

or for  $T = 1/K$ ,

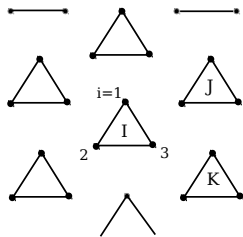
$$\boxed{\frac{dT}{dl} \simeq -\varepsilon T + \frac{1}{2} T^2}$$

$d = 1$  is the lower critical dimension of the Ising model.



# Approximation Methods

Niemeijer and van Leeuwen (1974)



- 2-d Ising model on a triangular lattice
- $-\mathcal{H} = \sum_{\langle i,j \rangle} K \sigma_i \sigma_j + h \sum_i \sigma_i$
- Projection (majority rule):

$$P(\mu_I; \sigma_{1I}, \sigma_{2I}, \sigma_{3I}) = P(\mu_I; \{\sigma_{Ii}\}) \\ = \delta(\mu_I, \text{sign}(\sum_i \sigma_{Ii}))$$

- Note that

$$\sum_{\mu_I = \pm 1} P(\mu_I; \{\sigma_{Ii}\}) = 1$$

$$\begin{aligned}
 Z &= \sum_{\{\sigma\}} e^{-\mathcal{H}(\{\sigma\}, K, h)} = \sum_{\{\sigma\}} \underbrace{\sum_{\{\mu\}} \prod_I P(\mu_I; \{\sigma_I\})}_{=1} e^{-\mathcal{H}(\{\sigma\}, K, h)} \\
 &= \sum_{\{\mu\}} \sum_{\{\sigma\}} \prod_I P(\mu_I; \{\sigma_I\}) e^{-\mathcal{H}(\{\sigma\}, K, h)} \equiv \sum_{\{\mu\}} e^{N' K'_0} e^{-\mathcal{H}'(\{\mu\}, K', h')}
 \end{aligned}$$

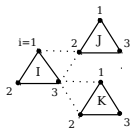
The main difficulty lies in evaluating  $\sum_{\sigma}$  to get  $\mathcal{H}'$ . Need to use an approximation. Write  $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$ , where

$$\begin{aligned}
 -\mathcal{H}_0 &= K \sum_I (\sigma_{1I} \sigma_{2I} + \sigma_{2I} \sigma_{3I} + \sigma_{3I} \sigma_{1I}) \\
 &\quad + h \sum_I (\sigma_{1I} + \sigma_{2I} + \sigma_{3I}),
 \end{aligned}$$

$$-\mathcal{V} = K \sum_{\langle I, J \rangle} \sum_{i,j} \sigma_{iI} \sigma_{jJ}, \text{ for example,}$$

$$-\mathcal{V}_{IJ} = K(\sigma_{1I} \sigma_{2J} + \sigma_{3I} \sigma_{2J}),$$

$$-\mathcal{V}_{IK} = K(\sigma_{3I} \sigma_{1K} + \sigma_{3I} \sigma_{2K})$$



Let

$$Z_0 = \sum_{\{\sigma\}} \prod_I P(\mu_I; \{\sigma_I\}) e^{-\mathcal{H}_0(\{\sigma\}, K, h)}.$$

Then

$$e^{N'K'_0} e^{-\mathcal{H}'(\{\mu\}, K', h')} = Z_0 \langle e^{-\mathcal{V}} \rangle,$$

where

$$\langle A \rangle = \frac{1}{Z_0} \sum_{\{\sigma\}} \prod_I P(\mu_I; \{\sigma_I\}) A e^{-\mathcal{H}_0(\{\sigma\}, K, h)}$$

Use a cumulant expansion and take the lowest order as a first approximation.

$$\langle e^{-\mathcal{V}} \rangle = \exp \left[ -\langle \mathcal{V} \rangle + \frac{1}{2} (\langle \mathcal{V}^2 \rangle - \langle \mathcal{V} \rangle^2) + \dots \right] \simeq e^{-\langle \mathcal{V} \rangle}$$

Need to evaluate  $Z_0$  and  $\langle \mathcal{V} \rangle$ .

