one-loop quantum physics of matter in curved spacetime

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or, how quantum matter "renormalizes" classical geometry

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one-loop renormalization as selective Gaussian integration

functional Gaussian integral via heat kernel

2d Weyl anomaly & s-wave Hawking radiation

Bogolyubov, Hawking, Unruh, and de Sitter

references

- **Wilson & Kogut, "The renormalization group and the ϵ-expansion,"**
  Physics Reports 12 (1974) 75-200.
- **McKean & Singer, "Curvature and the eigenvalues of the Laplacian, "**
  Journal of Differential Geometry, vol.1 (1967) 43-69.
- **Barvinsky's summary of an improved heat kernel expansion**
  http://www.scholarpedia.org/article/Heat_kernel_expansion_in_the_background_field_formalism
- **Misner, Thorne & Wheeler, "Gravitation," W.H. Freedman and Company.**
- **Birrel & Davies, "Quantum Fields in Curved Space," Cambridge University Press.**
- **Preskill's Lecture Note: Physics 236c, "Quantum Field Theory in Curved Spacetime"**
  http://www.theory.caltech.edu/~preskill/notes.html

one-loop renormalization as selective Gaussian integration

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2 y^2}$$

$$\epsilon = 1 \quad \Rightarrow \quad 4.96145$$

$$\epsilon = 0.1 \quad \Rightarrow \quad 6.02068$$

$$\epsilon = 0.01 \quad \Rightarrow \quad 6.25245$$

$$\epsilon = 0.001 \quad \Rightarrow \quad 6.28005$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2 y^2} = \int dx\; e^{-\frac{1}{2}x^2}\int dy\; e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

recall the Gaussian integration

$$\int_{-\infty}^{\infty} dy\, e^{-\frac{m}{2}y^2} = \sqrt{\frac{2}{m}} \int_{-\infty}^{\infty} d\tilde{y}\; e^{-\tilde{y}^2}$$

$$= \sqrt{\frac{2}{m}} \int_{0}^{\infty} d\tilde{y}^2\; \tilde{y}^{-1} e^{-\tilde{y}^2}$$

$$= \sqrt{\frac{2}{m}} \int_{0}^{\infty} ds\; s^{-1/2} e^{-s}$$

$$= \sqrt{\frac{2}{m}} \times \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2\pi}{m}}$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2 y^2} = \int dx\; e^{-\frac{1}{2}x^2}\int dy\; e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

$$= \int dx\; e^{-\frac{1}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{1+\epsilon x^2}}$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2 y^2} = \int dx\; e^{-\frac{1}{2}x^2}\int dy\; e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

$$= \int dx\; e^{-\frac{1}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{1+\epsilon x^2}}$$

$$= \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}x^2-\frac{1}{2}\log((1+\epsilon x^2))}$$

$$= \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}(1+\epsilon)x^2-O(\epsilon^2 x^4)}$$

textbook renormalization = selective integration + truncation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2y^2} = \int dx\; e^{-\frac{1}{2}x^2}\int dy\; e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

$$= \int dx\; e^{-\frac{1}{2}x^2}\times\frac{\sqrt{2\pi}}{\sqrt{1+\epsilon x^2}}$$

$1:$ bare

$$= \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}x^2-\frac{1}{2}\log((1+\epsilon x^2))}$$

$1+\epsilon:$ renormalized

$$= \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}(1+\epsilon)x^2-O(\epsilon^2x^4)}$$

$$\simeq \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}(1+\epsilon)x^2}$$

textbook renormalization = selective integration + truncation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2 y^2} \quad \simeq \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}(1+\epsilon)x^2}$$

$\epsilon = 1$	$\Rightarrow$	4.96145	4.44288
$\epsilon = 0.1$	$\Rightarrow$	6.02068	5.99078
$\epsilon = 0.01$	$\Rightarrow$	6.25245	6.25200
$\epsilon = 0.001$	$\Rightarrow$	6.28005	6.28005

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{M}{2}x^2-\frac{m}{2}y^2-\frac{\epsilon}{2}x^2y^2} = \int dx\; e^{-\frac{M}{2}x^2}\int dy\; e^{-\frac{m+\epsilon x^2}{2}y^2}$$

$$= \int dx\; e^{-\frac{M}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{m+\epsilon x^2}}$$

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{M}{2}x^2-\frac{1}{2}\log((1+\epsilon/m)x^2))}$$

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2-O((\epsilon^2/m^2)x^4)}$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{M}{2}x^2-\frac{m}{2}y^2-\frac{\epsilon}{2}x^2y^2} = \int dx\; e^{-\frac{M}{2}x^2}\int dy\; e^{-\frac{m+\epsilon x^2}{2}y^2}$$

$$= \int dx\; e^{-\frac{M}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{m+\epsilon x^2}}$$

$M\,:\,$ bare

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{M}{2}x^2-\frac{1}{2}\log((1+\epsilon/m)x^2))}$$

$M+\epsilon/m\,:\,$ renormalized

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2-O((\epsilon^2/m^2)x^4)}$$

textbook renormalization = selective integration + truncation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{M}{2}x^2-\frac{m}{2}y^2-\frac{\epsilon}{2}x^2y^2} = \int dx\; e^{-\frac{M}{2}x^2}\int dy\; e^{-\frac{m+\epsilon x^2}{2}y^2}$$

$$= \int dx\; e^{-\frac{M}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{m+\epsilon x^2}}$$

M : bare

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{M}{2}x^2-\frac{1}{2}\log((1+\epsilon/m)x^2))}$$

$M+\epsilon/m$: renormalized

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2-O((\epsilon^2/m^2)x^4)}$$

$$\simeq \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2}$$

q.f.t. renormalization = selective path-integral

$$\int [d\phi] e^{-\int \mathcal{L}_{renormalized}(\phi)} \equiv \int [d\phi] \int [d\psi] \; e^{-\int \mathcal{L}_{bare}(\phi;\psi)}$$

example : momentum shell integration

$$\int_{p^2<\mu^2} [d\phi] e^{-\int \mathcal{L}_\mu(\phi_\mu)} \equiv \int_{p^2<\Lambda^2} [d\phi]\, e^{-\int \mathcal{L}_\Lambda(\phi_\Lambda)}$$

$$= \int_{p^2<\mu^2} \int_{\mu^2<p^2<\Lambda^2} [d\phi]\, e^{-\int \mathcal{L}_\Lambda(\phi_\Lambda)}$$

example : U(1) gauge theory with massive charged field

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \tilde{D}_\mu \Phi^* \tilde{D}^\mu \Phi + m^2 |\Phi|^2 \right]$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$$

$$\tilde{D}_\mu = \partial_\mu - ig\tilde{A}_\mu$$

$$A_\mu = g\tilde{A}_\mu \quad \downarrow$$

$$D_\mu = \partial_\mu - iA_\mu$$

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2 \right]$$

U(1) gauge theory with massive charged field

$$\int \mathcal{L} = \int dx^4 \ \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]$$

$$A_\mu \quad \rightarrow \quad A_\mu + i\partial_\mu \Theta$$

$$\Phi \quad \rightarrow \quad e^{i\Theta}\Phi$$

$$F_{\mu\nu} \quad \rightarrow \quad F_{\mu\nu}$$

$$(\partial_\mu - iA_\mu)\Phi \quad \rightarrow \quad e^{i\Theta}(\partial_\mu - iA_\mu)\Phi$$

U(1) gauge theory with massive charged field

$$\int [dA][d\Phi][d\Phi^*] \; e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$\simeq \int [dA] e^{-\int \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}} \times \frac{1}{\mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)}$$

$$\int \Delta\mathcal{L}(m,\Lambda) = \log \mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)$$

$$= \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= \int [dA] e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \Delta\mathcal{L}(F; m, \Lambda)\right]}$$

functional determinant and functional trace

$$Q \equiv -(\partial - iA)^2 + m^2 \qquad\qquad Q|\psi_n\rangle = \lambda_n|\psi_n\rangle$$

$$\mathrm{Det}Q \equiv \prod_n \lambda_n$$

$$\log \mathrm{Det}Q = \log\left(\prod_n \lambda_n\right) = \sum_n \log(\lambda_n) = \mathrm{Tr}\log(Q)$$

log of determinant = trace of log

$$\log \mathrm{Det} Q = \mathrm{Tr} \log Q$$

$$
\begin{aligned}
-\int_{1/\Lambda^2}^{\infty} \frac{ds}{s}\; e^{-a\cdot s} \qquad &= -\int_{a/\Lambda^2}^{\infty} \frac{d(a\cdot s)}{a\cdot s}\; e^{-a\cdot s} \\
&= -\int_{a/\Lambda^2}^{\infty} \frac{dy}{y}\; e^{-y} \\
&= -\log(y) e^{-y}\Big|_{a/\Lambda^2}^{\infty} + \int_{a/\Lambda^2}^{\infty} dy\; \log(y)\; e^{-y} \\
&= +\log(a/\Lambda^2) + O(a/\Lambda^2)
\end{aligned}
$$

log of functional determinant of operator
= functional trace of log of operator

$$\log \det Q = \operatorname{tr} \log Q$$

$$= -\mathrm{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-sQ\right]$$

$$= -\sum_n \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \left\langle \psi_n \right| \exp\left[-sQ\right] \left| \psi_n \right\rangle$$

$$= -\int dx^d \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \left\langle x \right| \exp\left[-sQ\right] \left| x \right\rangle$$

U(1) gauge theory with massive charged field

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= -\mathrm{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-s(-(\partial - iA)^2 + m^2)\right]$$

$$= -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int dx^4 \, \langle x| \exp\left[-s(-(\partial - iA)^2 + m^2)\right] |x\rangle$$

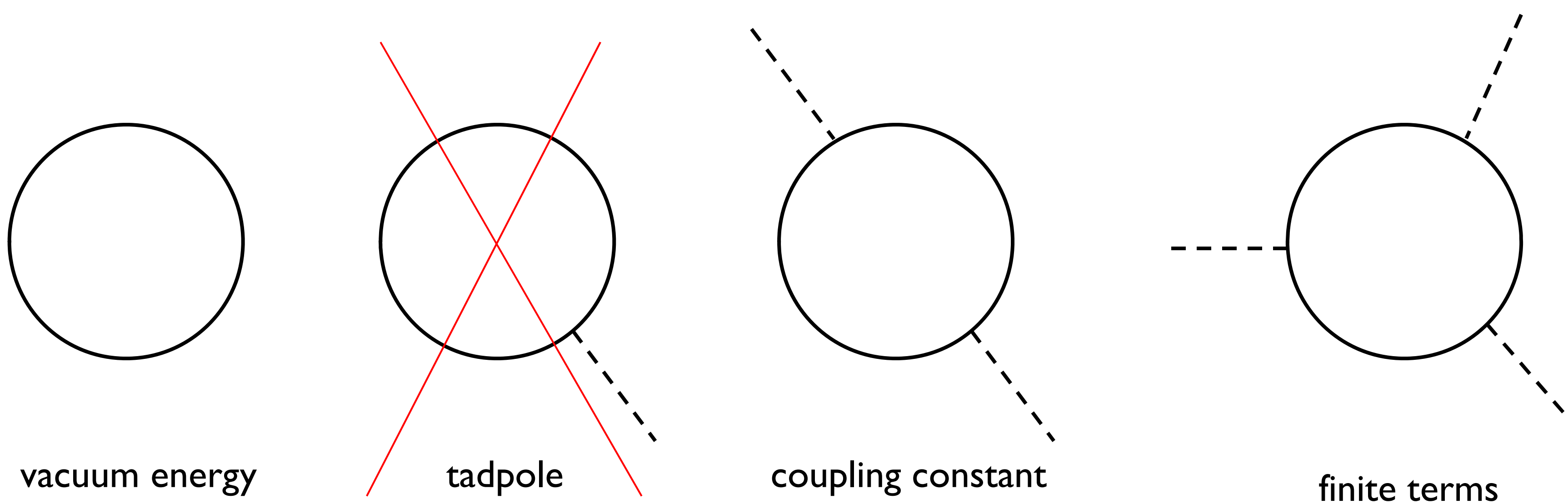
$$= -\frac{1}{16\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^3} \, e^{-m^2 s} \int dx^4$$

$$+ \frac{1}{192\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \, e^{-m^2 s} \times \int dx^4 F_{\mu\nu} F^{\mu\nu} \quad + \cdots$$

U(1) gauge theory with massive charged field

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= \Lambda^4 a(m/\Lambda) + b(m/\Lambda)\frac{1}{4}F^2 + \frac{c(m/\Lambda)}{\Lambda^2}F^3 + \cdots$$



vacuum energy renormalization

tadpole cancellation

coupling constant (also wavefunction-) renormalization

finite terms

U(1) gauge theory with massive charged field

$$\Lambda^{4-n} I_n(m/\Lambda) \equiv \int_{1/\Lambda^2}^{\infty} ds\ s^{n/2-3}\ e^{-m^2 s} = \Lambda^{4-n} \int_1^{\infty} d\tilde{s}\ \tilde{s}^{n/2-3}\ e^{-(m^2/\Lambda^2)\tilde{s}}$$

$$I_0(m/\Lambda) = 2\left(1 - m^2/\Lambda^2 + m^4/\Lambda^4 I_4(m/\Lambda)\right)$$

$$I_4(m/\Lambda) \simeq \begin{cases} -\gamma - \log(m^2/\Lambda^2) + O(m^2/\Lambda^2) & \text{when } m^2 \ll \Lambda^2 \\ e^{-m^2/\Lambda^2} & \text{when } m^2 \gg \Lambda^2 \end{cases}$$

U(1) gauge theory with massive charged field

$$\int [dA] e^{-W_{eff}(A)} \equiv \int [dA][d\Phi][d\Phi^*] \; e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$W_{eff} = \int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \Delta\mathcal{L}(F; m, \Lambda)\right]$$

$$= \int \left[\cdots + \left(\frac{1}{4g^2} + \frac{1}{192\pi^2} I_4(m/\Lambda)\right) F_{\mu\nu}F^{\mu\nu} + \cdots\right]$$

$$= \frac{1}{4g^2(0; m, \Lambda)_{ren}}$$

U(1) gauge theory with massive charged fields

$$\frac{1}{4g^2(0;m,\Lambda)_{ren}} = \frac{1}{4g^2} + \frac{1}{192\pi^2} I_4(m_{scalar}/\Lambda)$$

$$+ \frac{4}{192\pi^2} I_4(m_{spinor}/\Lambda)$$

U(1) gauge theory with massive charged fields

more generally we may wish to integrate out partially, say, $\mu^2 < p^2 < \Lambda^2$

$$\int_{p^2<\mu^2} [dA][d\Phi][d\Phi^*][\cdots] e^{-\int \frac{1}{4g^2_{ren}(\mu;m,\Lambda)} F^2+\cdots}$$

$$\equiv \int_{p^2<\Lambda^2} [dA][d\Phi][d\Phi^*][\cdots]\; e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}+D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2+\cdots\right]}$$

U(1) gauge theory with massive charged fields

more generally we may wish to integrate out partially, say, $\mu^2 < p^2 < \Lambda^2$

$$\frac{1}{4g^2(\mu; m, \Lambda)_{ren}} = \frac{1}{4g^2} + \frac{1}{192\pi^2}\tilde{I}_4(m_{scalar}; \mu, \Lambda) + \frac{4}{192\pi^2}\tilde{I}_4(m_{spinor}; \mu, \Lambda)$$

$$\tilde{I}_4(m; \mu, \Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\, e^{-m^2 s}$$

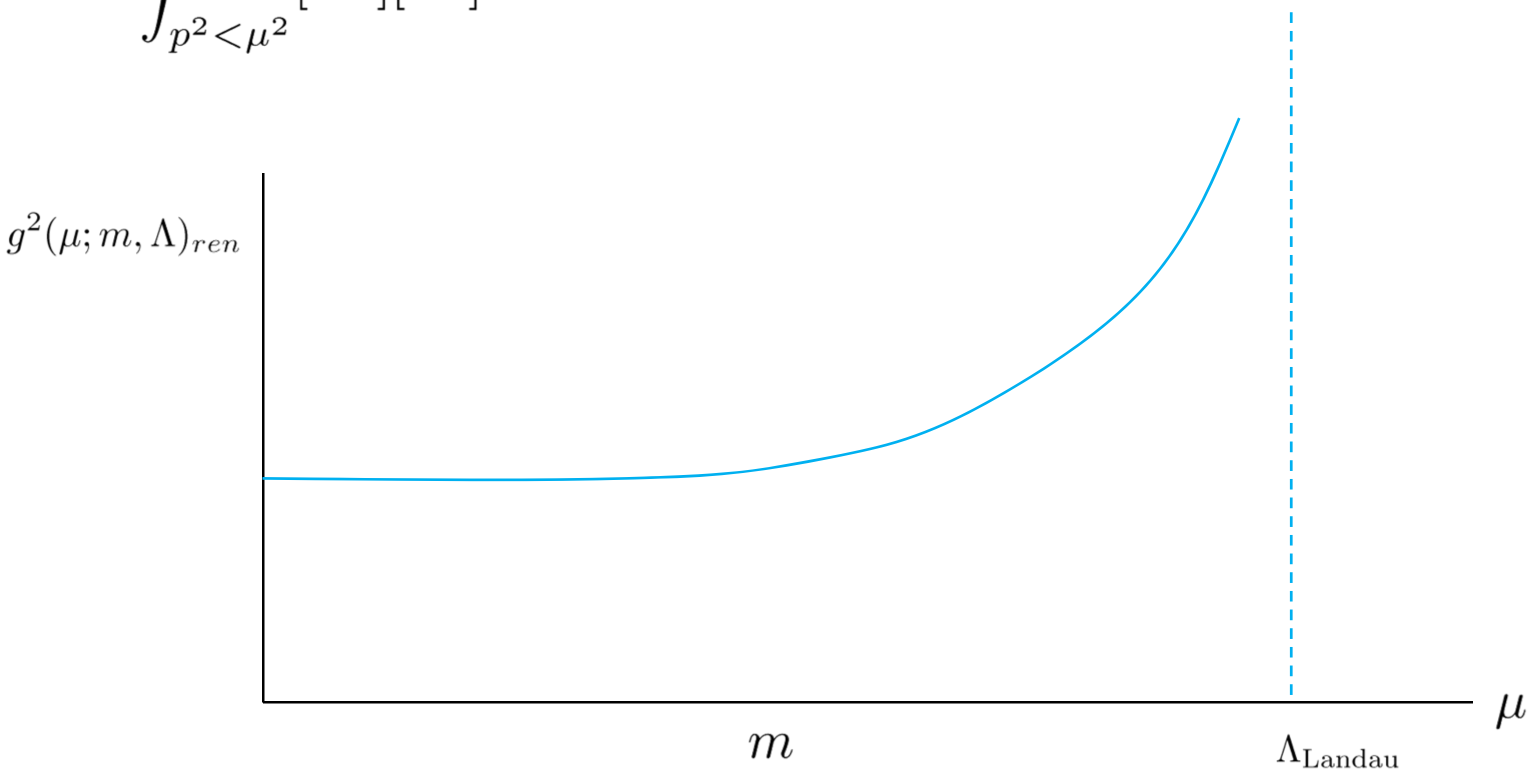
$$\sim -\log(m^2/\Lambda^2) \quad \text{when } \mu^2 < m^2 < \Lambda^2$$

$$\sim -\log(\mu^2/\Lambda^2) \quad \text{when } m^2 < \mu^2 < \Lambda^2$$

U(1) gauge theory with massive charged fields

$$\int_{p^2<\Lambda^2} [dA][\cdots]\, e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \cdots\right]}$$

$$= \int_{p^2<\mu^2} [dA][\cdots] e^{-\int \left[\cdots + \frac{1}{4g(\mu;m,\Lambda)^2_{ren}} F_{\mu\nu}F^{\mu\nu} + \cdots\right]}$$



functional Gaussian integral via heat kernel

U(1) gauge theory with a massive charged scalar

$$\begin{aligned}
\mathrm{Tr}_\Lambda \log Q &= -\mathrm{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-sQ\right] \\
&= -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int dx^4 \, \langle x| \exp\left[-sQ\right] |x\rangle \\
&= -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int dx^4 \, G_s(x;x)
\end{aligned}$$

$$G_s(x;y) \equiv \langle x|e^{-sQ}|y\rangle$$

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y)$$

U(1) gauge theory with a massive charged scalar

$$\mathrm{Tr}_\Lambda \log Q = -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int dx^4\, G_s(x;x)$$

$$G_s(x;y) \equiv \langle x|e^{-sQ}|y\rangle \qquad -\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \ = \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \; = \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$-\frac{\partial}{\partial\beta}G_\beta^{(0)}(x;y) = Q^{(0)}G_\beta^{(0)}(x;y)$$

$$G_\beta^{(0)}(x;y) = \langle x|e^{\beta\partial^2}|y\rangle = \frac{1}{(4\pi\beta)^{d/2}}e^{-(x-y)^2/4\beta}e^{-\beta m^2} \quad \text{for } R^d$$

$$\lim_{\beta\to 0} G_\beta^{(0)}(x;y) = \delta(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial \beta} G_\beta(x;y) = Q G_\beta(x;y) \;\; = \left(Q^{(0)} + Q^{(1)}\right) G_\beta(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$-\frac{\partial}{\partial \beta} G_\beta^{(n+1)} = Q^{(0)} G_\beta^{(n+1)} + Q^{(1)} G_\beta^{(n)}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = Q^{(0)}G_\beta^{(n+1)}(x;y) + Q^{(1)}G_\beta^{(n)}(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = Q^{(0)}G_\beta^{(n+1)}(x;y) + Q^{(1)}G_\beta^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = Q^{(0)}G_\beta^{(n+1)}(x;y) + Q^{(1)}G_\beta^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \; \frac{\partial}{\partial\beta}G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$- \lim_{s\to\beta}\int_z \; G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = Q^{(0)}G_\beta^{(n+1)}(x;y) + Q^{(1)}G_\beta^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \; \frac{\partial}{\partial\beta}G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$-Q^{(1)}(x)G_\beta^{(n)}(x;y)$$

$$\lim_{\beta\to 0} G_\beta^{(0)}(x;y) = \delta(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = Q^{(0)}G_\beta^{(n+1)}(x;y) + Q^{(1)}G_\beta^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = -Q^{(0)}\left[-\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)\right]$$

$$-Q^{(1)}(x)G_\beta^{(n)}(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = Q^{(0)}G_\beta^{(n+1)}(x;y) + Q^{(1)}G_\beta^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = -Q^{(0)}\left[-\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)\right]$$

$$-Q^{(1)}(x)G_\beta^{(n)}(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \ = \left(Q^{(0)} + Q^{(1)}\right) G_\beta(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \, G^{(0)}_{\beta-s}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$= (-1)^n \int_0^\beta ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \int_{z_1} \cdots \int_{z_n}$$

$$G^{(0)}_{\beta-s_1}(x;z_1)Q^{(1)}G_{s_1-s_2}(z_1;z_2)\cdots Q^{(1)}G^{(0)}_{s_n}(z_n;y)$$

heat kernel expansion: β power counting

1. each $G^{(0)}$ → $\beta^{-d/2}$
2. each x-integral → $\beta^{d/2}$
3. each s-integral → β
4. each derivative → $\beta^{-1/2}$
5. each x → $\beta^{1/2}$

$[x]^a[\partial]^b \text{ in } Q^{(1)}$ → $\beta^{1+(a-b)/2}$ at each iteration

heat kernel expansion for U(1) R.G.

$$Q \equiv -(\partial - iA)^2 + m^2$$

$$= -\partial^2 + m^2 + 2iA\cdot\partial + i(\partial\cdot A) + A^2$$

$$\simeq -\partial^2 + m^2 + iF_{\mu\nu}x^\mu\partial^\nu + \frac{1}{4}F_{\sigma\mu}F^{\sigma\nu}x^\mu x_\nu$$

$$Q^{(0)} \qquad Q^{(1)}$$

$$A_\mu \to \frac{1}{2}F_{\alpha\mu}x^\alpha$$

if derivative of field strength can be ignored or is irrelevant

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;\; = \left(Q^{(0)} + Q^{(1)}\right) G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G^{(0)}_\beta(x;y) = \frac{1}{(4\pi\beta)^{d/2}} e^{\,-(x-y)^2/4\beta} e^{-\beta m^2}$$

$$G^{(1)}_\beta(x;y) = -\int_0^\beta ds \int_z \; G^{(0)}_{\beta-s}(x;z) Q^{(1)} G^{(1)}_s(z;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;= \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G^{(1)}_\beta(x;y) \;= -\int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d z \; e^{-(x-z)^2/4(\beta-s)} Q^{(1)} e^{-(z-y)^2/4s} \right\}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \ = \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;y) \ = -\int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d z \ e^{-(x-z)^2/4(\beta-s)} Q^{(1)} e^{-(z-y)^2/4s} \right\}$$

we must now decide what is the most useful decomposition of Q

→ we will eventually take $G_s(x;x)$ for determinant,
so Q should be expanded around this position

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;= \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;x) \;= -\int_0^\beta ds\left\{\frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}}\int d^dz\; e^{-(x-z)^2/4(\beta-s)}Q^{(1)}\Big|_z e^{-(z-x)^2/4s}\right\}$$

$$= \int_0^\beta ds\left\{\frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}}\int d^dz\;\left[\frac{1}{2s}b_{\mu\nu}(z) - c_{\mu\nu}(z)\right](z-x)^\mu(z-x)^\nu e^{-\beta(z-x)^2/4s(\beta-s)}\right\}$$

$$= \int_0^\beta ds\left\{\frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}}\int d^d\tilde{z}\;\left[\frac{1}{2s}b_{\mu\nu}(z) - c_{\mu\nu}(z)\right]\Big|_{z=\tilde{z}+x}\tilde{z}^\mu\tilde{z}^\nu e^{-\beta\tilde{z}^2/4s(\beta-s)}\right\}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;\; = \left(Q^{(0)} + Q^{(1)}\right) G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;x) \;\; = -\int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d z \; e^{-(x-z)^2/4(\beta-s)} Q^{(1)}\Big|_z e^{-(z-x)^2/4s} \right\}$$

$$= \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d\tilde{z} \; \left[\frac{1}{2s}b_{\mu\nu}(z) - c_{\mu\nu}(z)\right]\Big|_{z=\tilde{z}+x} \tilde{z}^\mu \tilde{z}^\nu e^{-\beta\tilde{z}^2/4s(\beta-s)} \right\}$$

$$= \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d\tilde{z} \left[\frac{1}{2s}b_{\mu\nu} - c_{\mu\nu}\right]\Big|_x \delta_{\mu\nu}\frac{\tilde{z}^2}{d} e^{-\tilde{z}^2\beta/4s(\beta-s)} \right\} + O(\beta^2\partial\partial b, \beta^3\partial\partial c)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;\; = \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;x) \;\simeq \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d\tilde{z} \left[\frac{1}{2s}b_{\mu\nu} - c_{\mu\nu}\right]\Big|_x \delta_{\mu\nu}\frac{\tilde{z}^2}{d}e^{-\tilde{z}^2\beta/4s(\beta-s)} \right\}$$

$$\simeq \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \int d^d\tilde{z}\; \frac{\tilde{z}^2}{d}e^{-\tilde{z}^2\beta/4s(\beta-s)} \right\}$$

$$\simeq \int_0^\beta ds \left\{ \frac{4^{d/2+1}((\beta-s)s)e^{-\beta m^2}}{(4\pi)^d\beta^{d/2+1}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \int d^d\tilde{z}\; \frac{\tilde{z}^2}{d}e^{-\tilde{z}^2} \right\}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;= \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G^{(1)}_\beta(x;x) \;\simeq \int_0^\beta ds \left\{ \frac{4^{d/2+1}((\beta-s)s)e^{-\beta m^2}}{(4\pi)^d \beta^{d/2+1}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \int d^d\tilde{z}\; \frac{\tilde{z}^2}{d} e^{-\tilde{z}^2} \right\}$$

$$\simeq \int_0^\beta ds \left\{ \frac{4^{d/2+1}(\beta-s)se^{-\beta m^2}}{(4\pi)^d \beta^{d/2+1}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \frac{\pi^{d/2}}{2} \right\}$$

$$\simeq \frac{e^{-\beta m^2}}{(4\pi\beta)^{d/2}} \left[\int_0^\beta ds \left[\frac{\beta-s}{\beta}b_\mu{}^\mu - \frac{2s(\beta-s)}{\beta}c_\mu{}^\mu\right]\Big|_x\right]$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;\; = \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;x) \;= \frac{e^{-\beta m^2}}{(4\pi\beta)^{d/2}}\left[\frac{\beta}{2}b_\mu{}^\mu(x) - \frac{\beta^2}{3}c_\mu{}^\mu(x)\right] \; + O(\beta^2\partial\partial b, \beta^3\partial\partial c)$$

heat kernel expansion for U(1) R.G.

$$Q \equiv -(\partial - iA)^2 + m^2$$

$$= -\partial^2 + m^2 + 2iA\cdot\partial + i(\partial\cdot A) + A^2$$

$$\simeq -\partial^2 + m^2 + iF_{\mu\nu}x^\mu\partial^\nu + \frac{1}{4}F_{\sigma\mu}F^{\sigma\nu}x^\mu x_\nu$$

$Q^{(0)}$ $\quad$ $Q^{(1)}$

$$A_\mu \to \frac{1}{2}F_{\alpha\mu}x^\alpha$$

if derivative of field strength can be ignored or is irrelevant after one more iteration

→ one-loop R.G. of U(1) gauge theory with massive charged scalar

$$\int \Delta\mathcal{L}(F;m,\Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= -\mathrm{Tr}\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-s(-(\partial - iA)^2 + m^2)\right]$$

$$= -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int dx^4 \, \langle x| \exp\left[-s(-(\partial - iA)^2 + m^2)\right] |x\rangle$$

$$= -\frac{1}{16\pi^2}\int_{1/\Lambda^2}^{\infty} \frac{ds}{s^3}\, e^{-m^2 s} \int dx^4$$

$$+ \frac{1}{192\pi^2}\int_{1/\Lambda^2}^{\infty} \frac{ds}{s}\, e^{-m^2 s} \times \int dx^4 F_{\mu\nu}F^{\mu\nu} \quad + \cdots$$

excursion: quantum one-loop as a functional Gaussian integral

how to derive the Asymptotic Freedom, or Yang-Mills beta function

Yang-Mills theory renormalization & asymptotic freedom

$$\int \mathcal{L} = \int dx^4 \; \left[\frac{1}{2}\mathrm{tr}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right]$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - ig[\tilde{A}_\mu, \tilde{A}_\nu]$$

$$A_\mu = g\tilde{A}_\mu \quad \downarrow$$

$$\int \mathcal{L} = \int dx^4 \; \left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \;\left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A = A^\dagger \qquad B = B^\dagger$$

$$(i[A,B])^\dagger = i^*(AB - BA)^\dagger = -i(B^\dagger A^\dagger - A^\dagger B^\dagger) = -i[B,A] = i[A,B]$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

a canonical example

$$A_\mu = \sum_{a=1,2,3} A^a_\mu \frac{\sigma^a}{2}$$

$$F_{\mu\nu} = \sum_{a=1,2,3} F^a_{\mu\nu} \frac{\sigma^a}{2}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\mu A^c_\nu$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \; \left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

more generally

$$A_\mu = \sum_a A_\mu^a T^a$$

$$F_{\mu\nu} = \sum_a F_{\mu\nu}^a T^a$$

$$[T^a, T^b] = f_{abc} T^c$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - i f_{abc} A_\mu^b A_\nu^c$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \;\left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad\qquad A_\mu \quad \rightarrow \quad U^\dagger A_\mu U + iU^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad\qquad F_{\mu\nu} \quad \rightarrow \quad U^\dagger F_{\mu\nu} U$$

$$U \in \mathcal{U}(N)$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad\qquad A_\mu \quad \rightarrow \quad U^\dagger A_\mu U + i U^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad\qquad F_{\mu\nu} \quad \rightarrow \quad U^\dagger F_{\mu\nu} U$$

$$\mathrm{tr} F_{\mu\nu} = 0 = \mathrm{tr} A_\mu \qquad\qquad U \in \mathcal{SU}(N)$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad A_\mu \quad \rightarrow \quad U^\dagger A_\mu U + iU^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad F_{\mu\nu} \quad \rightarrow \quad U^\dagger F_{\mu\nu} U$$

$$A_\mu^T I = I A_\mu \qquad U \in \mathcal{O}(N)$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad\qquad A_\mu \quad \to \quad U^\dagger A_\mu U + iU^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad\qquad F_{\mu\nu} \quad \to \quad U^\dagger F_{\mu\nu} U$$

$$A_\mu^T I = I A_\mu \qquad\qquad U \in \mathcal{SO}(N)$$

$$\mathrm{tr}F_{\mu\nu} = 0 = \mathrm{tr}A_\mu$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad A_\mu \quad \rightarrow \quad U^\dagger A_\mu U + i U^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad F_{\mu\nu} \quad \rightarrow \quad U^\dagger F_{\mu\nu} U$$

$$A_\mu^T J = J A_\mu \qquad U \in \mathcal{SP}(N/2) = USp(N)$$

$$J^T = -J, \quad J^2 = -1$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \ \left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$$A_\mu = \sum_a A^a_\mu T^a$$

$$F_{\mu\nu} = \sum_a F^a_{\mu\nu} T^a$$

$$[T^a, T^b] = f_{abc} T^c$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - i f_{abc} A^b_\mu A^c_\nu$$

Yang-Mills theory with matter fields

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2 + \cdots \right]$$

$$D_\mu \Phi^k = \partial_\mu \Phi^k - i A_\mu^a (t^a)^k{}_l \Phi^l$$

$$(t^a)^k{}_l, \quad k, l = 1, \ldots, n$$

$$[t^a, t^b] = f_{abc} t^c$$

for example $\mathcal{SU}(2) = \mathcal{SP}(1)$

$$t^a_{(s)} = J^a_{spin=s}$$

$$s = 0, 1/2, 1, 3/2, \ldots$$

$$n = 1, 2, 3, 4, \ldots$$

Yang-Mills theory with matter fields

$$\int [dA][d\Phi][d\Phi^*]\ e^{-\int \left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}+D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}} \times \int [d\Phi][d\Phi^*]\ e^{-\int\left[D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}} \times \frac{1}{\mathrm{Det}(-(\partial - iA)^2 + m^2)}$$

$$= \int [dA] e^{-\int \left[\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+\Delta\mathcal{L}(F;m,\Lambda)\right]}$$

$$\int \Delta\mathcal{L}(F;m,\Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

Yang-Mills theory with matter fields

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= -\mathrm{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-s(-(\partial - iA)^2 + m^2)\right]$$

$$= - \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \, \mathrm{tr} \int dx^4 \, \langle x | \exp\left[-s(-(\partial - iA)^2 + m^2)\right] | x \rangle$$

$$= -\frac{1}{16\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^3} \, e^{-m^2 s} \times n$$

$$+ \frac{1}{192\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \, e^{-m^2 s} \times F_{\mu\nu}F^{\mu\nu} \times \mathcal{T}_2(t) \quad + \cdots$$

heat kernel expansion for U(1) R.G.

$$Q_{scalar} \equiv -(\partial - iA)^2 + m^2$$

$$= \underbrace{(-\partial^2 + m^2)}_{Q^{(0)}_{scalar}} + \underbrace{2iA \cdot \partial + i(\partial \cdot A) + A^2}_{Q^{(1)}_{scalar}}$$

$$Q_{spinor} \equiv -\left[\gamma^\mu(\partial_\mu - iA_\mu)\right]^2 + m^2$$

$$= 1_{2^{d/2} \times 2^{d/2}} \times \left[-(\partial - iA)^2 + m^2\right] + iF_{\mu\nu}\gamma^{\mu\nu}$$

$$= \underbrace{1_{2^{d/2} \times 2^{d/2}} Q^{(0)}_{scalar}}_{Q^{(0)}_{spinor}} + \underbrace{1_{2^{d/2} \times 2^{d/2}} Q^{(1)}_{scalar} + iF \cdot \gamma}_{Q^{(1)}_{spinor}}$$

heat kernel expansion for Yang-Mills R.G.

$$Q_{scalar} \equiv -(1_{n\times n}\partial - iA)^2 + m^2 1_{n\times n}$$

$$= \underbrace{1_{n\times n}(-\partial^2 + m^2)}_{Q^{(0)}_{scalar}} + \underbrace{2iA\cdot\partial + i(\partial\cdot A) + A^2}_{Q^{(1)}_{scalar}}$$

$$Q_{spinor} \equiv -\left[\gamma^\mu(\partial_\mu - iA_\mu)\right]^2 + m^2$$

$$= 1_{2^{d/2}\times 2^{d/2}} \times \left[-(1_{n\times n}\partial - iA)^2 + m^2 1_{n\times n}\right] + iF_{\mu\nu}\gamma^{\mu\nu}$$

$$= \underbrace{1_{2^{d/2}\times 2^{d/2}} Q^{(0)}_{scalar}}_{Q^{(0)}_{spinor}} + \underbrace{1_{2^{d/2}\times 2^{d/2}} Q^{(1)}_{scalar} + iF\cdot\gamma}_{Q^{(1)}_{spinor}}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;= \left(Q^{(0)} + Q^{(1)}\right)G_\beta(x;y)$$

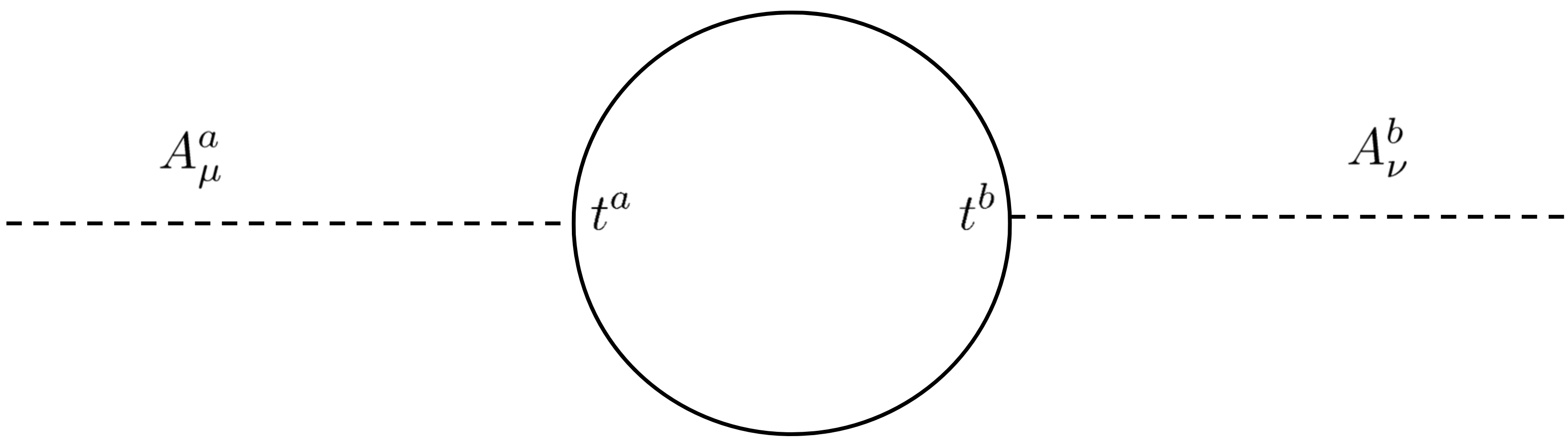
$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$G_\beta^{(n)}(x;y) = -\int_0^\beta ds \int_z \; G^{(0)}_{\beta-s}(x;z)Q^{(1)}G_s^{(n-1)}(z;y)$$

$$= (-1)^n \int_0^\beta ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \int_{z_1} \cdots \int_{z_n}$$

$$G^{(0)}_{\beta-s_1}(x;z_1)Q^{(1)}G_{s_1-s_2}(z_1;z_2)\cdots Q^{(1)}G^{(0)}_{s_n}(z_n;y)$$

Yang-Mills theory with matter fields



$$\mathrm{tr}\ t^a t^b = \delta^{ab} \mathcal{T}_2(t)$$

Yang-Mills theory with matter fields

$$W_{eff} = \cdots + \int \frac{1}{2g_{ren}^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{2g^2(m, \Lambda; \mu)_{ren}} = \frac{1}{2g^2} + \frac{1}{96\pi^2} \tilde{I}_4(m_{scalar}; \mu, \Lambda) \mathcal{T}_2(t_{scalar})$$

$$+ \frac{4}{96\pi^2} \tilde{I}_4(m_{spinor}; \mu, \Lambda) \mathcal{T}_2(t_{spinor})$$

$$\tilde{I}_4(m; \mu, \Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s} \, e^{-m^2 s}$$

Yang-Mills theory with matter fields

Yang-Mills fields interact among themselves,
so there are additional contribution

$$\int_{p^2<\mu^2} [dA][d\Phi][d\Phi^*][\cdots] e^{-W_{eff}(A,\cdots)} \equiv$$

$$\int_{p^2<\Lambda^2} [dA][d\Phi][d\Phi^*] \; e^{-\int \left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi^* D^\mu\Phi + m^2|\Phi|^2\right]}$$

Yang-Mills theory with matter fields

Yang-Mills fields interact among themselves,
so there are additional contribution

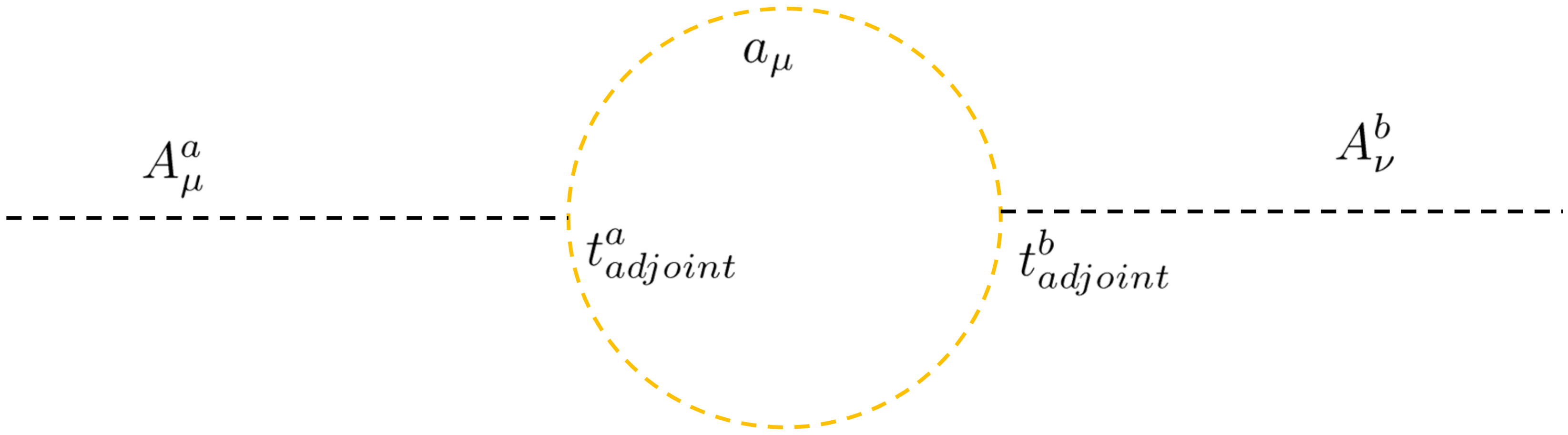
$$\int \mathcal{L} = \int dx^4 \ \left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right]$$

$$\Big\downarrow \quad A_\mu \ \to \ A_\mu + a_\mu$$

$$\int dx^4 \ \left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right] \ + \ \left[\frac{1}{2g^2}\mathrm{tr}(D_\mu a_\nu - D_\nu a_\mu)^2 + \cdots\right]$$

Yang-Mills theory with matter fields

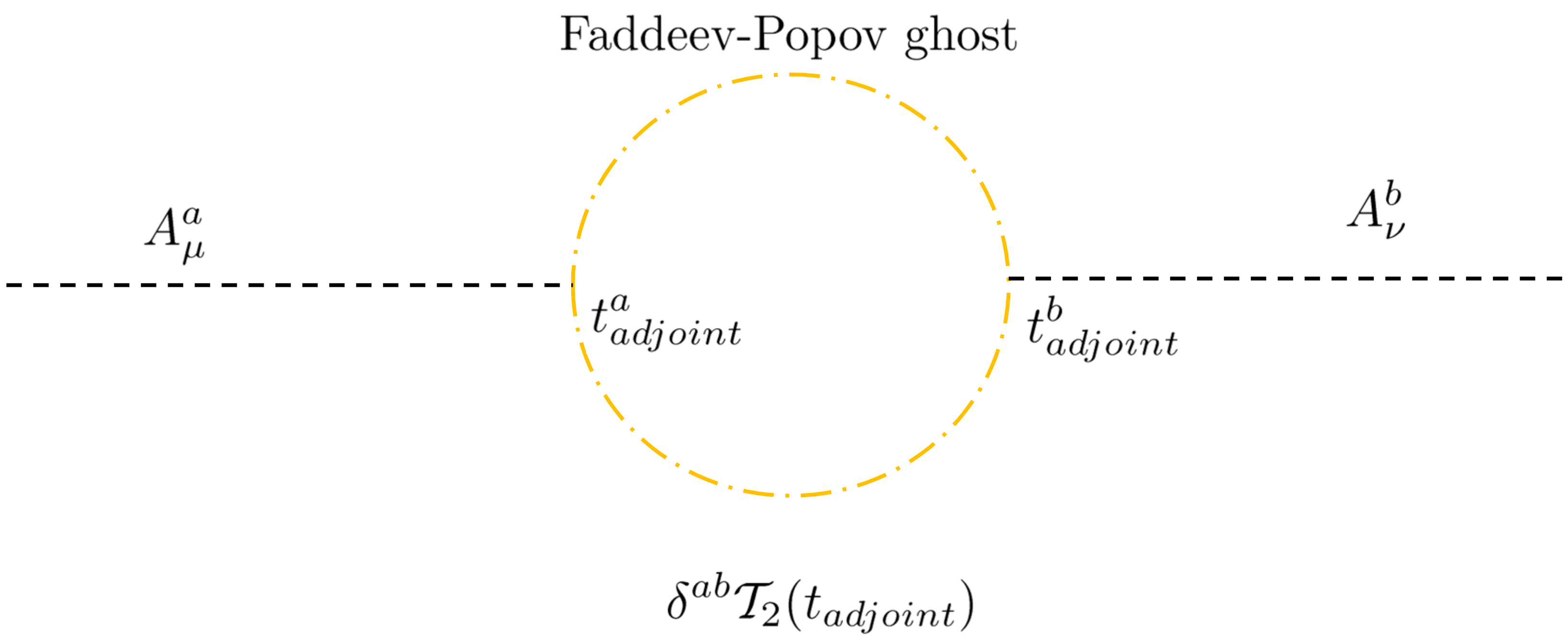
Yang-Mills fields interact among themselves,
so there are additional contribution



$$\delta^{ab}\mathcal{T}_2(t_{adjoint})$$

Yang-Mills theory with matter fields

gauge-fixing introduces further contribution from Faddeev-Popov ghost, which is like minus of a single complex scalar in the adjoint representation



Yang-Mills theory with matter fields

$$W_{eff} = \cdots + \int \frac{1}{2g^2_{ren}} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{2g^2(m,\Lambda;\mu)_{ren}} = \frac{1}{2g^2} + \frac{1}{96\pi^2} \tilde{I}_4(m_{scalar};\mu,\Lambda)\mathcal{T}_2(t_{scalar})$$