- $\mu_I = +1$

$\sigma_{1I}$	$\sigma_{2I}$	$\sigma_{3I}$	$e^{-\mathcal{H}_0}$
+	+	+	$e^{3K+3h}$
+	+	-	$e^{-K+h}$
+	-	+	$e^{-K+h}$
-	+	+	$e^{-K+h}$

$$\sum_{\sigma_I} e^{-\mathcal{H}_0} = e^{3K+3h} + 3e^{-K+h}$$

- $\mu_I = -1$

$\sigma_{1I}$	$\sigma_{2I}$	$\sigma_{3I}$	$e^{-\mathcal{H}_0}$
-	-	-	$e^{3K-3h}$
-	-	+	$e^{-K-h}$
+	+	-	$e^{-K-h}$
+	-	-	$e^{-K-h}$

$$\sum_{\sigma_I} e^{-\mathcal{H}_0} = e^{3K-3h} + 3e^{-K-h}$$

The above results can be summarized as  $e^{A(K,h)+B(K,h)\mu_l}$  with

$$e^{A+B} = e^{3K+3h} + 3e^{-K+h}, \quad e^{A-B} = e^{3K-3h} + 3e^{-K-h}.$$

For  $\langle \mathcal{V} \rangle$ , note that in  $\mathcal{H}_0$ , there is no mixing between  $\sigma_{il}$  and  $\sigma_{jJ}$  for  $l \neq J$ . Therefore,

$$-\langle \mathcal{V} \rangle = K \sum_{\langle l,J \rangle} \sum_{i,j} \langle \sigma_{il} \rangle \langle \sigma_{jJ} \rangle$$

Note that  $\langle \sigma_{il} \rangle$  is indep. of  $i$  for given  $l$ .

- $\mu_l = +1$

$$\langle \sigma_{il} \rangle = \frac{e^{3K+3h} + 2e^{-K+h} - e^{-K+h}}{e^{3K+3h} + 3e^{-K+h}} \equiv C(K, h) + D(K, h)$$

- $\mu_l = -1$

$$\langle \sigma_{il} \rangle = \frac{-e^{3K-3h} - 2e^{-K-h} + e^{-K-h}}{e^{3K-3h} + 3e^{-K-h}} \equiv C(K, h) - D(K, h)$$

This can be summarized as  $\langle \sigma_{il} \rangle = C(K, h) + D(K, h)\mu_l$



We have

$$Z_0 = \prod_I e^{A+B\mu_I} = e^{N'A+B\sum_I \mu_I}.$$

and

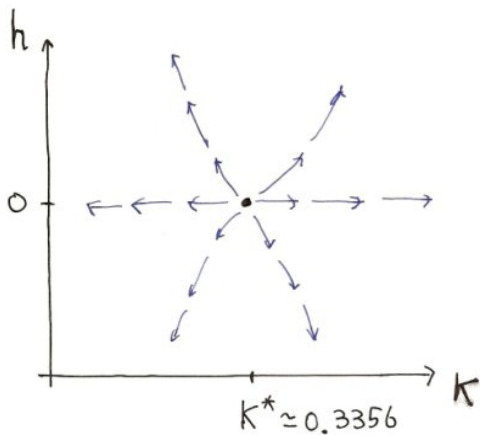
$$\begin{aligned} -\langle \mathcal{V} \rangle &= 2K \sum_{\langle I,J \rangle} (C + D\mu_I)(C + D\mu_J) \\ &= 2K \left[ C^2 \frac{N'z}{2} + CDz \sum_I \mu_I + D^2 \sum_{\langle I,J \rangle} \mu_I \mu_J \right], \end{aligned}$$

where  $N' = N/3$  and  $z = 6$  is the number of nearest neighbors. The factor 2 comes from the two ways of connecting neighboring blocks. We finally have the recursion relations