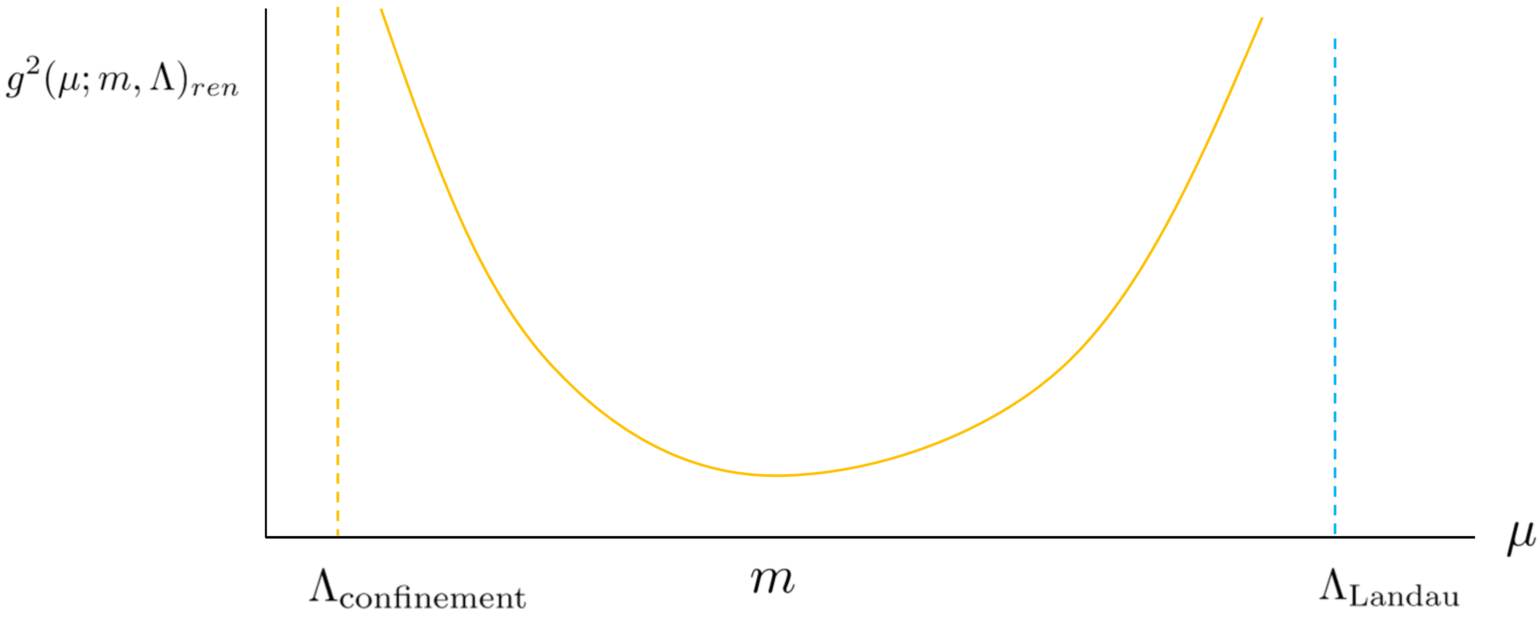
$$+ \frac{4}{96\pi^2} \tilde{I}_4(m_{spinor};\mu,\Lambda)\mathcal{T}_2(t_{spinor})$$

$$- \frac{11}{96\pi^2} \tilde{I}_4(0;\mu;\Lambda)\mathcal{T}_2(t_{adjoint})$$

$$\tilde{I}_4(m;\mu,\Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\ e^{-m^2 s}$$

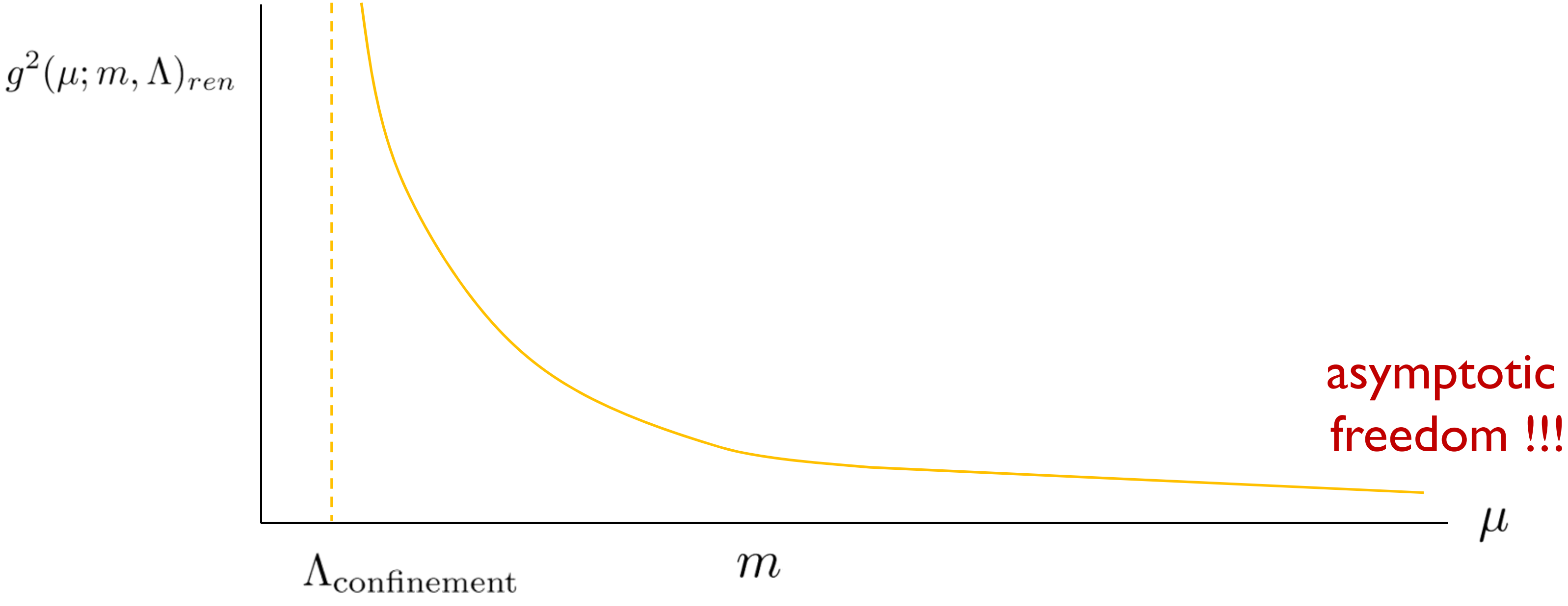
Yang-Mills theory with matter fields

$$\sum \mathcal{T}_2(t_{scalar}) + 4\sum \mathcal{T}_2(t_{spinor}) > 11\mathcal{T}_2(t_{adjoint})$$



Yang-Mills theory with matter fields

$$\sum \mathcal{T}_2(t_{scalar}) + 4\sum \mathcal{T}_2(t_{spinor}) < 11\mathcal{T}_2(t_{adjoint})$$



renormalization is selective path-integral

$$\int [dA][d\Phi][d\Phi^*] \; e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}} \times \int [d\Phi][d\Phi^*] \; e^{-\int \left[D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$\downarrow \quad \int \Delta\mathcal{L}(m, \Lambda) = \log \mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)$$

$$= \int [dA] e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \Delta\mathcal{L}(F; m, \Lambda)\right]}$$

usual textbook renormalization is selective path-integral + truncation

$$\int [dA][d\Phi][d\Phi^*]\; e^{-\int \left[\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}} \times \int [d\Phi][d\Phi^*]\; e^{-\int\left[D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2\right]}$$

$$\int \Delta\mathcal{L}(m,\Lambda) = \log \mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)$$

$$\simeq \#\log(\Lambda/m)F_{\mu\nu}F^{\mu\nu} + \cdots$$

$$\simeq \int [dA] e^{-\int\left[\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+\#\log(\Lambda/m)F_{\mu\nu}F^{\mu\nu}\right]}$$

but there is far more to such renormalization processes than mere replacement of numbers

$$\int [dA][d\Phi][d\Phi^*]\ e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}} \times \int [d\Phi][d\Phi^*]\ e^{-\int \left[D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$\int \Delta\mathcal{L}(m,\Lambda) = \log \mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)$$

$$= \int [dA] e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \Delta\mathcal{L}(F;m,\Lambda)\right]}$$

for example, what kind of things happen if quantum matter "renormalize" geometry?

$$\int [dg][d\Phi][d\Phi^*]\; e^{\int \sqrt{g}\left[\frac{1}{16\pi G_N} R(g) - D_\mu \Phi^* D^\mu \Phi - m^2 |\Phi|^2\right]}$$

$$= \int [dg] e^{\int \frac{1}{16\pi G_N} \sqrt{g} R(g)} \times \int [d\Phi][d\Phi^*]\; e^{-\int \sqrt{g}\left[D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

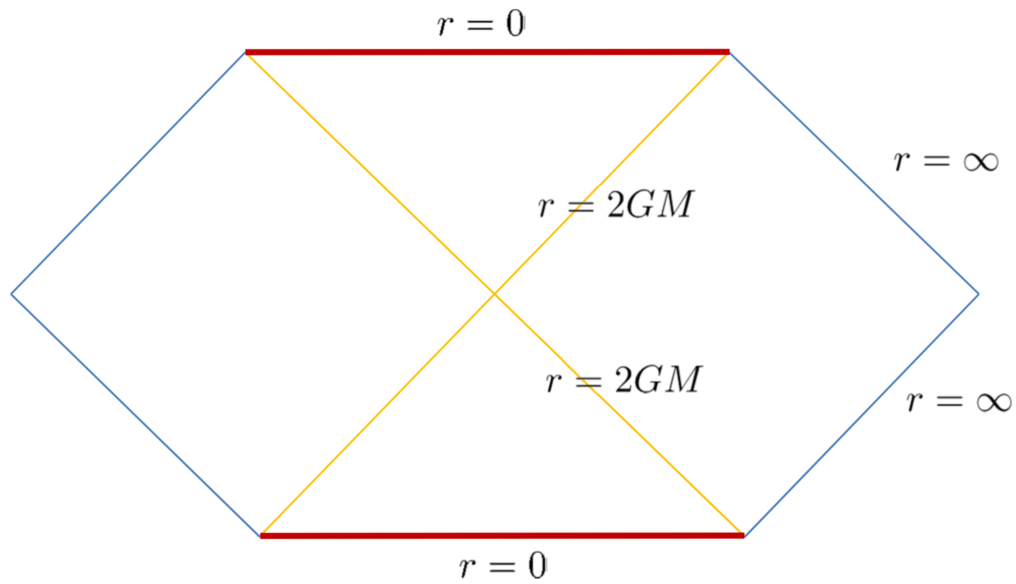
$$\int \Delta \mathcal{L}(g; m, \Lambda) = \log \mathrm{Det}_\Lambda(-\nabla^2 + m^2)$$

$$= \int [dg] e^{\int \left[\frac{1}{16\pi G_N} \sqrt{g} R(g) - \Delta \mathcal{L}(g; m, \Lambda)\right]}$$

2d Weyl anomaly and s-wave Hawking radiation

Schwarzschild black holes

$$g^{(4)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right]$$



Schwarzschild black holes

$$g^{(4)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right]$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$0 = \delta \int dx^4 \; \frac{1}{16\pi G} \sqrt{-g^{(4)}} R^{(4)}$$

Schwarzschild black holes

$$e^{-2\Phi} = r^2 \qquad g^{(2)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

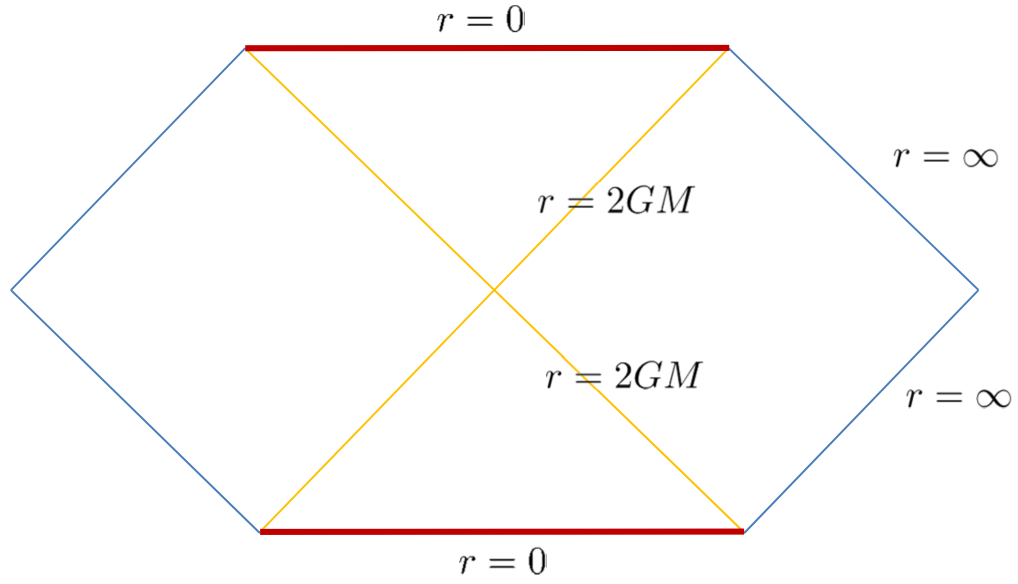
$$0 = \delta \int dx^2 \frac{1}{16\pi G} \sqrt{-g^{(2)}} e^{-2\Phi} \left[R^{(2)} + 2(\nabla\Phi)^2 + 2e^{2\Phi}\right]$$

$$g^{(4)} = g^{(2)} + e^{-2\Phi} \left[d\theta^2 + \sin^2\theta d\phi^2\right]$$

$$0 = \delta \int dx^4 \, \frac{1}{16\pi G} \sqrt{-g^{(4)}} R^{(4)}$$

Schwarzschild black holes

$$g^{(2)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

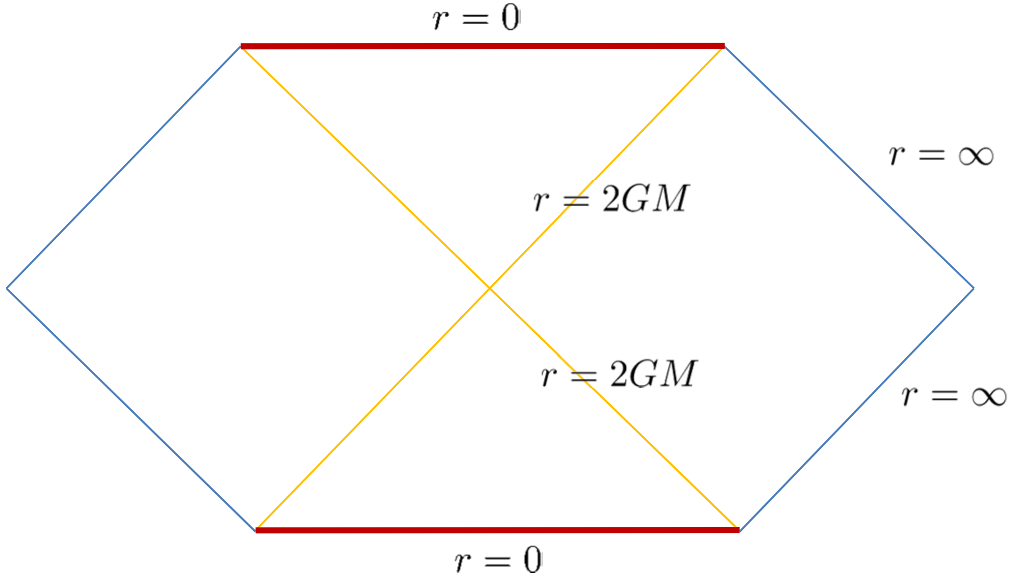


Schwarzschild black holes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dt^2 + dr_*^2\right]$$

$$r_* = \int \frac{r}{r - r_H} dr$$

$$= r + r_H \log(r/r_H - 1)$$



in d=2, metric can always be put in such a conformal form

$$g^{(2)} = e^{2\rho(x)} dx_i dx^i$$

a 2d real scalar field coupled to 2d "gravity"

$$\int [dg][d\Phi]e^{-S(g,\Phi)} \int [dX]\; e^{-\int dx^2\; \sqrt{g}[g^{ij}\partial_i X \partial_j X + m^2 X^2]}$$

$$\simeq \int [dg][d\Phi]e^{-S(g,\Phi)} \frac{1}{\sqrt{\mathrm{Det}\left(-\nabla^2 + m^2\right)}}$$

$$\downarrow \quad W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$= \int [dg][d\Phi]e^{-S(g,\Phi)-W(g;m)}$$

a 2d real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$= -\frac{1}{2}\int_{\epsilon=1/\Lambda^2}^{\infty}\frac{ds}{s}\ \mathrm{Tr}\left[e^{s\nabla^2 - sm^2}\right]$$

$$= -\frac{1}{2}\int_{\epsilon}^{\infty}\frac{ds}{s}\ e^{-sm^2}\mathrm{Tr}\left[e^{s\nabla^2}\right]$$

a 2d real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$= -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\, e^{-sm^2}\mathrm{Tr}\left[e^{s\nabla^2}\right]$$

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$g \to \tilde{g}_{ij} = e^{2f}g_{ij}$$

$$\tilde{\nabla}^2 = \frac{e^{-2f}}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$W(e^{2f}g;m) = -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\, e^{-sm^2}\mathrm{Tr}\left[e^{se^{-2f}\nabla^2}\right]$$

a 2d real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$= -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\, e^{-sm^2}\mathrm{Tr}\left[e^{s\nabla^2}\right]$$

$$g \to \tilde{g}_{ij} = e^{2f}g_{ij}$$

$$\tilde{\nabla}^2 = \frac{e^{-2f}}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$\frac{\delta}{\delta f(x)}W(\tilde{g};m) \quad = \int_\epsilon^\infty ds\; e^{-sm^2}\mathrm{Tr}\left[\delta_x\tilde{\nabla}^2 e^{s\tilde{\nabla}^2}\right]$$

a 2d real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$= -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\, e^{-sm^2}\mathrm{Tr}\left[e^{s\nabla^2}\right]$$

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$\downarrow \quad g \to \tilde{g}_{ij} = e^{2f}g_{ij}$$

$$\tilde{\nabla}^2 = \frac{e^{-2f}}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$\frac{\delta}{\delta f(x)}W(\tilde{g};m) \quad = \int_\epsilon^\infty ds\, e^{-sm^2}\mathrm{Tr}\left[\delta_x\frac{\partial}{\partial s}e^{s\tilde{\nabla}^2}\right]$$

a 2d real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

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$$g \to \tilde{g}_{ij} = e^{2f}g_{ij}$$

$$\tilde{\nabla}^2 = \frac{e^{-2f}}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$\frac{\delta}{\delta f(x)}W(\tilde{g};m) = \int_\epsilon^\infty ds\, e^{-sm^2}\frac{\partial}{\partial s}\mathrm{Tr}\left[\delta_x e^{s\tilde{\nabla}^2}\right]$$

a 2d real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$= -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\, e^{-sm^2}\mathrm{Tr}\left[e^{s\nabla^2}\right]$$

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$g \to \tilde{g}_{ij} = e^{2f}g_{ij}$$

$$\tilde{\nabla}^2 = \frac{e^{-2f}}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$\frac{\delta}{\delta f(x)}W(\tilde{g};m) = e^{-sm^2}\mathrm{Tr}\left[\delta_x e^{\epsilon\tilde{\nabla}^2}\right]\Big|_\epsilon^\infty - \int_\epsilon^\infty ds\,\frac{de^{-sm^2}}{ds}\mathrm{Tr}\left[\delta_x e^{s\tilde{\nabla}^2}\right]$$

a 2d real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$= -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\, e^{-sm^2}\mathrm{Tr}\left[e^{s\nabla^2}\right]$$

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$g \to \tilde{g}_{ij} = e^{2f}g_{ij}$$

$$\tilde{\nabla}^2 = \frac{e^{-2f}}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$\frac{\delta}{\delta f(x)}W(\tilde{g};m) = -e^{-\epsilon m^2}\mathrm{Tr}\left[\delta_x e^{\epsilon\tilde{\nabla}^2}\right] + \int_\epsilon^\infty ds\; m^2 e^{-sm^2}\mathrm{Tr}\left[\delta_x e^{s\tilde{\nabla}^2}\right]$$

a 2d real massless scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right) \qquad \nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

$$2g_{ij}(x)\frac{\delta}{\delta g^{ij}(x)}W(g;m=0) \;=\; -\frac{\delta}{\delta f(x)}W(\tilde{g};m=0)\Big|_{f=1}$$

$$= \mathrm{Tr}\left[\delta_x e^{\epsilon\nabla^2}\right]$$

Taylor expansion of metric

$$g_{jk}(x + \delta x) = g_{jk}(x) + \frac{\partial g_{jk}}{\partial x^p}\delta x^p + \frac{1}{2}\frac{\partial^2 g_{jk}}{\partial x^p \partial x^q}\delta x^p \delta x^q + O(\delta x^3)$$

with geodesic normal coordinate system

$$g_{jk}(x+\delta x) = g_{jk}(x) + \frac{\partial g_{jk}}{\partial x^p}\delta x^p + \frac{1}{2}\frac{\partial^2 g_{jk}}{\partial x^p \partial x^q}\delta x^p \delta x^q + O(\delta x^3)$$

$$= \delta_{jk} + \frac{1}{3}R_{pjkq}(x)\delta x^p \delta x^q + O(\delta x^3)$$

with geodesic normal coordinate system

$$g_{jk}(x+\delta x) = \delta_{jk} + \frac{1}{3} R_{pjkq}(x) \delta x^p \delta x^q + O(\delta x^3)$$

$$\begin{aligned}
\Gamma_{mjk}(x+\delta x) &= \frac{1}{6} \Big(-R_{mjkp}(x)\delta x^p - R_{pjkm}(x)\delta x^p + R_{kmjp}(x)\delta x^p \\
&\quad + R_{pmjk}(x)\delta x^p + R_{jmkp}(x)\delta x^p + R_{pmkj}(x)\delta x^p \Big) + O(\delta x^2) \\
&= \frac{1}{3} \left(R_{pkmj}(x) - R_{pjkm}(x) \right) \delta x^p + O(\delta x^2)
\end{aligned}$$

$$\begin{aligned}
R_{mjkl}(x) = \left[\partial_k \Gamma_{mjl} - \partial_l \Gamma_{mjk} \right] \Big|_x &= \frac{1}{3} \left(R_{klmj}(x) - R_{kjlm}(x) - R_{lkmj}(x) + R_{ljkm}(x) \right) \\
&= \frac{1}{3} \left(2R_{klmj}(x) + R_{lmjk}(x) + R_{ljkm}(x) \right) \\
&= \frac{1}{3} \left(2R_{klmj}(x) - R_{lkmj}(x) \right) = R_{klmj}(x) = R_{mjkl}(x)
\end{aligned}$$

with geodesic normal coordinate system

$$g_{jk}(x+\delta x) = \delta_{jk} + \frac{1}{3}R_{pjkq}(x)\delta x^p \delta x^q + O(\delta x^3)$$

$$\Downarrow$$

$$g^{mk} \simeq \delta^{mk} + \frac{1}{3}R_{mpkq}(x)\delta x^p \delta x^q$$

$$\sqrt{g} \simeq 1 - \frac{1}{6}R_{pq}(x)\delta x^p \delta x^q$$

$$\frac{1}{\sqrt{g}} \simeq 1 + \frac{1}{6}R_{pq}(x)\delta x^p \delta x^q$$

with geodesic normal coordinate system

$$g_{jk}(x) = \delta_{jk} + \frac{1}{3}R_{pjkq}(0)x^p x^q + O(x^3)$$

$$\Downarrow$$

$$\nabla^2 \simeq \left(1 + \frac{1}{6}R_{ij}x^i x^j\right)\partial^m \left(\delta_{mk}\left(1 - \frac{1}{6}R_{pq}x^p x^q\right) + \frac{1}{3}R_{mpkq}x^p x^q\right)\partial^k$$

$$\simeq \; \partial^2 - \frac{1}{3}R_{kq}x^q\partial^k + \frac{1}{3}R_{mpkq}\partial^m x^p x^q \partial^k$$

$$\simeq \; \partial^2 - \frac{1}{3}R_{kq}x^q\partial^k + \frac{1}{3}R_{mpkq}\partial^m x^p x^q \partial^k$$

with geodesic normal coordinate system

$$g_{jk}(x) = \delta_{jk} + \frac{1}{3} R_{pjkq}(0) x^p x^q + O(x^3)$$

$$Q = -\nabla^2 = -\frac{1}{\sqrt{g}} \partial_m \sqrt{g} g^{mk} \partial_k \simeq \boxed{-\partial^2} + \underbrace{\boxed{\frac{1}{3} R_{kq} x^q \partial^k - \frac{1}{3} R_{mpkq} \partial^m x^p x^q \partial^k}}_{Q^{(1)}}$$

with geodesic normal coordinate system

$$g_{jk}(x) = \delta_{jk} + \frac{1}{3} R_{pjkq}(0) x^p x^q + O(x^3)$$

$$\Downarrow$$

$$Q^{(1)} G_s(x;0) = \left[\frac{1}{3} R_{mk}(0) x^m \partial^k - \frac{1}{3} R_{mpkq}(0) \partial^p x^m x^k \partial^q \right] G_s^{(0)}(x;0)$$

$$= \left[-\frac{1}{6s} R_{mk}(0) x^m x^k + \frac{1}{6s} R_{mpkq}(0) \partial^p x^m x^k x^q \right] G_s^{(0)}(x;0)$$

$$= \left[-\frac{1}{6s} R_{mk}(0) x^m x^k \right] G_s^{(0)}(x;0) \quad = \left[\frac{1}{3} R_{mk}(0) x^m \partial^k \right] G_s^{(0)}(x;0)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;= \left(Q^{(0)} + Q^{(1)}\right) G_\beta(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;x) \;= \frac{e^{-\beta m^2}}{(4\pi\beta)^{d/2}}\left[\frac{\beta}{2}b_\mu{}^\mu(x) - \frac{\beta^2}{3}c_\mu{}^\mu(x)\right] \;+O(\beta^2\partial\partial b, \beta^3\partial\partial c)$$

with geodesic normal coordinate system at x

$$g_{jk}(x+\delta x) = \delta_{jk} + \frac{1}{3} R_{pjkq}(x)\delta x^p \delta x^q + O(x^3)$$

$$\Downarrow$$

$$\langle x|e^{s\nabla^2}|x\rangle \;= \frac{1}{(4\pi s)^{d/2}}\left(1+\frac{s}{6}R(x)+\cdots\;\right)$$

$$= \frac{\sqrt{g(x)}}{(4\pi s)^{d/2}}\left(1+\frac{s}{6}R(x)+\cdots\;\right)$$

a 2d massless real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$2g_{ij}(x)\frac{\delta}{\delta g^{ij}(x)}W(g;0) \quad = \mathrm{Tr}\left[\delta_x e^{\epsilon\nabla^2}\right] \quad = \langle x|e^{\epsilon\nabla^2}|x\rangle$$

$$= \frac{\sqrt{g}}{4\pi\epsilon} + \frac{1}{24\pi}\sqrt{g}R^{(2)} + O(\epsilon R^2)$$

a 2d massless real scalar field coupled to 2d "gravity"

$$W(g;m) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2 + m^2\right)$$

$$2g_{ij}(x)\frac{\delta}{\delta g^{ij}(x)}W(g;0) \quad = \mathrm{Tr}\left[\delta_x e^{\nabla^2/\Lambda^2}\right] \; = \langle x|e^{\nabla^2/\Lambda^2}|x\rangle$$

$$= \frac{\sqrt{g}}{4\pi}\Lambda^2 + \frac{1}{24\pi}\sqrt{g}R^{(2)} + O(R^2/\Lambda^2)$$

this result is widely known as Weyl or 2d conformal anomaly

$$W(g;0) \equiv \frac{1}{2}\mathrm{Tr}\log\left(-\nabla^2\right) \;=-\frac{1}{2}\log\left[\int [dX]\; e^{-\int dx^2\;\sqrt{g}\left[g^{ij}\partial_i X \partial_j X\right]}\right]$$

$$g_{ij} = e^{2\rho}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$=-\frac{1}{2}\log\left[\int [dX]\; e^{-\int dx^2\;\left[\delta^{ij}\partial_i X \partial_j X\right]}\right]$$

$$2g_{ij}\frac{\delta}{\delta g^{ij}}W(g;0) \;\;= \frac{\sqrt{g}}{4\pi}\Lambda^2 + \frac{1}{24\pi}\sqrt{g}R^{(2)} + O(R^2/\Lambda^2)$$

integrating Weyl anomaly

$$2g_{ij}\frac{\delta}{\delta g^{ij}}W(g=e^{2\rho}\delta;0) \;=\; \frac{\sqrt{g}}{4\pi}\Lambda^2+\frac{1}{24\pi}\sqrt{g}R^{(2)}+\cdots$$

$$-\frac{\delta}{\delta\rho}W(g=e^{2\rho}\delta;0) \;=\; \frac{\sqrt{g}}{4\pi}\Lambda^2+\frac{1}{24\pi}\left[-2\partial^2\rho\right]+\cdots$$

$$W(g=e^{2\rho}\delta;0) \;=\; -\frac{\sqrt{g}}{4\pi}\Lambda^2+\frac{1}{24\pi}\left[\rho\,\partial^2\rho\right]+\cdots$$

$$= -\frac{\sqrt{g}}{4\pi}\Lambda^2+\frac{1}{96\pi}\left[\sqrt{g}\,R^{(2)}\frac{1}{\nabla^2}R^{(2)}\right]+\cdots$$

quantizing a 2d massless real scalar "renormalize" 2d gravity as

$$\int [dg][d\Phi] e^{-S(g,\Phi)} \int [dX]\ e^{-\int dx^2\ \sqrt{g}\left[g^{ij}\partial_i X \partial_j X\right]}$$

$$\simeq \int [dg][d\Phi] e^{-S(g,\Phi)} \frac{1}{\sqrt{\mathrm{Det}\,(-\nabla^2)}}$$

$$\simeq \int [dg][d\Phi]\ \mathrm{Exp}\left(-S(g,\Phi) - \frac{1}{96\pi}\int \sqrt{g}\ R^{(2)} \frac{1}{\nabla^2} R^{(2)}\right)$$

quantizing 2d massless real scalars "renormalize" 2d gravity as

$$\int [dg][d\Phi] e^{-S(g,\Phi)} \int \prod_{m=1}^{N} [dX^m]\, e^{-\int d\sigma^2\, \sqrt{g}[g^{ij}\partial_i X^m \partial_j X^m]}$$

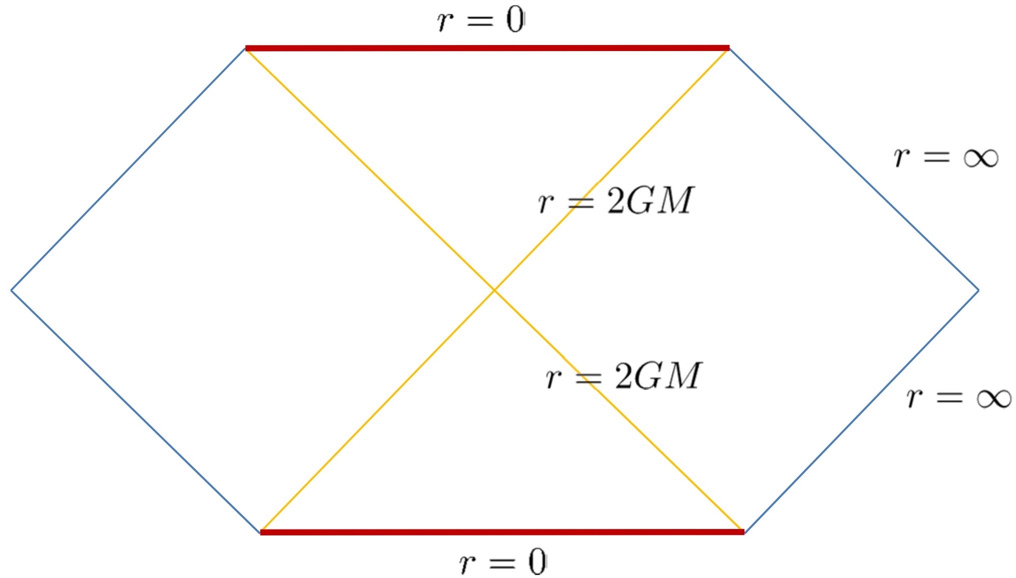
$$\simeq \int [dg][d\Phi] e^{-S(g,\Phi)} \left[\frac{1}{\sqrt{\mathrm{Det}\,(-\nabla^2)}} \right]^N$$

$$\simeq \int [dg][d\Phi]\ \mathrm{Exp}\left(-S(g,\Phi) - \frac{N}{96\pi} \int \sqrt{g}\ R^{(2)} \frac{1}{\nabla^2} R^{(2)} \right)$$

s-wave Hawking radiation from black holes

back to Schwarzschild black holes

$$g^{(4)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right]$$



Schwarzschild black holes

$$e^{-2\Phi} = r^2 \qquad g^{(2)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

↑

$$0 = \delta \int dx^2 \frac{1}{16\pi G} \sqrt{-g^{(2)}} e^{-2\Phi} \left[R^{(2)} + 2(\nabla\Phi)^2 + 2e^{2\Phi}\right]$$

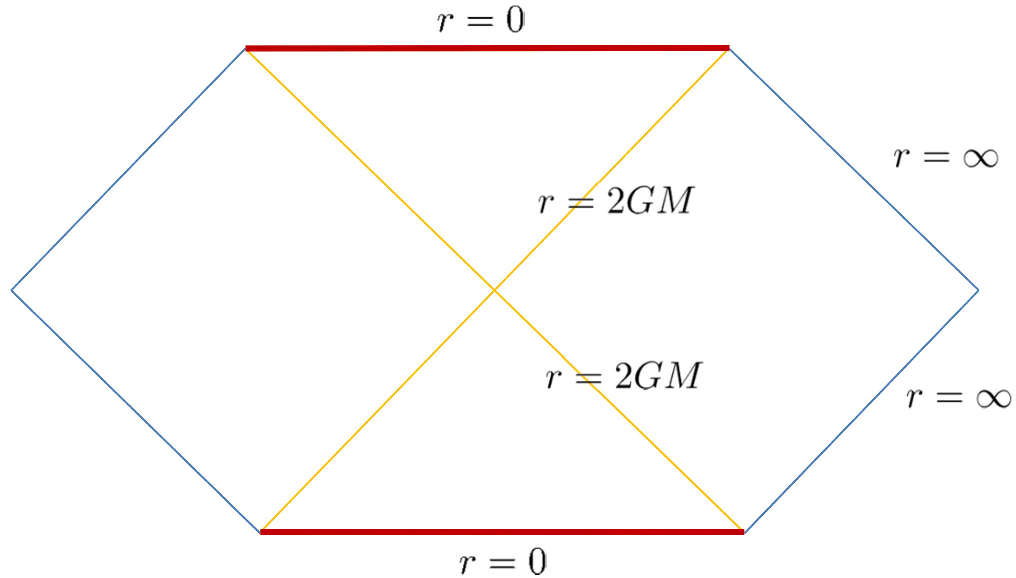
↑

$$g^{(4)} = g^{(2)} + e^{-2\Phi} \left[d\theta^2 + \sin^2\theta d\phi^2\right]$$

$$0 = \delta \int dx^4 \, \frac{1}{16\pi G} \sqrt{-g^{(4)}} R^{(4)}$$

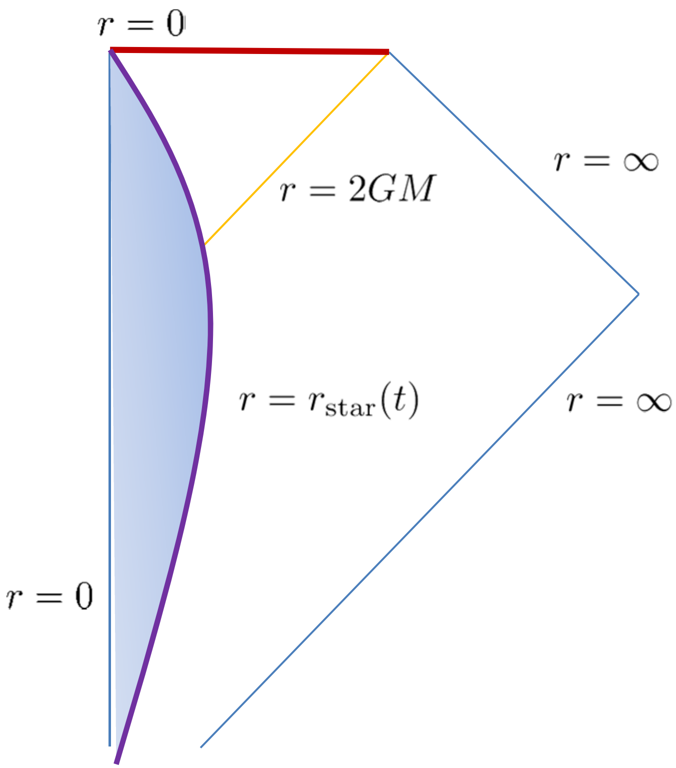
Schwarzschild black holes

$$g^{(2)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$



black holes from collapsing stars

$$g^{(2)}\Big|_{r>r_{\rm star}} = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2$$

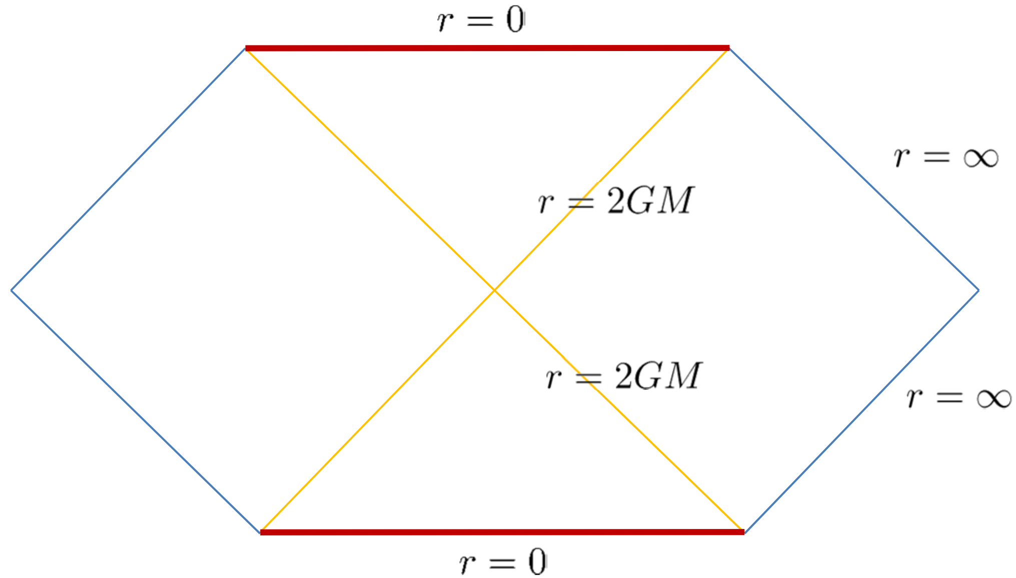


Schwarzschild black holes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dt^2 + dr_*^2\right]$$

$$r_* = \int \frac{r}{r - r_H} dr$$

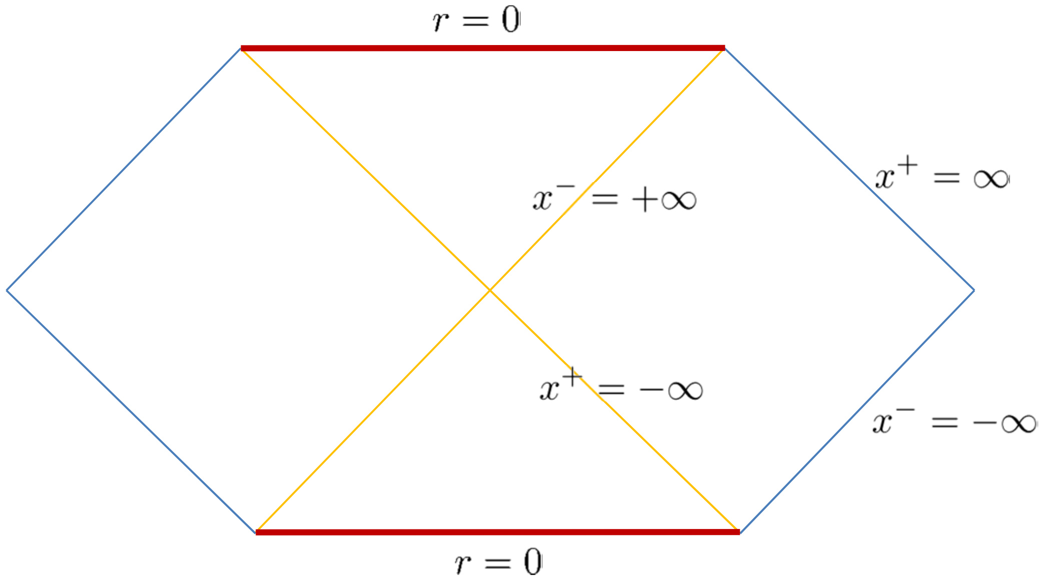
$$= r + r_H \log(r/r_H - 1)$$



Schwarzschild black holes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right] \equiv e^{2\rho(r_*)}\left[dx^- dx^+\right]$$

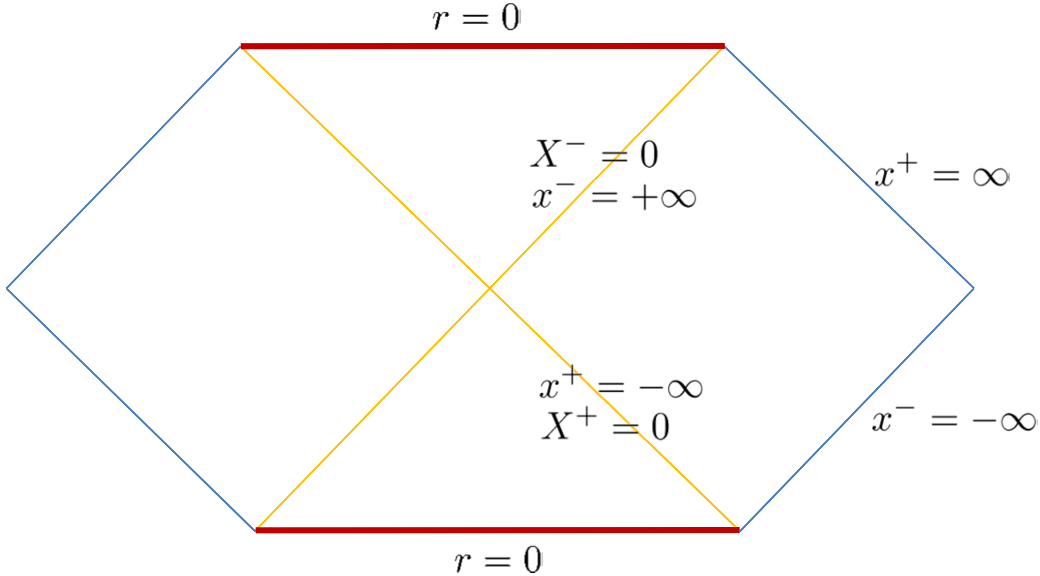
$$x^{\pm} = t \pm r_* \qquad e^{r_*/r_H} = \left(\frac{r}{r_H} - 1\right) e^{r/r_H}$$



Schwarzschild black holes

$$g^{(2)} = \frac{4r_H}{r} e^{-r/r_H} \left[-dX^- dX^+ \right] \qquad X^\pm = \pm r_H e^{\pm x^\pm / 2r_H}$$

$$x^\pm = t \pm r_* \qquad e^{r_*/r_H} = \left(\frac{r}{r_H} - 1 \right) e^{r/r_H}$$



black holes from collapsing stars

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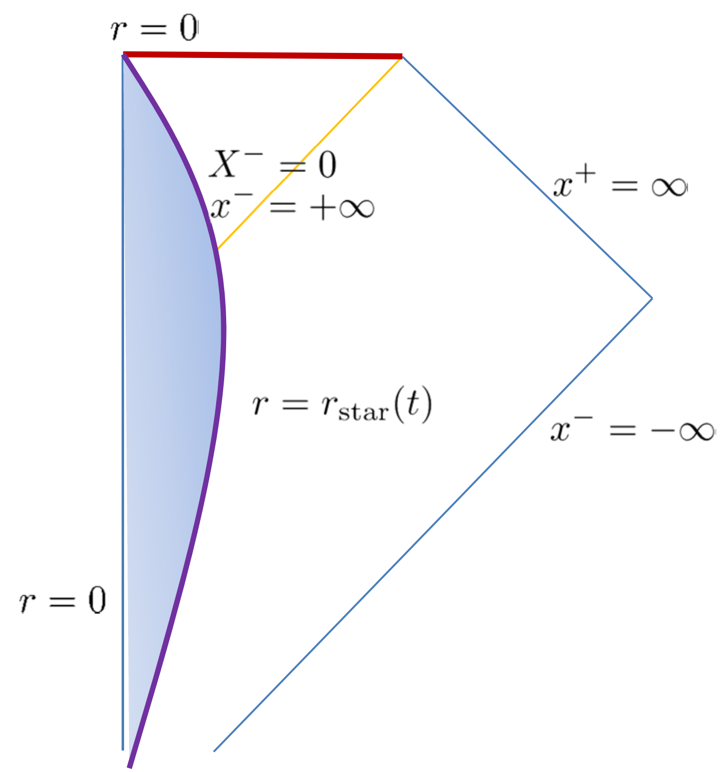

Schwarzschild black holes

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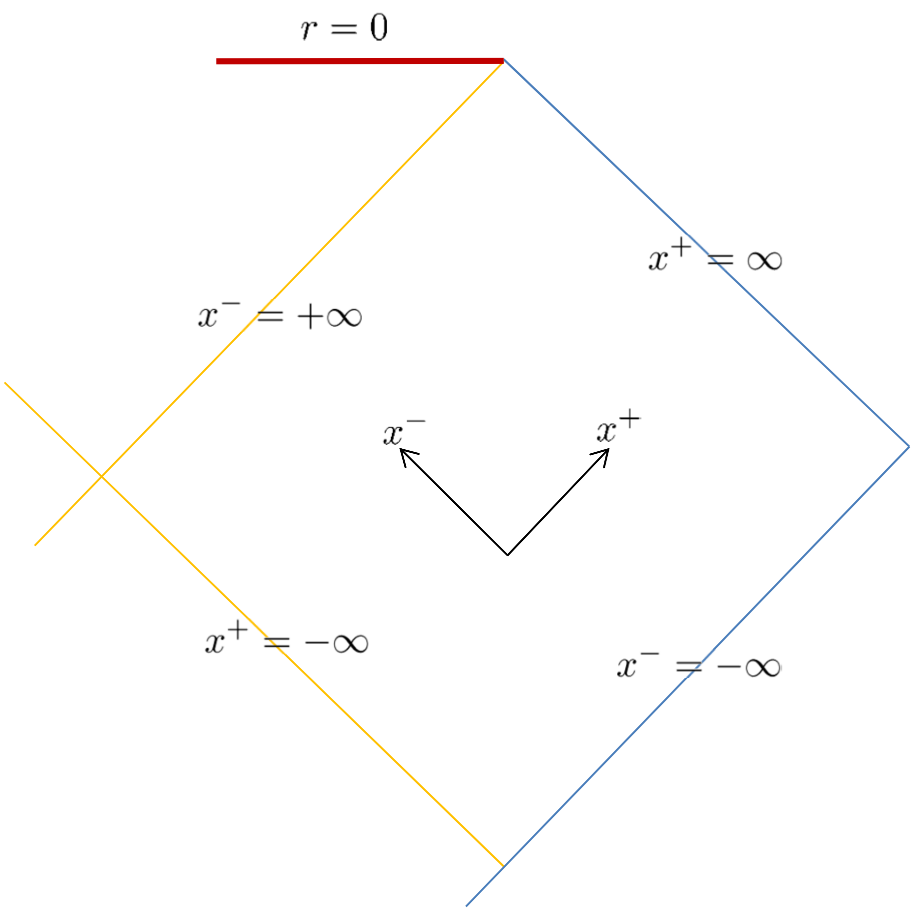


$$x^{\pm} = t \pm r_*$$

$$e^{r_*/r_H} = \left(\frac{r}{r_H} - 1\right) e^{r/r_H}$$

s-waves of N scalar fields, minimally coupled to spherically symmetric geometry

$$S_{\text{classical}} = S_{\text{gravity}} + S_{\text{matter}} =$$

$$\int dx^2 \frac{1}{16\pi G}\sqrt{-g^{(2)}}e^{-2\Phi}\left[R^{(2)} + 2(\nabla\Phi)^2 + 2e^{2\Phi}\right] - \sum_{a=1}^{N}\frac{1}{2}\int dx^2 \sqrt{-g^{(2)}}(\nabla\Psi_a)^2$$

$$\Downarrow$$

$$S_{\text{renomalized}} = S_{\text{gravity}} + W_{\text{eff}}(g^{(2)}) =$$

$$\int dx^2 \frac{1}{16\pi G}\sqrt{-g^{(2)}}e^{-2\Phi}\left[R^{(2)} + 2(\nabla\Phi)^2 + 2e^{2\Phi}\right] + \int dx^2 \frac{N}{96\pi}\left[\sqrt{-g^{(2)}}\,R^{(2)}\frac{1}{\nabla^2}R^{(2)}\right] + \cdots$$

quantized S-waves of N scalar fields, minimally coupled to spherically symmetric geometry

$$S_{\text{classical}} = S_{\text{gravity}} + S_{\text{matter}} =$$

$$\int dx^2 \frac{1}{16\pi G}\sqrt{-g^{(2)}}e^{-2\Phi}\left[R^{(2)} + 2(\nabla\Phi)^2 + 2e^{2\Phi}\right] - \sum_{a=1}^{N}\frac{1}{2}\int dx^2 \sqrt{-g^{(2)}}(\nabla\Psi_a)^2$$

$$g^{ij}T_{ij}^{\text{classical}} = \frac{2g^{ij}}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}S_{\text{matter}} = 0$$

$$S_{\text{renomalized}} = S_{\text{gravity}} + W_{\text{eff}}(g^{(2)}) =$$

$$\int dx^2 \frac{1}{16\pi G}\sqrt{-g^{(2)}}e^{-2\Phi}\left[R^{(2)} + 2(\nabla\Phi)^2 + 2e^{2\Phi}\right] + \int dx^2 \frac{N}{96\pi}\left[\sqrt{-g^{(2)}}\,R^{(2)}\frac{1}{\nabla^2}R^{(2)}\right] + \cdots$$

$$g^{ij}\langle T_{ij}\rangle = \frac{2g^{ij}}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}W_{\text{eff}} = \frac{N}{24\pi}R^{(2)}$$

quantized S-waves of N scalar fields, minimally coupled to spherically symmetric geometry

$$g^{ij}\langle T_{ij}\rangle = \frac{2g^{ij}}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}W_{\text{eff}} = \frac{N}{24\pi}R^{(2)} = -\frac{N}{12\pi}e^{-2\rho}\partial_{r_*}^2\rho \neq 0$$