$$K' = 2KD^2(K, h)$$

$$h' = B(K, h) + 12KC(K, h)D(K, h)$$

$$K'_0 = A(K, h) + 6KC^2(K, h)$$



$$K_c^{\text{exact}} \approx 0.27465$$

$$K_c^{\text{MH}} = \frac{1}{2} = \frac{1}{6}$$

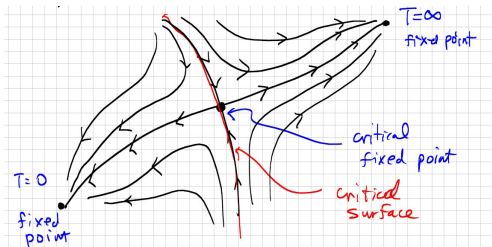
# General Theory

- RG transform  $\mathcal{R}_b : \{K\} \rightarrow \{K'\}$  generates a flow in a multi-dimensional parameter space
- Length scale shrinks down at each step

$$\xi(K') = \xi(K)/b$$

- Fixed Points:

$$\mathcal{R}_b\{K^*\} = K^*, \quad \xi(K^*) = \xi(K^*)/b, \quad \xi = 0, \text{ or } \xi = \infty \text{ (critical)}$$



- Linearize  $\mathcal{R}_b$  near critical fixed point:

$$K'_a - K_a^* \sim \sum_b T_{ab}(K_b - K_b^*)$$

- Solve eigenvalue equation

$$\sum_a \phi_a^{(i)} T_{ab} = \lambda^{(i)} \phi_b^{(i)}$$

- Multiply the linearized flow eq. by  $\phi_a^{(i)}$  and sum over  $a$

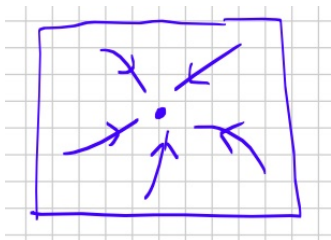
$$\sum_a \phi_a^{(i)} (K'_a - K_a^*) = \lambda^{(i)} \sum_b \phi_b^{(i)} (K_b - K_b^*)$$

or

$$u'_i = \lambda^{(i)} u_i,$$

where  $u_i \equiv \sum_a \phi_a^{(i)} (K_a - K_a^*)$  is the scaling variable.

- Classification: Let  $\lambda^{(i)} \equiv b^{y_i}$  ( $y_i$  scaling exponent).
  - Relevant variable:  $y_i > 0$  ( $u_i$  flows away from the fixed point)
  - Irrelevant variable:  $y_i < 0$  ( $u_i \rightarrow 0$  under RG flow)
  - Marginal variable:  $y_i = 0$
- Critical surface: a surface spanned by **irrelevant** variables



- One has to adjust **relevant** variables to be on the critical surface.
- Ising universality class: 2 relevant scaling variables
  - 1 thermal,  $u_t \sim t \equiv (T - T_c)/T_c$ ,  $y_t > 0$
  - 1 magnetic,  $u_h \sim h \equiv \beta h$ ,  $y_h > 0$

## Scaling theory for Ising universality class

- Free energy per site:  $f(\{K\}) = -(1/N) \ln Z(\{K\})$ . Invariance of partition function gives

$$Nf(\{K\}) = N'f(\{K'\}) + Ng(\{K\}), \quad N' = N/b^d$$

where  $g(\{K\})$  is a constant term. The singular part behaves as

$$f_s(\{K\}) = b^{-d} f_s(\{K'\})$$

$$f_s(t, h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h) = \dots = b^{-nd} f_s(b^{ny_t} t, b^{ny_h} h)$$

$t$  and  $h$  are growing. Stop at  $|b^{ny_t} t| \sim t_0 = O(1)$ .

$$f_s(t, h) = \left| \frac{t}{t_0} \right|^{d/y_t} \Phi \left( \frac{h}{|t/t_0|^{y_h/y_t}} \right)$$

- Correlation function:  $G(\mathbf{r}_1 - \mathbf{r}_2; t)$

$$\frac{\delta^2 Z}{Z} = \sum_{\text{sites}} \delta h(\mathbf{r}_1) \delta h(\mathbf{r}_2) G(\mathbf{r}_1 - \mathbf{r}_2; t) = \sum_{\text{blocks}} \delta h'(\mathbf{r}'_1) \delta h'(\mathbf{r}'_2) G(\mathbf{r}'_1 - \mathbf{r}'_2; t')$$