$+$ $\nabla^i\langle T_{ij}\rangle = 0$ $\Rightarrow$ $\langle T_{km}\rangle - \frac{1}{2}g_{km}\langle g^{ij}T_{ij}\rangle \neq 0$

$$S_{\text{renomalized}} = S_{\text{gravity}} + W_{\text{eff}}(g^{(2)}) =$$

$$\int dx^2 \frac{1}{16\pi G}\sqrt{-g^{(2)}}e^{-2\Phi}\left[R^{(2)} + 2(\nabla\Phi)^2 + 2e^{2\Phi}\right] + \int dx^2 \frac{N}{96\pi}\left[\sqrt{-g^{(2)}}\,R^{(2)}\frac{1}{\nabla^2}R^{(2)}\right] + \cdots$$

energy-momentum of N quantized scalar s-wave modes

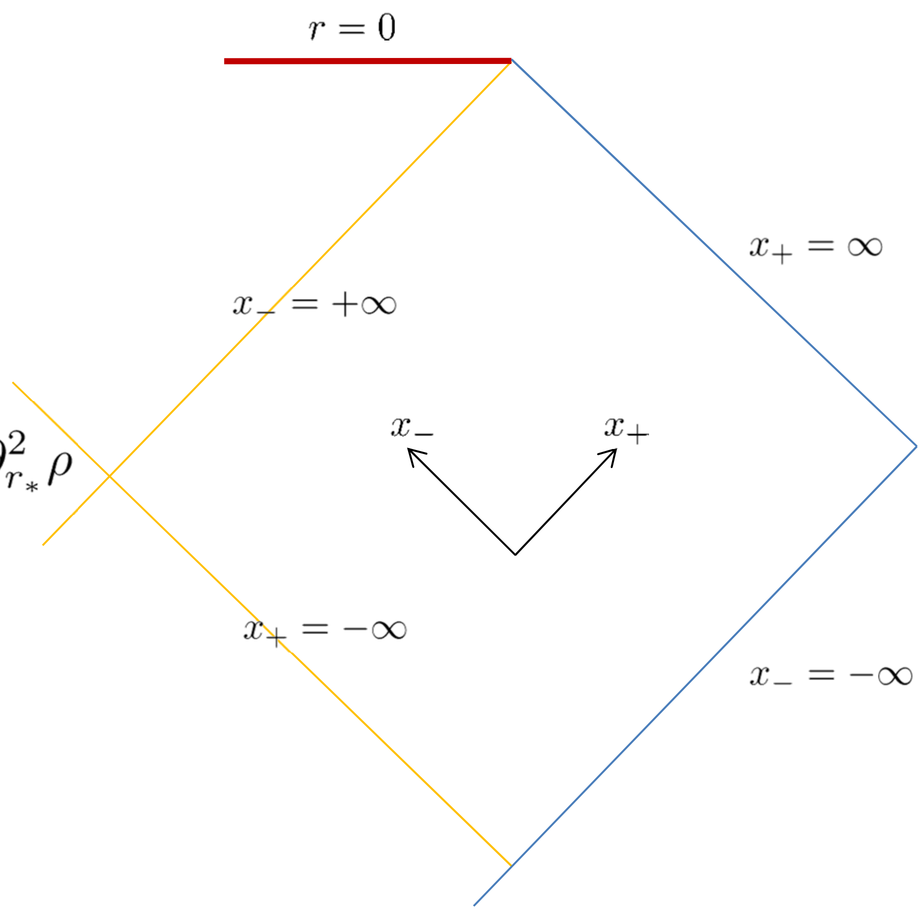
$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right]$$

$$\langle T_{ij} \rangle$$

$$\langle T_{+-} \rangle = \langle T^k{}_k \rangle / 2g^{+-} = \frac{N}{48\pi} \partial^2_{r_*} \rho$$

$\langle T_{--} \rangle$ outflow energy density

$\langle T_{++} \rangle$ inflow energy density



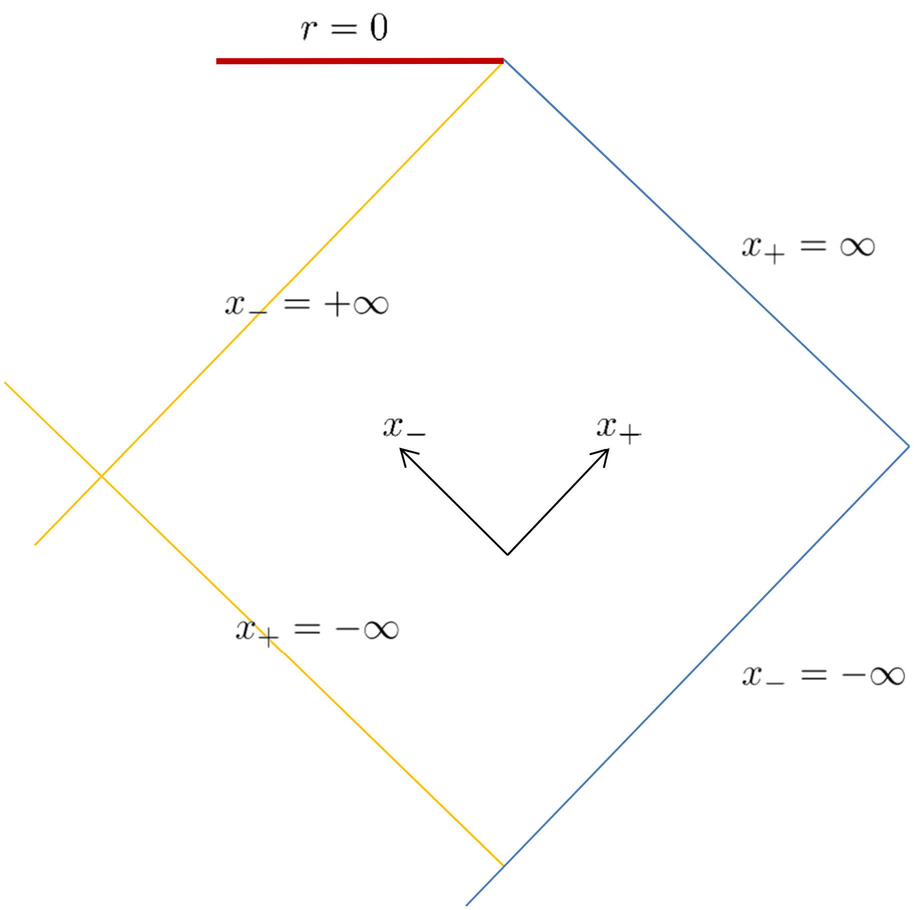
energy-momentum of N quantized scalar s-wave modes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right]$$

$$g^{ki}\nabla_k \langle T_{ij}\rangle = 0$$

$$\nabla_+ \langle T_{--}\rangle = -\nabla_- \langle T_{+-}\rangle$$

$$\partial_+ \langle T_{--}\rangle = -\nabla_- \langle T_{+-}\rangle$$



energy-momentum of N quantized scalar s-wave modes

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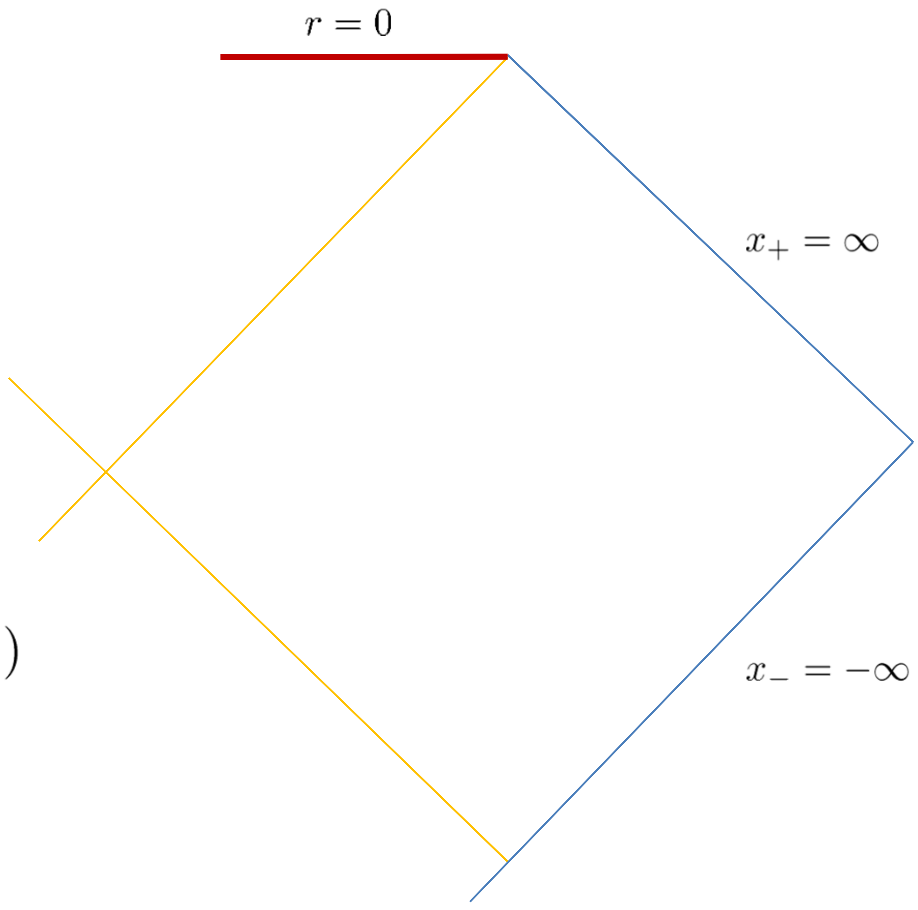


$$\langle T_{--}\rangle = t_{--}(x^-) + \Sigma(x^+, x^-)$$

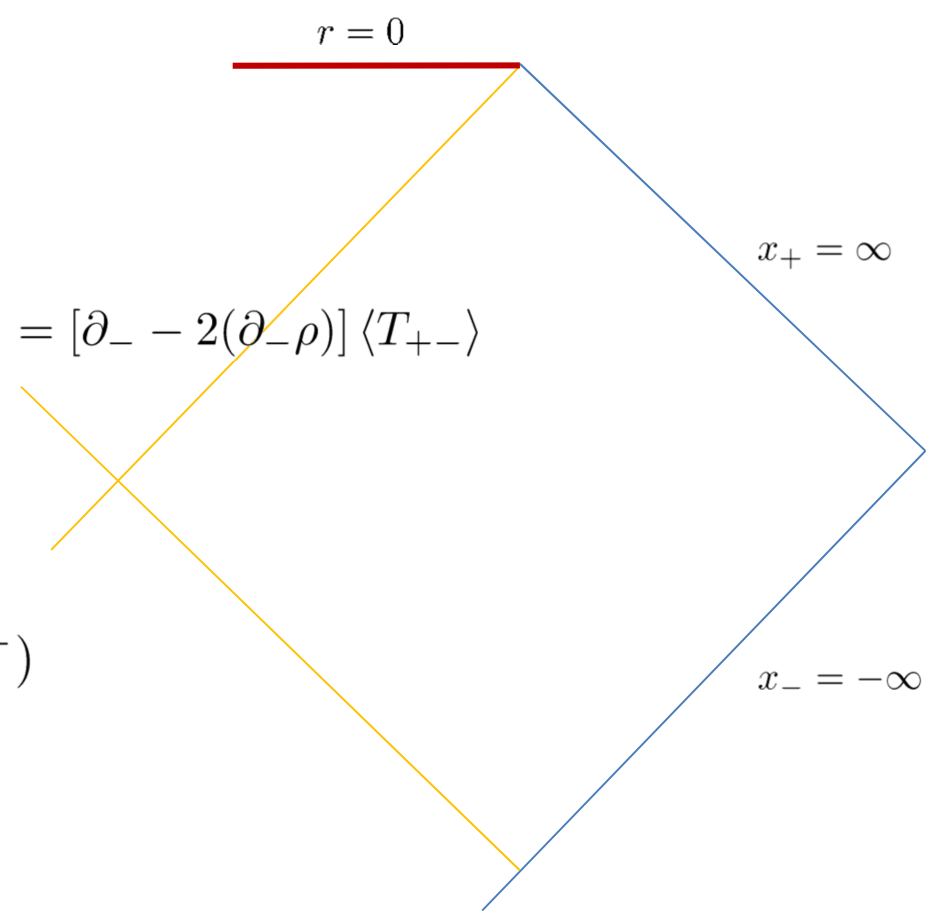
$$\Sigma \equiv -\int^{x^+} dx^+ \, \nabla_- \langle T_{+-}\rangle \Big|_{x^-}$$

energy-momentum of N quantized scalar s-wave modes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right]$$

$$= e^{2\rho(r_*)}\left[-dx^- dx^+\right]$$

$$\nabla_- \langle T_{+-}\rangle = \left[\partial_- - \Gamma^-{}_{--}\right]\langle T_{+-}\rangle \; = \left[\partial_- - 2(\partial_-\rho)\right]\langle T_{+-}\rangle$$



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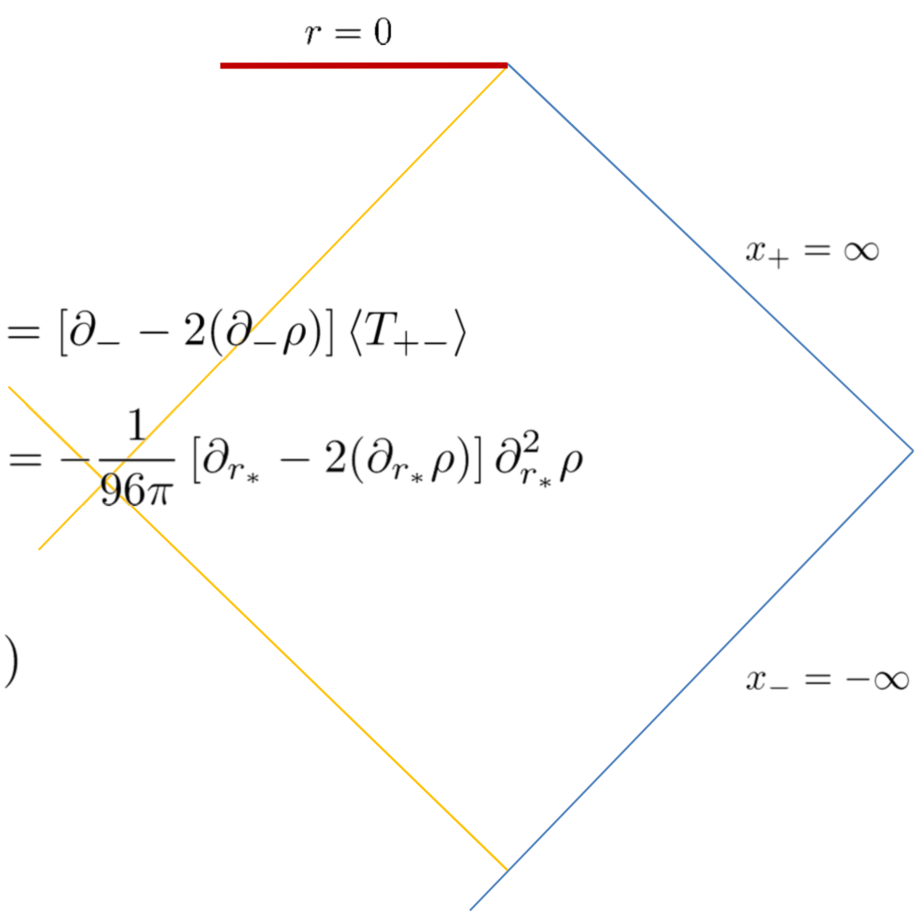
$$= -\frac{1}{96\pi}\left[\partial_{r_*} - 2(\partial_{r_*}\rho)\right] \partial^2_{r_*}\rho$$



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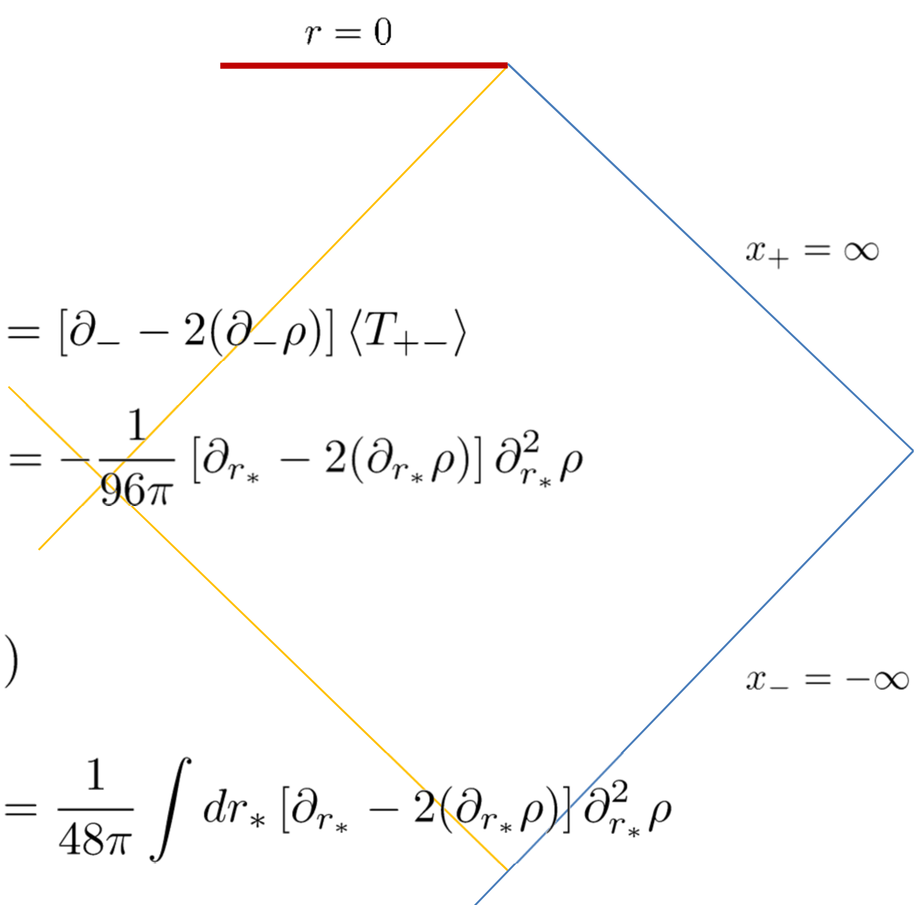


$$\langle T_{--}\rangle = t_{--}(x^-) + \Sigma(x^+, x^-)$$

$$\Sigma \equiv -\int^{x^+} dx^+ \,\nabla_- \langle T_{+-}\rangle\Big|_{x^-} = \frac{1}{48\pi}\int dr_* \left[\partial_{r_*} - 2(\partial_{r_*}\rho)\right]\partial^2_{r_*}\rho$$

$$= \frac{1}{192\pi}\left[2F(r)F''(r) - F'(r)^2\right]$$

energy-momentum of N quantized scalar s-wave modes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right]$$

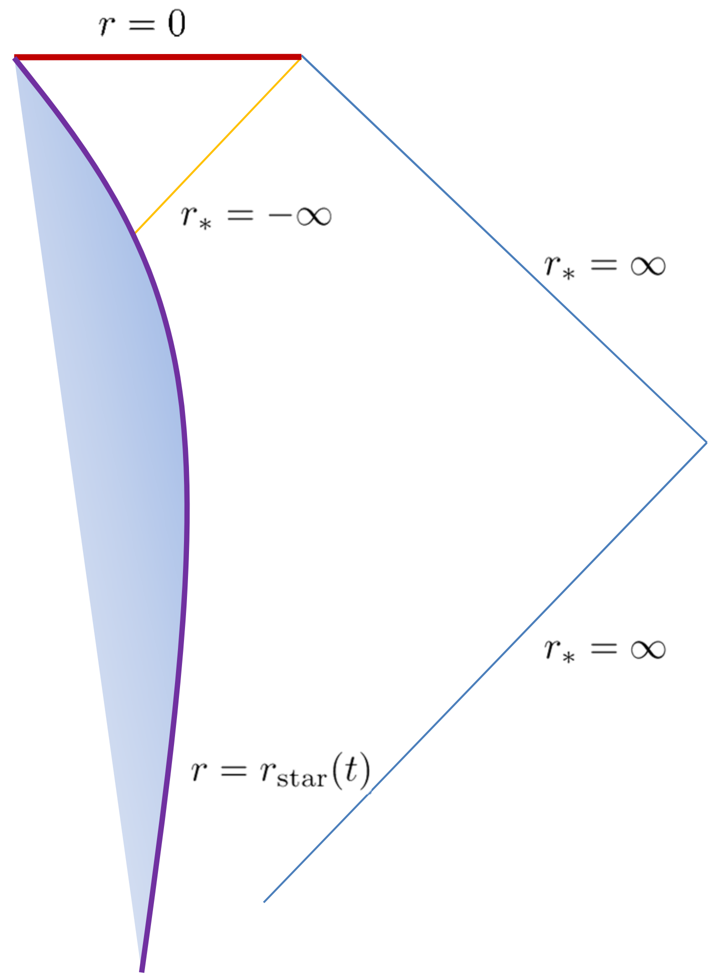
$$g^{ki}\nabla_k \langle T_{ij}\rangle = 0$$