Note that  $\delta h'(\mathbf{r}') = b^{y_h} \delta h(\mathbf{r})$ ,  $\mathbf{r}' = \mathbf{r}/b$ , and  $b^d$  spins in each block.

$$G(\mathbf{r}, t) = b^{2y_h - 2d} G(\mathbf{r}/b, b^{y_t} t) = b^{2n(y_h - d)} G(\mathbf{r}/b^n, b^{ny_t} t)$$

Stop at  $|b^{ny_t} t| \sim 1$ .

$$G(\mathbf{r}, t) = |t|^{\frac{2(d-y_h)}{y_t}} \Psi \left( \frac{\mathbf{r}}{|t|^{-\frac{1}{y_t}}} \right)$$

- Correlation length  $\xi \sim |t|^{-1/y_t} \equiv |t|^{-\nu}$ .  $y_t = 1/\nu$
- At  $T = T_c$  ( $t = 0$ ), stop at  $r/b^n \sim 1$ .

$$G(\mathbf{r}, 0) \sim r^{2(y_h - d)} \equiv r^{-(d - 2 + \eta)},$$

$$y_h = \frac{1}{2}(d + 2 - \eta)$$

- Specific heat

$$c \sim \left. \frac{\partial^2 f}{\partial t^2} \right|_{h=0} \sim |t|^{\frac{d}{y_t} - 2} \equiv |t|^{-\alpha}, \quad \boxed{\alpha = 2 - d\nu}$$

- Spontaneous magnetization

$$m \sim \left. \frac{\partial f}{\partial h} \right|_{h=0} \sim (-t)^{\frac{d}{y_t} - \frac{y_h}{y_t}} \equiv (-t)^\beta, \quad \boxed{\beta = \nu \left( \frac{d - 2 + \eta}{2} \right)}$$

- Susceptibility

$$\chi \sim \left. \frac{\partial m}{\partial h} \right|_{h=0} \sim |t|^{(d - 2y_h)/y_t} \equiv |t|^{-\gamma}, \quad \boxed{\gamma = \nu(2 - \eta)}$$



- $m$ - $h$  curve at  $T = T_c$

$$m(h) \sim \frac{\partial f}{\partial h} = |t|^{\frac{d-y_h}{y_t}} \Phi' \left( \frac{h}{|t|^{y_h/y_t}} \right)$$

As  $t \rightarrow 0$ , this must be finite. So we require that  $\Phi'(x) \sim x^{d/y_h-1}$  as  $x \rightarrow \infty$  so that as  $t \rightarrow 0$

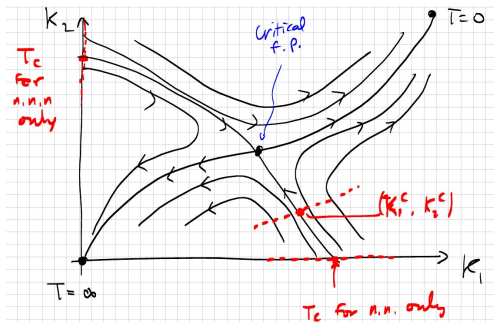
$$m(h) \sim |t|^{\frac{d-y_h}{y_t}} |t|^{(-d/y_h+1)(y_h/y_t)} h^{d/y_h-1} \sim h^{d/y_h-1} \equiv h^{1/\delta}$$

$$\delta = \frac{d+2-\eta}{d-2+\eta}$$

## Example

$$-\mathcal{H} = K_1 \sum_{\text{n.n.}} \sigma_i \sigma_j + K_2 \sum_{\text{n.n.n.}} \sigma_i \sigma_j,$$

where n.n.=nearest neighbor and n.n.n.=next nearest neighbor.



- Models with  $K_2 = 0$  or  $K_1 = 0$  or  $K_1 \neq 0, K_2 \neq 0$  are all in the same universality class.
- Behavior near the critical fixed point controls the critical exponents **not** initial conditions.