⬇

$$\langle T_{--}\rangle = t_{--}(x^-) + \Sigma(r_*)$$

physical initial condition for $\langle T_{--}\rangle$ is

$$\langle T_{--}\rangle\Big|_{r=r_{\rm star}(t)} = 0$$



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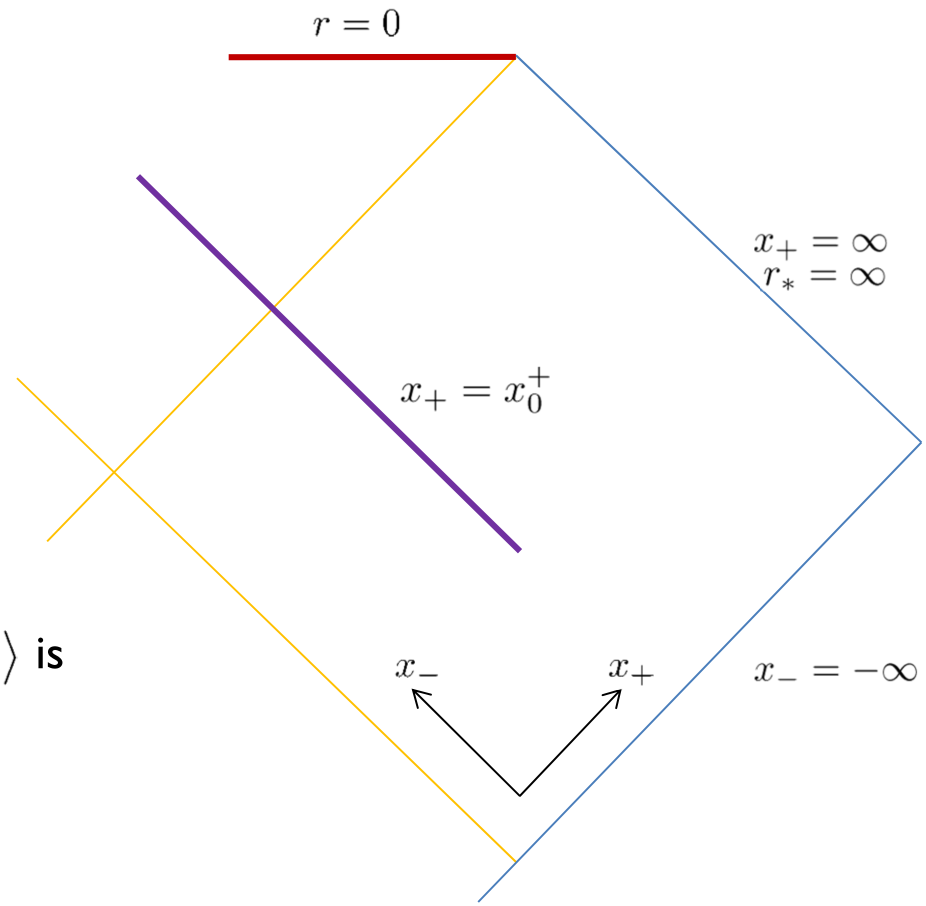
⬇

$$\langle T_{--}\rangle = t_{--}(x^-) + \Sigma(r_*)$$

physical initial condition for $\langle T_{--}\rangle$ is

$$\langle T_{--}\rangle\Big|_{x^+=x_0^+} = 0$$



no radiation near horizon implies net outgoing radiation far away

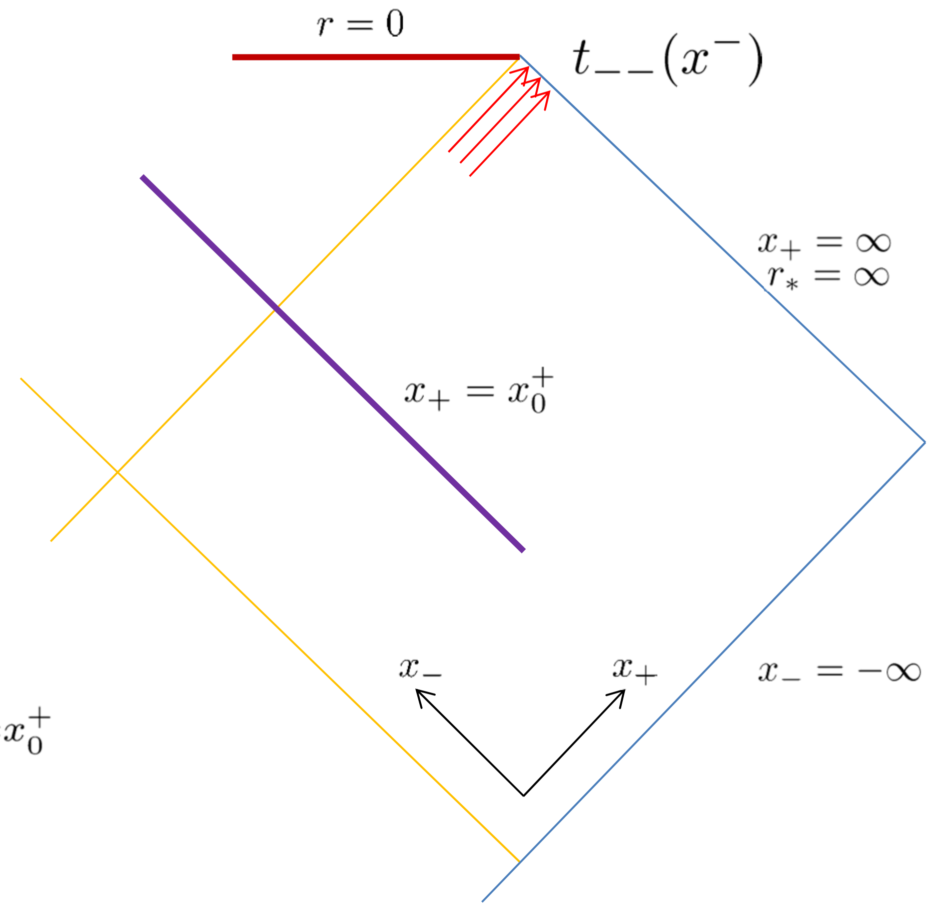
$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right]$$

$$g^{ki}\nabla_k \langle T_{ij}\rangle = 0$$

$$g^{ij}\langle T_{ij}\rangle = \frac{N}{24\pi}R^{(2)} \neq 0$$

⬇

$$\langle T_{--}\rangle\Big|_{x^+\to\infty} = -\Sigma(r_*)\Big|_{x^+=x_0^+}$$



late-time (s-wave) Hawking radiation

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right]$$

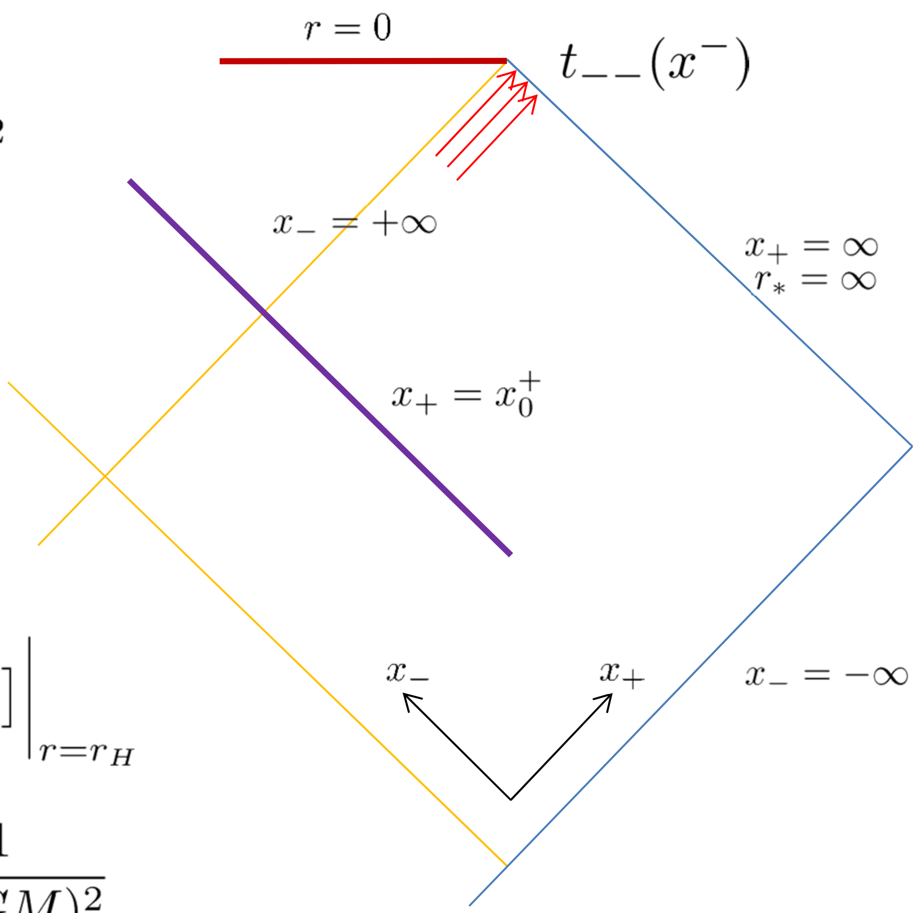
$$= -F(r)dt^2 + F(r)^{-1}dr^2$$

$$\langle T_{--}\rangle\Big|_{x^+=\infty;x^-\to\infty}$$

$$= -\Sigma(r = r_H)$$

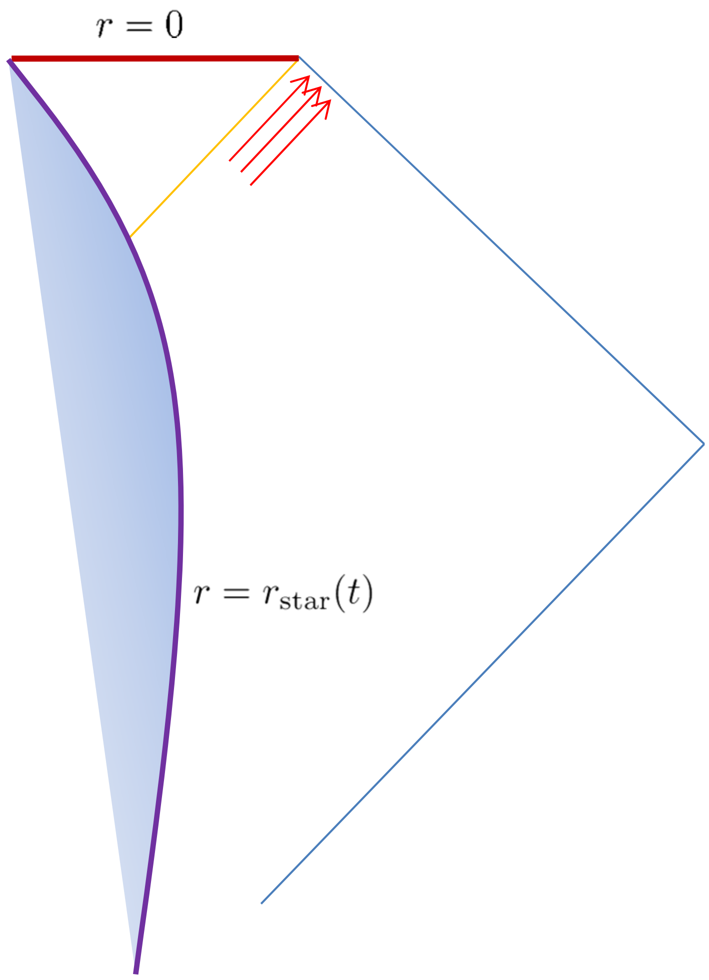
$$= \frac{N}{192\pi}\left[F'(r)^2 - 2F(r)F''(r)\right]\Big|_{r=r_H}$$

$$= \frac{N}{192\pi}\frac{1}{r_H^2} \quad \propto \quad T_{\rm BH}^2 = \frac{1}{(8\pi GM)^2}$$



Bogolyubov, Hawking, Unruh, and de Sitter

black holes from collapsing stars

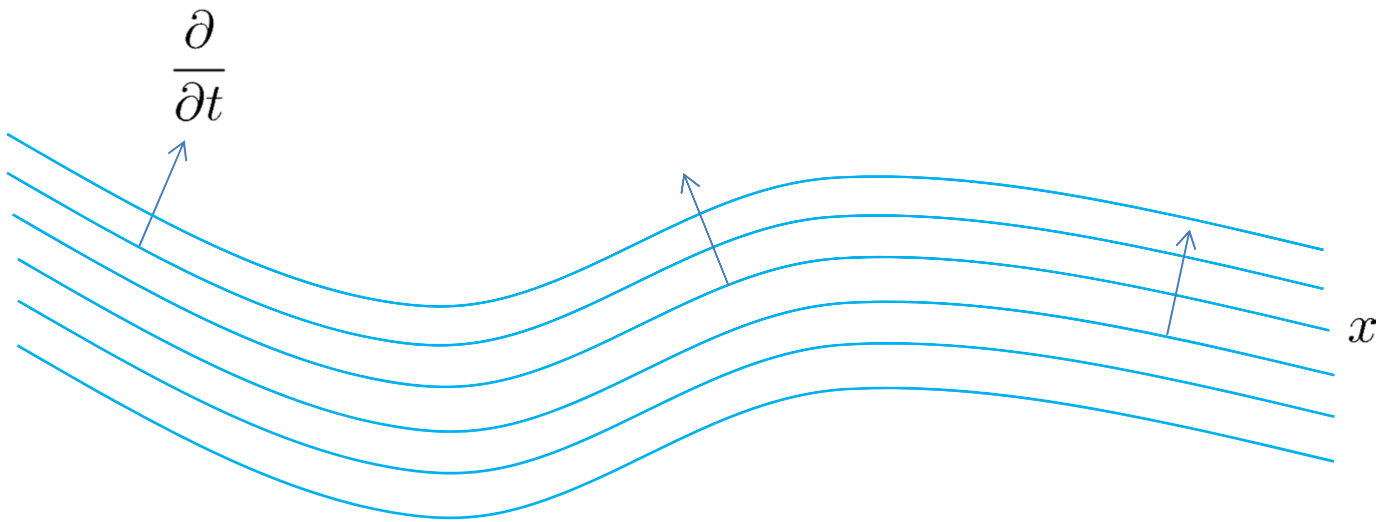


how do we understand such radiation “out of nothing”
via direct quantum analysis of the scalar fields ?

how does one define
a quantum vacuum for a free field in curved space-time ?

how does one define
a quantum state in curved spacetime ?

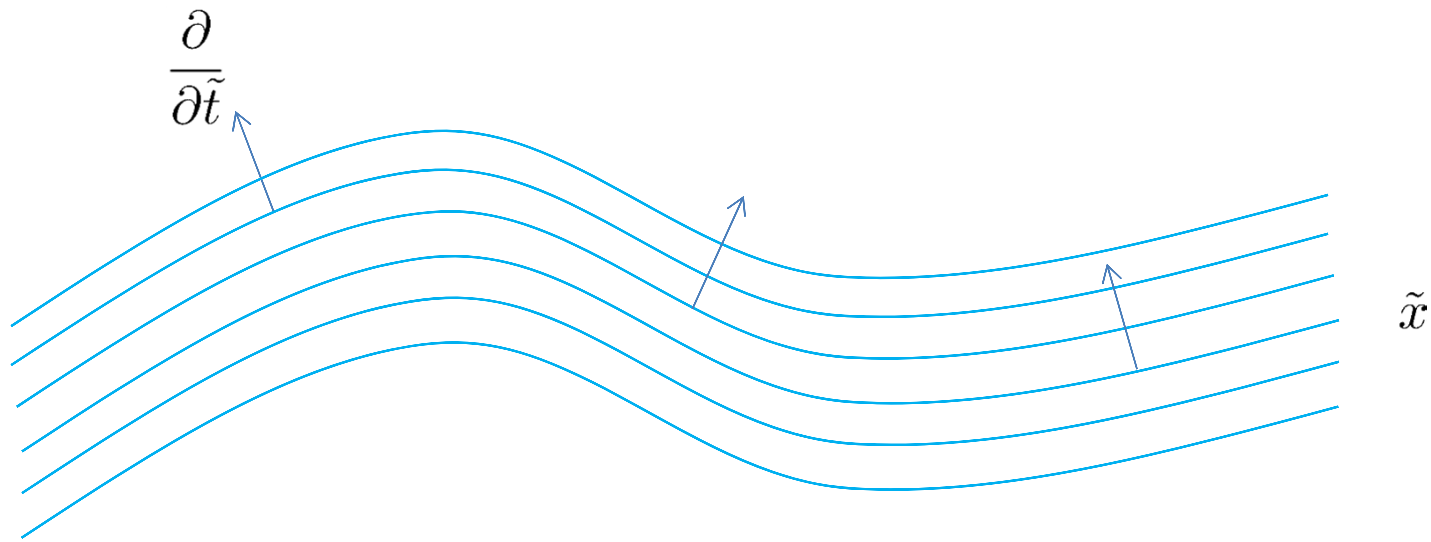
time-slices/coordinates are needed to define a Hilbert space



$$\Psi(t;x) = \sum a_{\vec{n}} e^{i\omega_{\vec{n}} t} f_{\vec{n}}(x) + a^{\dagger}_{\vec{n}} e^{-i\omega_{\vec{n}} t} f^{*}_{\vec{n}}(x)$$

$$a_{\vec{n}}|0\rangle_t = 0 \qquad |\{\vec{n}\}\rangle_t = \prod_{\{\vec{n}\}} a^{\dagger}_{\vec{n}}|0\rangle_t$$

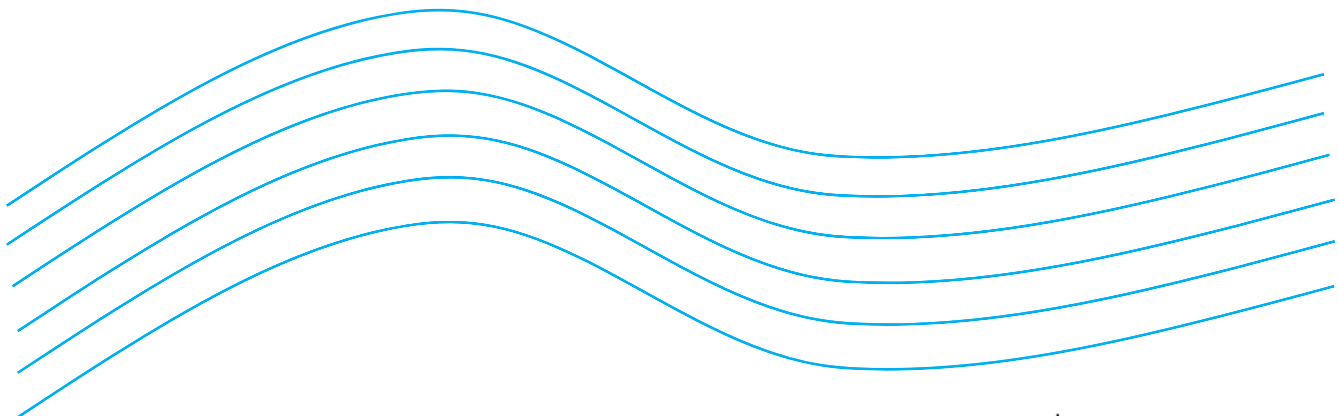
time-slices/coordinates are needed to define a Hilbert space



$$\Psi(\tilde{t};\tilde{x}) = \sum \tilde{a}_{\vec{n}} e^{i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}_{\vec{n}}(\tilde{x}) + \tilde{a}^{\dagger}_{\vec{n}} e^{-i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}^{*}_{\vec{n}}(\tilde{x})$$

$$\tilde{a}_{\vec{n}}|0\rangle_{\tilde{t}} = 0 \qquad\qquad |\{\vec{n}\}\rangle_{\tilde{t}} = \prod_{\{\vec{n}\}} \tilde{a}^{\dagger}_{\vec{n}}|0\rangle_{\tilde{t}}$$

we must have an inner product of wavefunctions, which is invariant under time-shift / coordinate changes

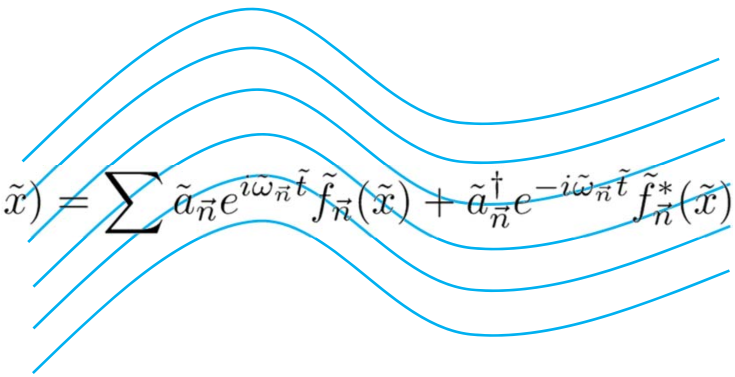


$$(f,g)_t = i\int dS^{\mu}\left[f^*\nabla_{\mu}g - g\nabla_{\nu}f^*\right]\Big|_t$$

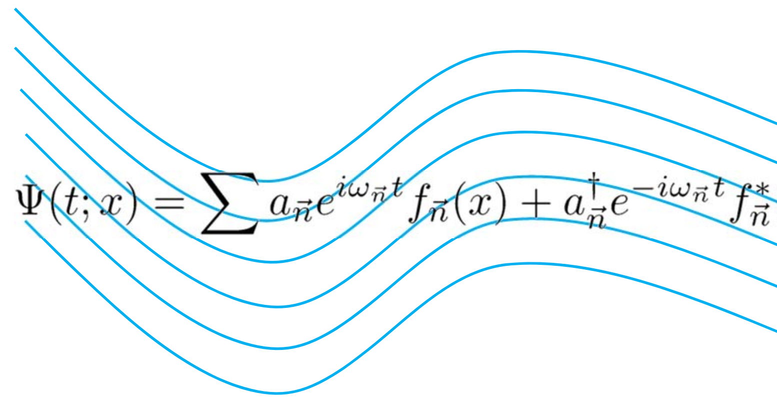
$$(f,g)_t - (f,g)_{t'} = i\int_{t'}^{t} dV\,\nabla^{\mu}\left[f^*\nabla_{\mu}g - g\nabla_{\nu}f^*\right] = i\int_{t'}^{t} dV\,\left[f^*\nabla^2 g - g\nabla^2 f^*\right] = 0$$

bases related by Bogolyubov transformation

$$|0\rangle_{\tilde{t}} \neq |0\rangle_{t}$$



$$\Psi(\tilde{t};\tilde{x}) = \sum \tilde{a}_{\vec{n}} e^{i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}_{\vec{n}}(\tilde{x}) + \tilde{a}^{\dagger}_{\vec{n}} e^{-i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}^{*}_{\vec{n}}(\tilde{x})$$



$$\Psi(t;x) = \sum a_{\vec{n}} e^{i\omega_{\vec{n}}t} f_{\vec{n}}(x) + a^{\dagger}_{\vec{n}} e^{-i\omega_{\vec{n}}t} f^{*}_{\vec{n}}(x)$$

$$\begin{pmatrix} e^{i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}_{\vec{n}} \\ \\ e^{-i\tilde{\omega}_{\vec{n}}} \tilde{f}^{*}_{\vec{n}} \end{pmatrix} = \sum_{\vec{k}} \begin{pmatrix} \alpha_{\vec{n};\vec{k}} & \beta_{\vec{n};\vec{k}} \\ \\ \beta^{*}_{\vec{n};\vec{k}} & \alpha^{*}_{\vec{n};\vec{k}} \end{pmatrix} \begin{pmatrix} e^{i\omega_{\vec{n}}t} f_{\vec{n}} \\ \\ e^{-i\omega_{\vec{n}}} f^{*}_{\vec{n}} \end{pmatrix}$$

bases related by Bogolyubov transformation

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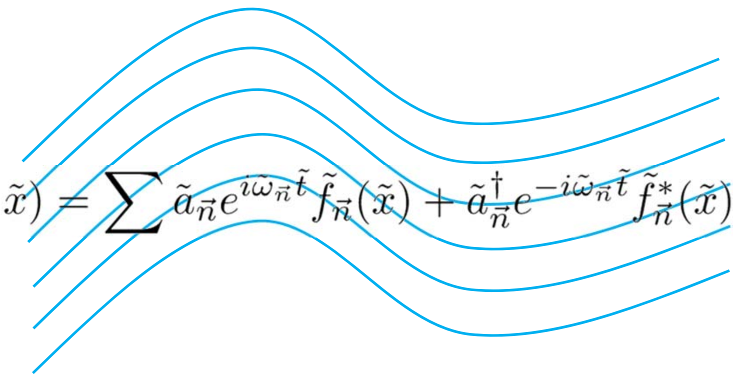
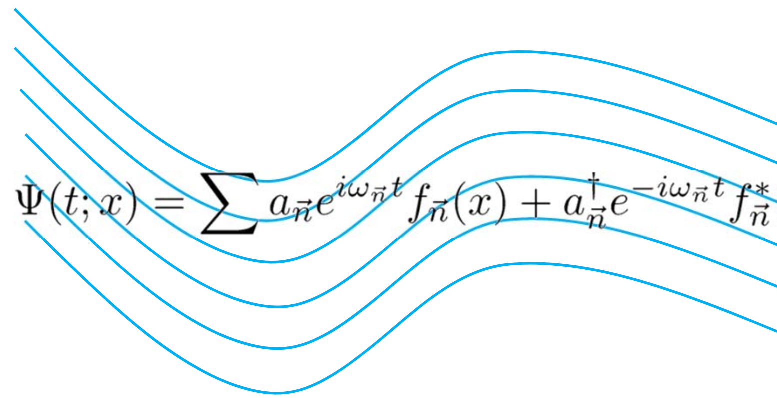


$$\Psi(\tilde{t};\tilde{x}) = \sum \tilde{a}_{\vec{n}} e^{i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}_{\vec{n}}(\tilde{x}) + \tilde{a}^{\dagger}_{\vec{n}} e^{-i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}^{*}_{\vec{n}}(\tilde{x})$$



$$\Psi(t;x) = \sum a_{\vec{n}} e^{i\omega_{\vec{n}}t} f_{\vec{n}}(x) + a^{\dagger}_{\vec{n}} e^{-i\omega_{\vec{n}}t} f^{*}_{\vec{n}}(x)$$

$$\left(e^{i\tilde{\omega}_{\vec{n}}\tilde{t}}\tilde{f}_{\vec{n}}, e^{i\tilde{\omega}_{\vec{m}}}\tilde{f}_{\vec{m}}\right) = \delta_{\vec{n},\vec{m}}$$

$$\left(e^{i\tilde{\omega}_{\vec{n}}\tilde{t}}\tilde{f}_{\vec{n}}, e^{-i\tilde{\omega}_{\vec{m}}}\tilde{f}^{*}_{\vec{m}}\right) = 0$$

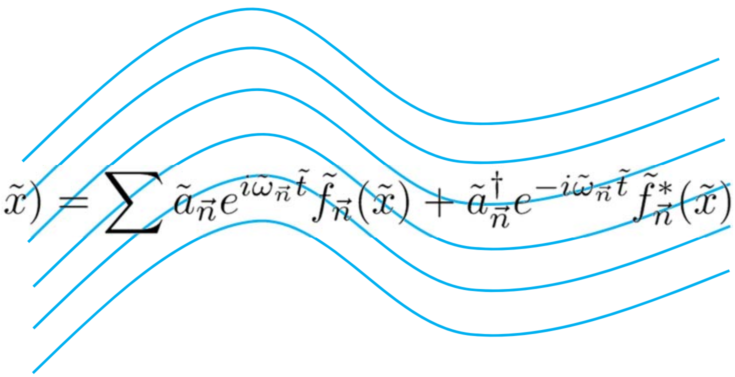
$$\Rightarrow$$

$$\alpha\alpha^{\dagger} - \beta\beta^{\dagger} = I$$

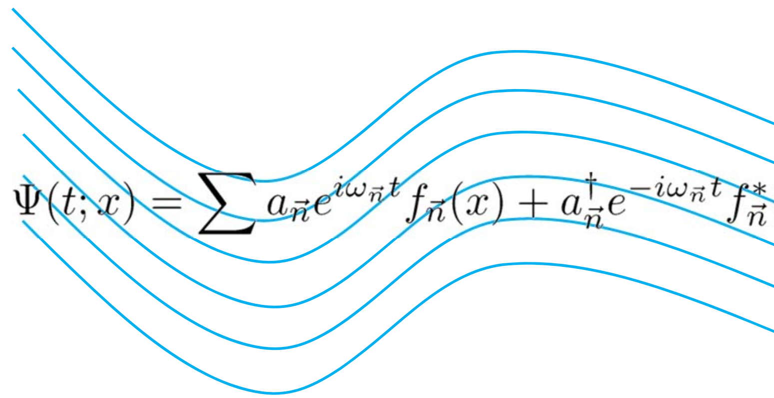
$$\alpha\beta^{T} - \beta\alpha^{T} = 0$$

→ operators related by Bogolyubov transformation

$$|0\rangle_{\tilde{t}} \neq |0\rangle_{t}$$



$$\Psi(\tilde{t};\tilde{x}) = \sum \tilde{a}_{\vec{n}} e^{i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}_{\vec{n}}(\tilde{x}) + \tilde{a}^{\dagger}_{\vec{n}} e^{-i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}^{*}_{\vec{n}}(\tilde{x})$$

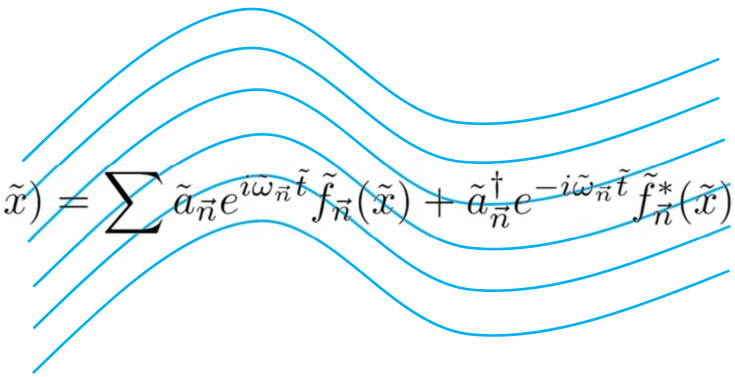


$$\Psi(t;x) = \sum a_{\vec{n}} e^{i\omega_{\vec{n}}t} f_{\vec{n}}(x) + a^{\dagger}_{\vec{n}} e^{-i\omega_{\vec{n}}t} f^{*}_{\vec{n}}(x)$$

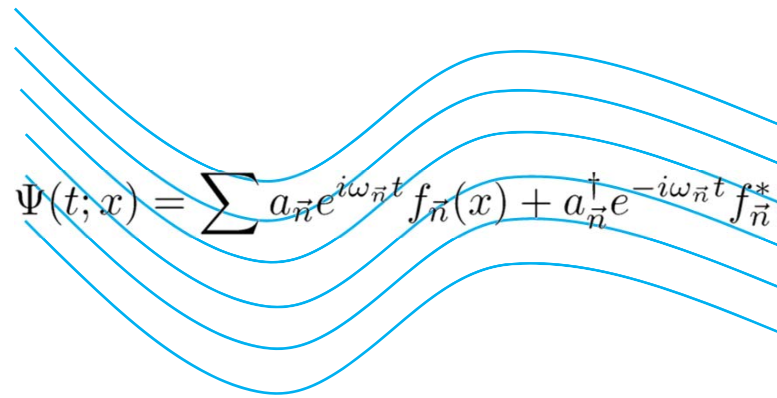
$$\begin{pmatrix} \tilde{a}_{\vec{n}} \\ \tilde{a}^{\dagger}_{\vec{n}} \end{pmatrix} = \sum_{\vec{k}} \begin{pmatrix} \alpha^{*}_{\vec{n};\vec{k}} & -\beta^{*}_{\vec{n};\vec{k}} \\ -\beta_{\vec{n};\vec{k}} & \alpha_{\vec{n};\vec{k}} \end{pmatrix} \begin{pmatrix} a_{\vec{k}} \\ a^{\dagger}_{\vec{k}} \end{pmatrix}$$

with operators related by Bogolyubov transformation

$$|0\rangle_{\tilde{t}} \neq |0\rangle_t$$



$$\Psi(\tilde{t};\tilde{x}) = \sum \tilde{a}_{\vec{n}} e^{i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}_{\vec{n}}(\tilde{x}) + \tilde{a}^\dagger_{\vec{n}} e^{-i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}^*_{\vec{n}}(\tilde{x})$$



$$\Psi(t;x) = \sum a_{\vec{n}} e^{i\omega_{\vec{n}}t} f_{\vec{n}}(x) + a^\dagger_{\vec{n}} e^{-i\omega_{\vec{n}}t} f^*_{\vec{n}}(x)$$

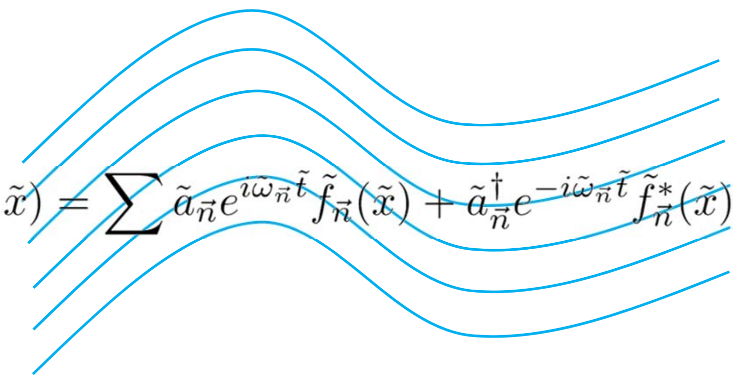
$$\tilde{a}_{\vec{n}} = (e^{i\tilde{\omega}_{\vec{n}}t}\tilde{f}_{\vec{n}}(x), \Psi) \quad = \quad \left(e^{i\tilde{\omega}_{\vec{n}}t}\tilde{f}_{\vec{n}}(x), \sum a_{\vec{k}} e^{i\omega_{\vec{k}}t} f_{\vec{k}}(x) + a^\dagger_{\vec{k}} e^{-i\omega_{\vec{k}}t} f^*_{\vec{k}}(x)\right)$$

$$= \sum_{\vec{k}} a_{\vec{k}} \boxed{\left(e^{i\tilde{\omega}_{\vec{n}}t}\tilde{f}_{\vec{n}}(x), e^{i\omega_{\vec{k}}t} f_{\vec{k}}(x)\right)} = \alpha^*_{\vec{n};\vec{k}}$$

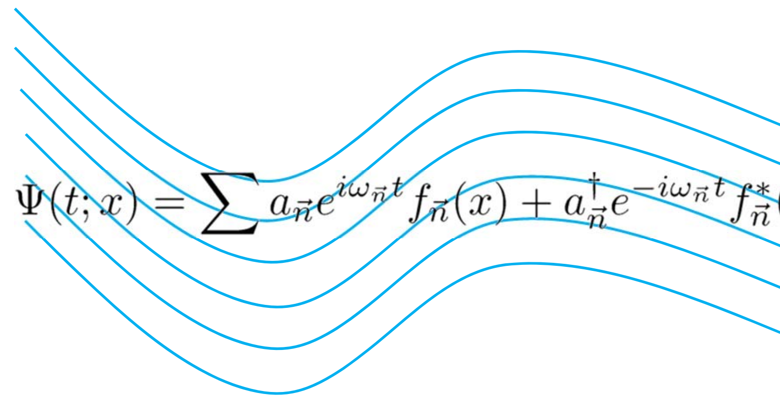
$$+ \sum_{\vec{k}} a^\dagger_{\vec{k}} \boxed{\left(e^{i\tilde{\omega}_{\vec{n}}t}\tilde{f}_{\vec{n}}(x), e^{-i\omega_{\vec{k}}t} f^*_{\vec{k}}(x)\right)} = -\beta^*_{\vec{n};\vec{k}}$$

the vacuum of one Hilbert space is not the vacuum of another

$$|0\rangle_{\tilde{t}} \neq |0\rangle_t$$



$$\Psi(\tilde{t};\tilde{x}) = \sum \tilde{a}_{\vec{n}} e^{i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}_{\vec{n}}(\tilde{x}) + \tilde{a}^{\dagger}_{\vec{n}} e^{-i\tilde{\omega}_{\vec{n}}\tilde{t}} \tilde{f}^{*}_{\vec{n}}(\tilde{x})$$



$$\Psi(t;x) = \sum a_{\vec{n}} e^{i\omega_{\vec{n}}t} f_{\vec{n}}(x) + a^{\dagger}_{\vec{n}} e^{-i\omega_{\vec{n}}t} f^{*}_{\vec{n}}(x)$$

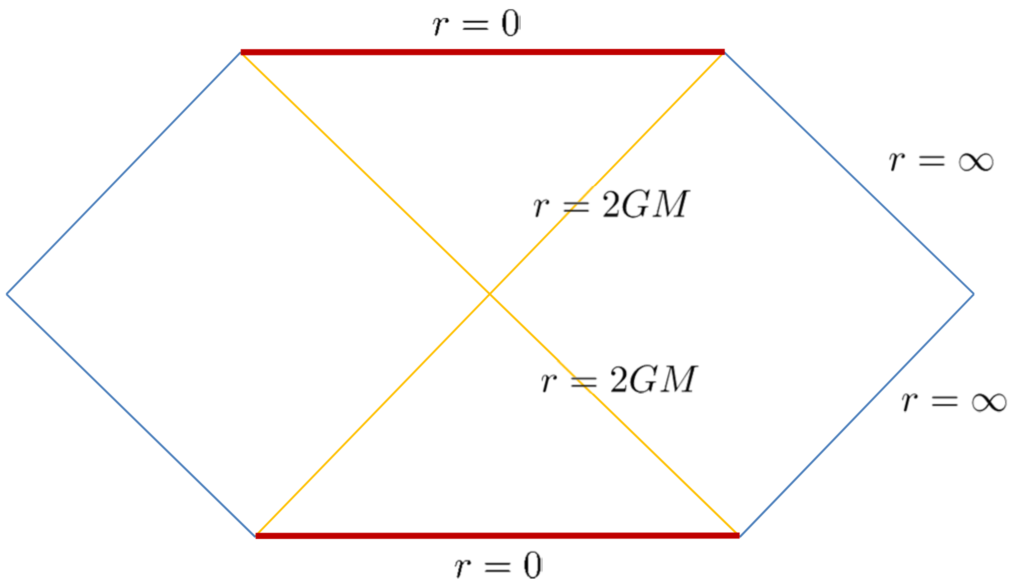
$$\langle \tilde{N}_{\vec{n}} \rangle_t = {}_t\langle 0|\tilde{a}^{\dagger}_{\vec{n}} \tilde{a}_{\vec{n}}|0\rangle_t = \sum_{\vec{k}} \beta_{\vec{n};\vec{k}} \beta^{*}_{\vec{n};\vec{k}}$$

$$a_{\vec{n}}|0\rangle_t = 0$$

$$\left[a^{\dagger}_{\vec{n}}, a_{\vec{m}}\right] = \delta_{\vec{n},\vec{m}}$$

Schwarzschild black holes

$$g^{(4)} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right]$$

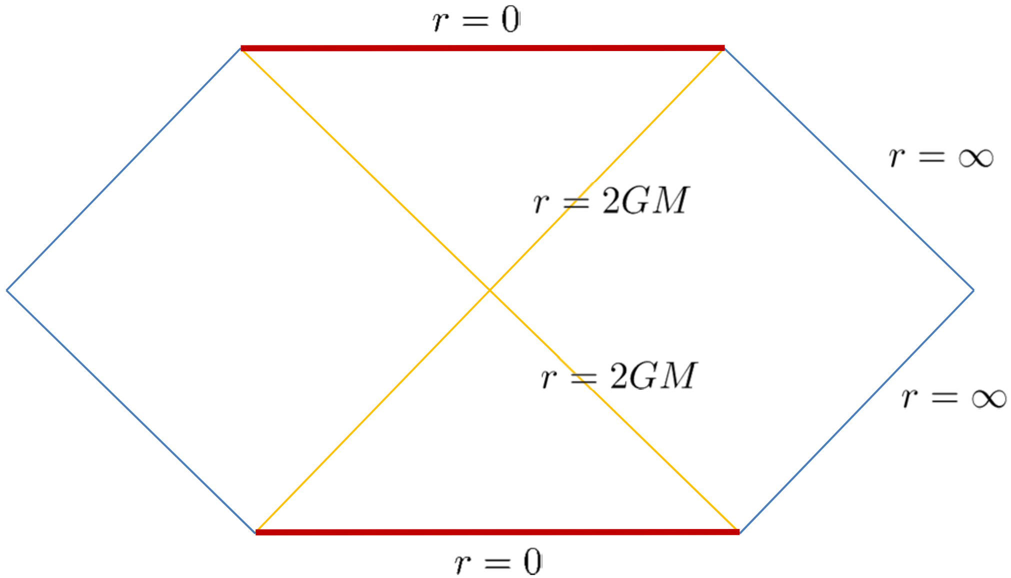


Schwarzschild black holes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dt^2 + dr_*^2\right]$$

$$r_* = \int \frac{r}{r - r_H} dr$$

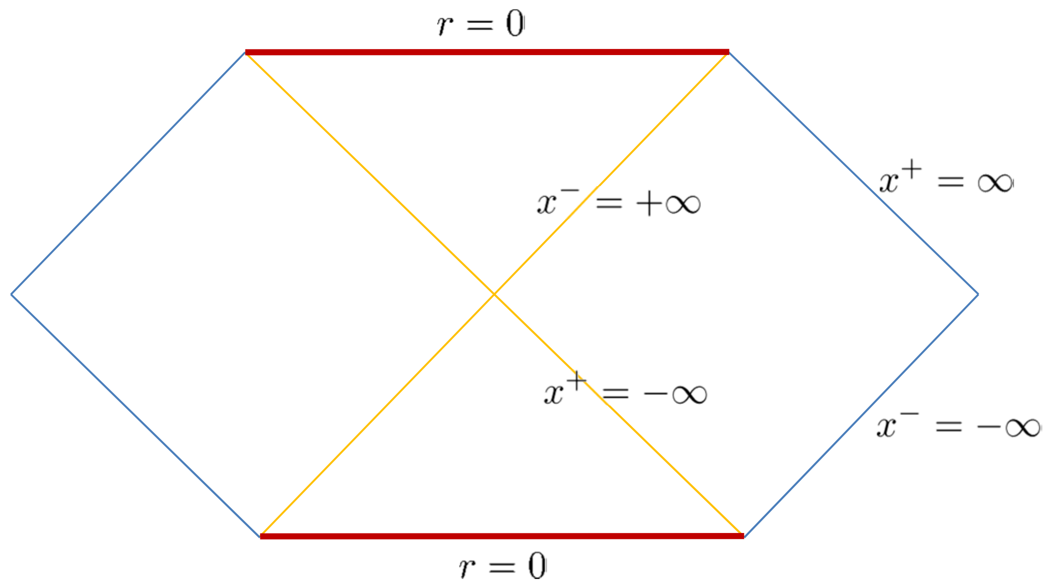
$$= r + r_H \log(r/r_H - 1)$$



Schwarzschild black holes

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)\left[-dx^- dx^+\right] \equiv e^{2\rho(r_*)}\left[dx^- dx^+\right]$$

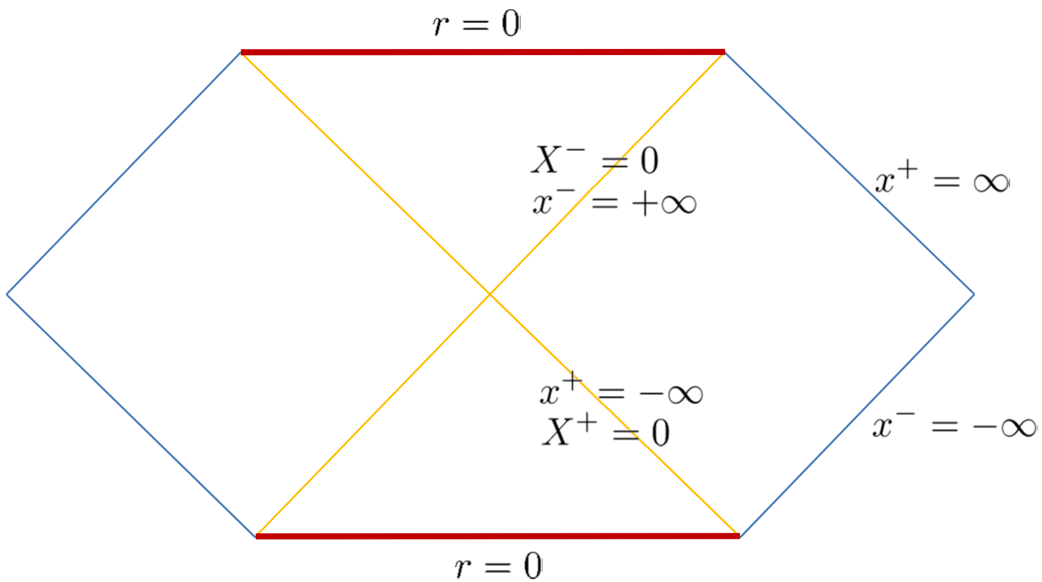
$$x^{\pm} = t \pm r_* \qquad e^{r_*/r_H} = \left(\frac{r}{r_H} - 1\right) e^{r/r_H}$$



Schwarzschild black holes

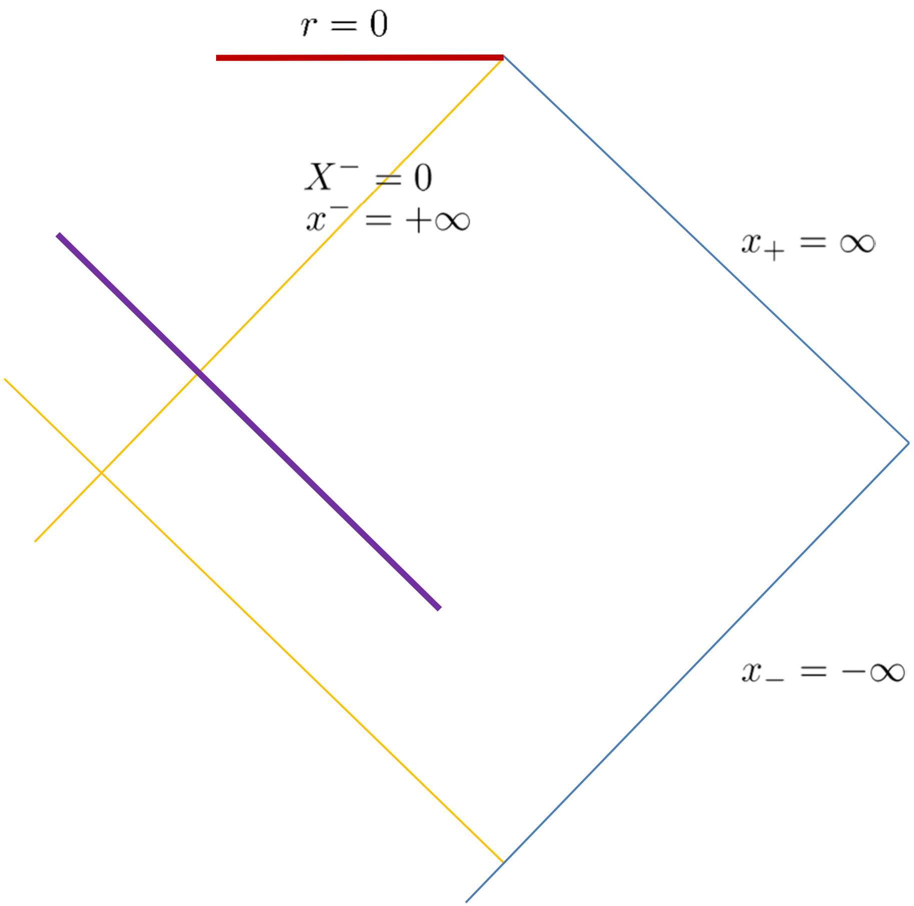
$$g^{(2)} = \frac{4r_H}{r} e^{-r/r_H} \left[-dX^- dX^+\right] \qquad X^\pm = \pm r_H e^{\pm x^\pm / 2r_H}$$

$$x^\pm = t \pm r_* \qquad e^{r_*/r_H} = \left(\frac{r}{r_H} - 1\right) e^{r/r_H}$$



back to the Schwarzschild black hole

$$g^{(2)} = \left(1 - \frac{r_H}{r}\right)[-dx_- dx_+] = \frac{4r_H}{r} e^{-r/r_H} [-dX^- dX^+]$$

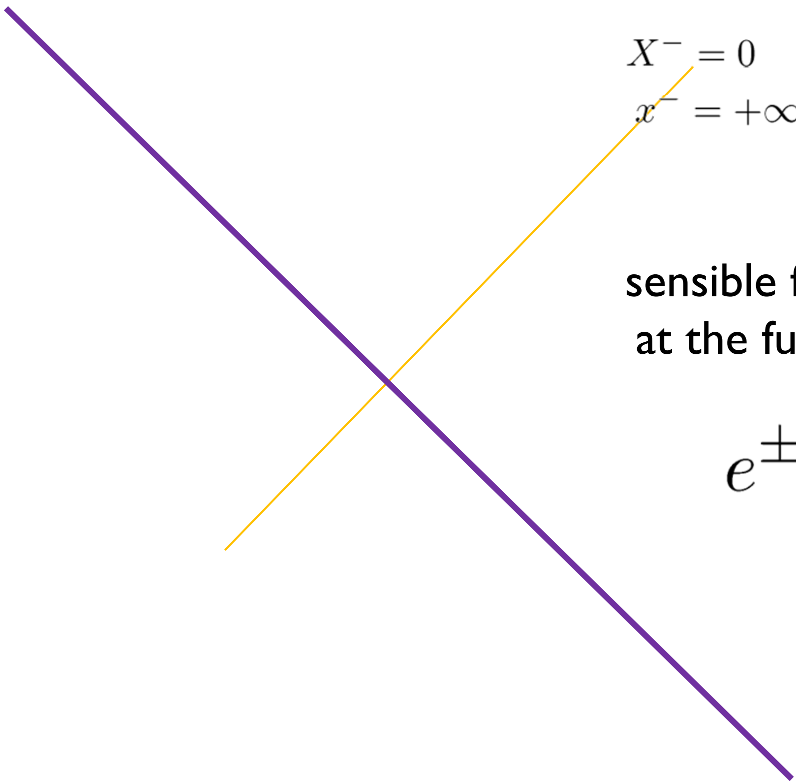


$$X^\pm = \pm r_H e^{\pm x^\pm / 2r_H}$$

$$x^\pm = t \pm r_*$$

back to the Schwarzschild black hole

$$X^{\pm} = \pm r_H e^{\pm x^{\pm}/2r_H} \qquad x^{\pm} = t \pm r_*$$



sensible for observers
at the future horizon

$$e^{\pm i\Omega X^-}$$

vs.

sensible for observers
far away from black hole

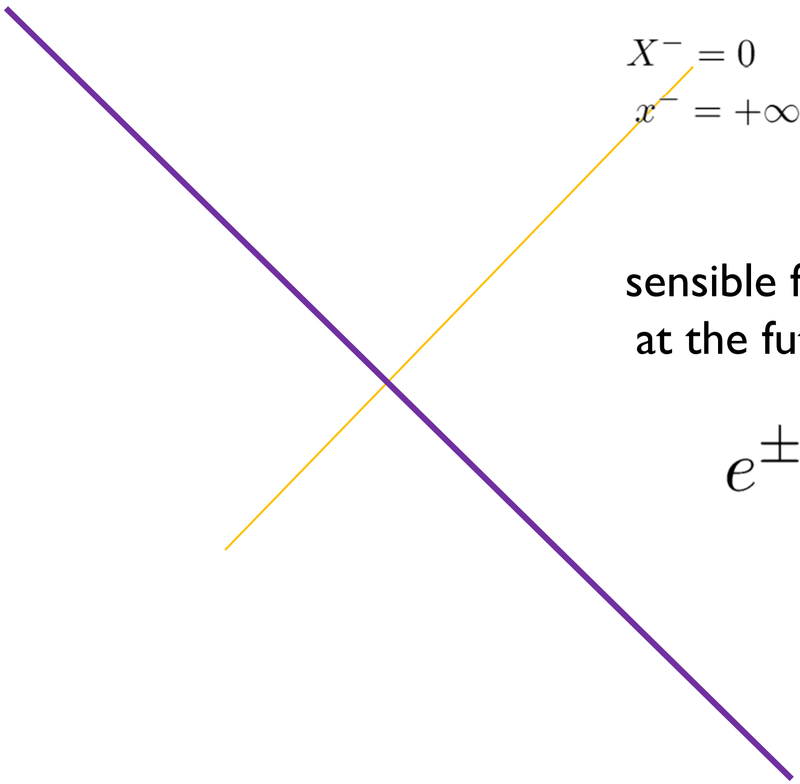
$$e^{\pm i\omega x^-}$$

$$\sim e^{\pm i\omega(t-r)}$$

modes moving toward infinity

back to the Schwarzschild black hole

$$X^{\pm} = \pm r_H e^{\pm x^{\pm}/2r_H} \qquad x^{\pm} = t \pm r_*$$



sensible for observers
at the future horizon

$$e^{\pm i\Omega X^-}$$

vs.

sensible for observers
far away from black hole

$$e^{\pm i\omega x^-}\Theta(-X^-)$$

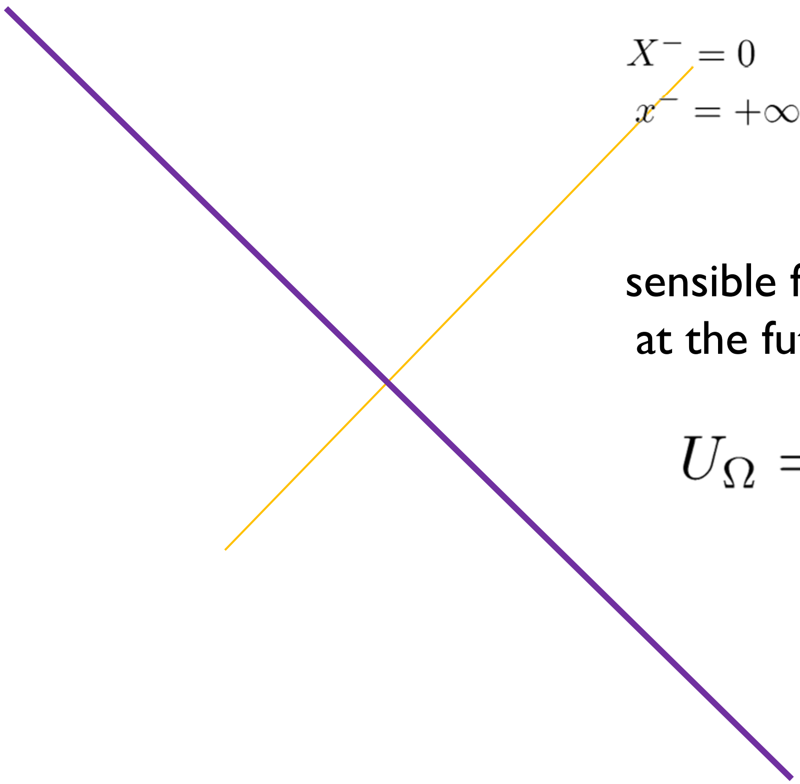
+

invisible but necessary

$$e^{\pm i\omega x^-}\Theta(X^-)$$

back to the Schwarzschild black hole

$$X^{\pm}_{\text{out}} = \pm r_H e^{\pm x^{\pm}_{\text{out}}/2r_H} \qquad x^{\pm}_{\text{out}} = t \pm r_*$$



sensible for observers
at the future horizon

$$U_\Omega = e^{i\Omega X^-}$$

vs.

sensible for observers
far away from black hole

$$v^{\text{out}}_\omega = e^{i\omega x^-}\Theta(-X^-)$$

+

invisible but necessary

$$v^{\text{in}}_\omega = e^{\pm i\omega x^-}\Theta(X^-)$$

what if the field is massive ?

what about higher angular momentum mode ?

what if the field has intrinsic spin ?

mode expansion of massive scalars in tortoise coordinate

$$-\nabla^2 + m^2 = -\frac{1}{\sqrt{g}}\partial_i \sqrt{g} g^{ij} \partial_j + m^2$$

$$= -g^{ij}\partial_i \partial_j + \cdots + m^2$$

$$= \left(1 - \frac{r_H}{r}\right)^{-1} \left[\partial_t^2 - \partial_{r_*}^2\right] + \frac{L^2}{r^2} + \cdots + m^2$$

$$= \left(1 - \frac{r_H}{r}\right)^{-1} \left[\partial_+ \partial_-\right] + \cdots + m^2$$

mode expansion of massive scalars in tortoise coordinate

$$\left(-\nabla^2 + m^2\right)\Psi = 0$$

$$\partial_+\partial_-\Psi + \left(1 - \frac{r_H}{r}\right)\left[m^2 + \cdots\right]\Psi = 0$$

$$\left(1 - \frac{r_H}{r}\right) \simeq e^{r_*/r_H} = e^{-|r_*|/r_H} \quad \text{near } r = r_H$$

mode expansion in tortoise coordinate

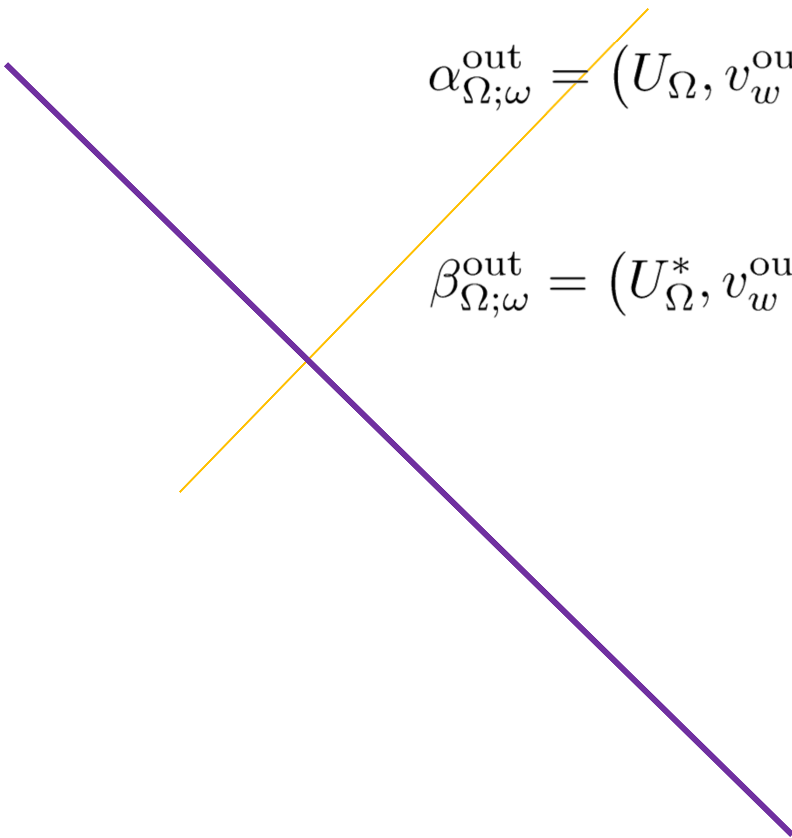
intrinsic spin,
angular momentum,
curvature effect, ….

$$\partial_+\partial_-\Psi + \left(1 - \frac{r_H}{r}\right)\left[m^2 + \cdots\right]\Psi = 0$$

$$\left(1 - \frac{r_H}{r}\right) \simeq e^{r_*/r_H} = e^{-|r_*|/r_H} \quad \text{near } r = r_H$$

back to the Schwarzschild black hole

$$X^{\pm}_{\rm out} = \pm r_H e^{\pm x^{\pm}_{\rm out}/2r_H} \qquad x^{\pm}_{\rm out} = \pm 2 r_H \log(\pm X^{\pm}/r_H)$$

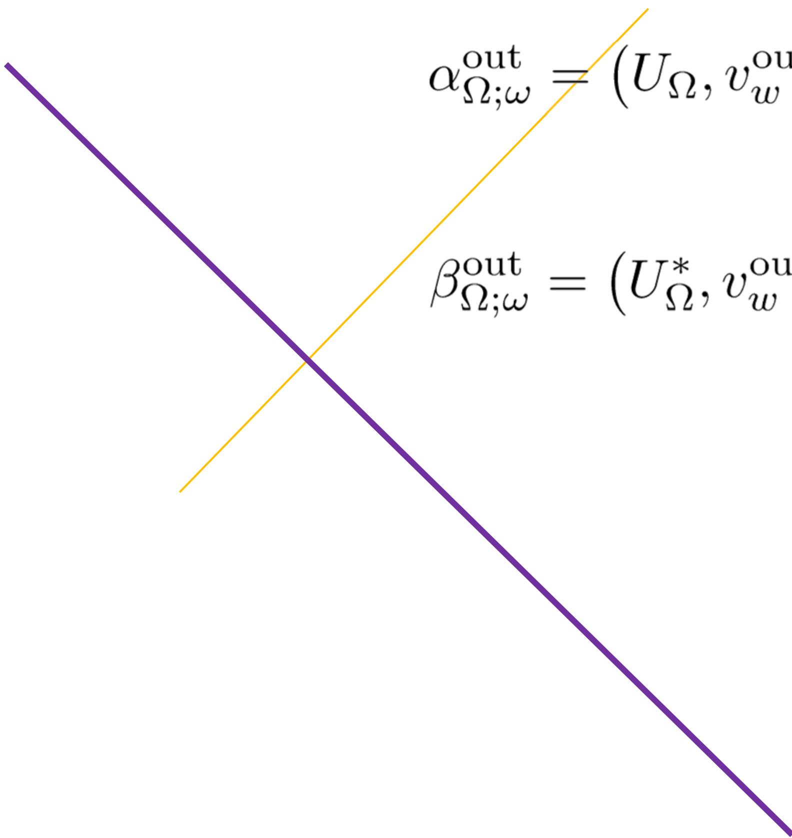


$$\alpha^{\rm out}_{\Omega;\omega} = \left(U_\Omega, v^{\rm out}_w\right) \sim \int_{-\infty}^{0} dX^- \; e^{-i\Omega X^-} e^{-i\omega \cdot 2r_H \log(-X^-/r_H)}$$

$$\beta^{\rm out}_{\Omega;\omega} = \left(U^*_\Omega, v^{\rm out}_w\right) \sim \int_{-\infty}^{0} dX^- \; e^{i\Omega X^-} e^{-i\omega \cdot 2r_H \log(-X^-/r_H)}$$

back to the Schwarzschild black hole

$$X^{\pm}_{\text{out}} = \pm r_H e^{\pm x^{\pm}_{\text{out}}/2r_H} \qquad x^{\pm}_{\text{out}} = \pm 2r_H \log(\pm X^{\pm}/r_H)$$



$$\alpha^{\text{out}}_{\Omega;\omega} = \left(U_\Omega, v^{\text{out}}_w\right) \sim \int_{-\infty}^{0} dX^- \; e^{-i\Omega X^-} e^{-i\omega\cdot 2r_H \log(-X^-/r_H)}$$

$$\beta^{\text{out}}_{\Omega;\omega} = \left(U^*_\Omega, v^{\text{out}}_w\right) \sim \int_{-\infty}^{0} dX^- \; e^{i\Omega X^-} e^{-i\omega\cdot 2r_H \log(-X^-/r_H)}$$

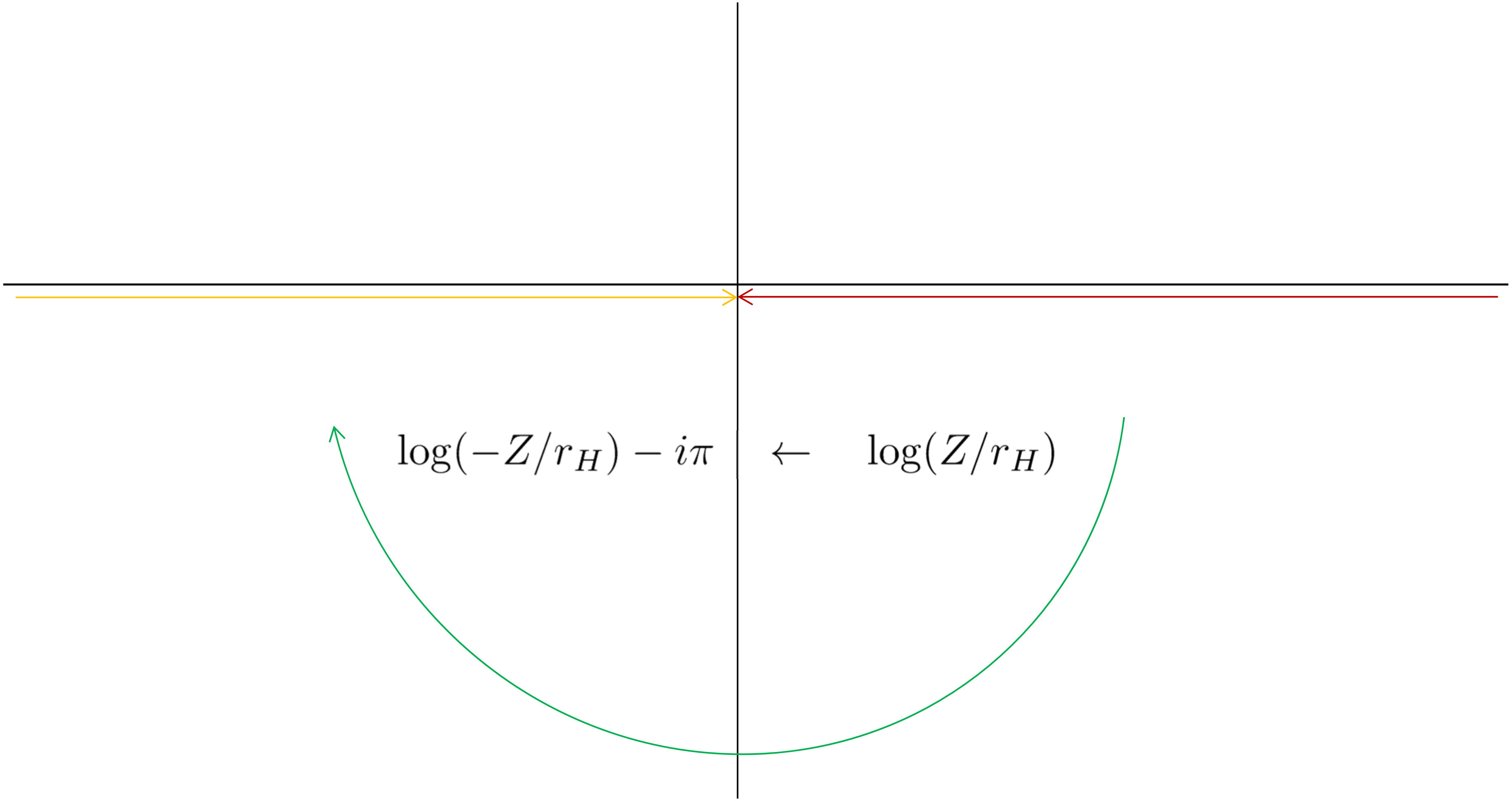
$$= -\int_{\infty}^{0} dZ \; e^{-i\Omega Z} e^{-i\omega\cdot 2r_H \log(Z/r_H)}$$

$$= -\int_{-\infty}^{0} dZ \; e^{-i\Omega Z} e^{-i\omega\cdot 2r_H \log(Z/r_H)}$$

$$= -e^{-2\pi r_H \omega} \int_{-\infty}^{0} dZ \; e^{-i\Omega Z} e^{-i\omega\cdot 2r_H \log(-Z/r_H)}$$

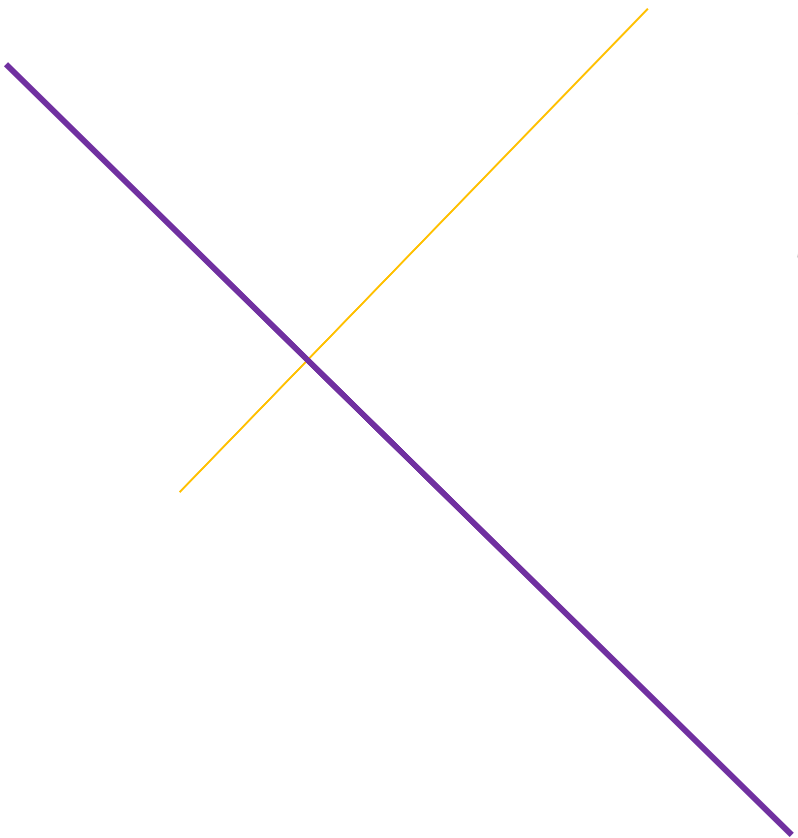
$$= -e^{-4\pi G M w} \alpha^{\text{out}}_{\Omega;\omega}$$

as we have to do an analytic continuation in the lower half plane



back to the Schwarzschild black hole

$$X^{\pm}_{\text{out}} = \pm r_H e^{\pm x^{\pm}_{\text{out}}/2r_H} \qquad x^{\pm} = t \pm r_*$$



$$\beta^{\text{out}}_{\Omega,w} = -e^{-4\pi GMw}\alpha_{\Omega;\omega}$$

$$\beta^{\text{in}}_{\Omega,w} = -e^{-4\pi GMw}\alpha_{\Omega;\omega}$$

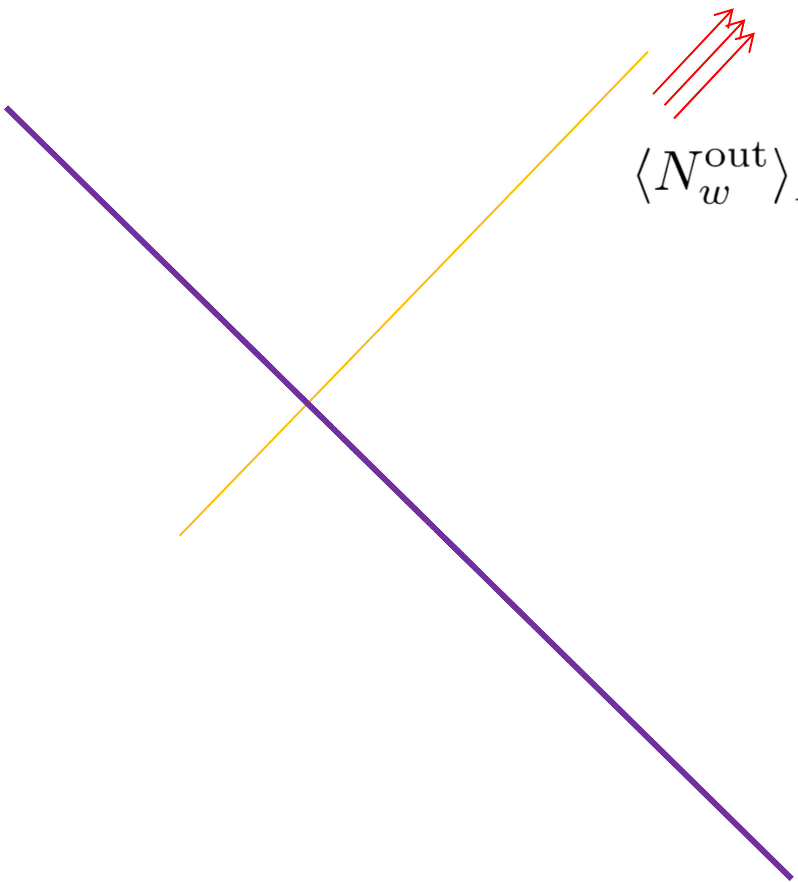
$$\delta_{w,w'} = \sum_{\Omega} \alpha^*_{\Omega;\omega}\alpha_{\Omega;\omega'} - \beta^*_{\Omega,\omega}\beta_{\Omega,\omega'}$$

$$= \sum_{\Omega} (e^{8\pi GMw} - 1)\beta^*_{\Omega,\omega}\beta_{\Omega,\omega'}$$

$$\sum_{\Omega} \beta^*_{\Omega,\omega}\beta_{\Omega,\omega'} = \frac{1}{e^{8\pi GMw} - 1}\delta_{\omega,\omega'}$$

back to the Schwarzschild black hole

$$X^{\pm}_{\text{out}} = \pm r_H e^{\pm x^{\pm}_{\text{out}}/2r_H} \qquad x^{\pm} = t \pm r_*$$



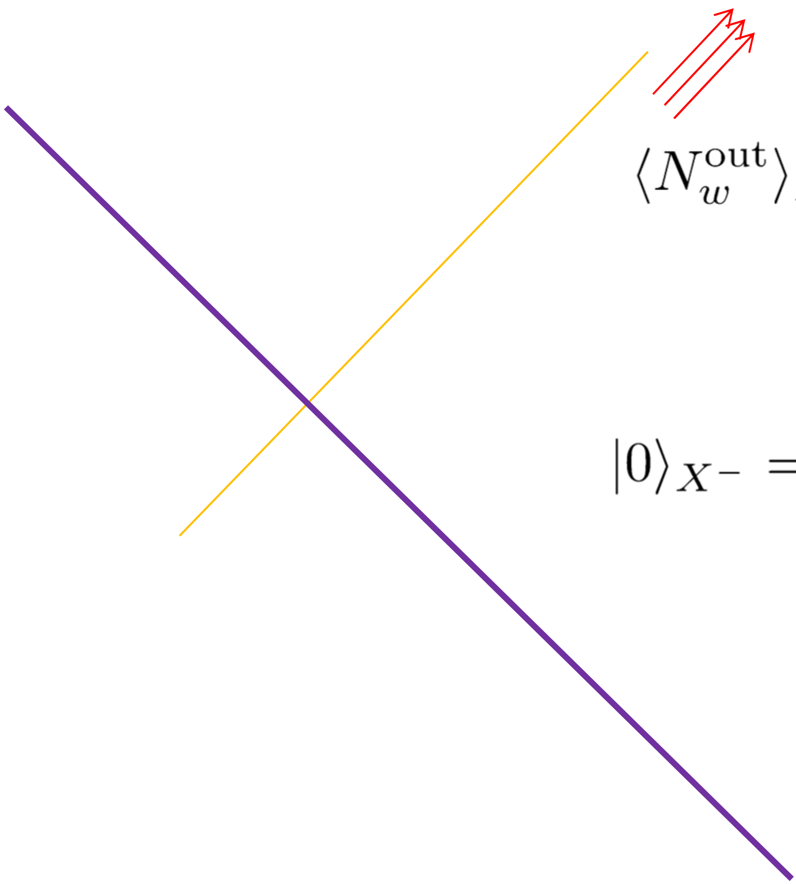
$$\langle N^{\text{out}}_w \rangle_{X^-} = \sum_{\Omega} \beta^{\text{out}*}_{\Omega,\omega} \beta^{\text{out}}_{\Omega,\omega} = \frac{1}{e^{8\pi G M w} - 1}$$

~ bosonic thermal radiation
at temperature

$$T_{\text{BH}} = \frac{1}{8\pi G M}$$

back to the Schwarzschild black hole

$$X^{\pm}_{\rm out} = \pm r_H e^{\pm x^{\pm}_{\rm out}/2r_H} \qquad x^{\pm} = t \pm r_*$$



$$\langle N^{\rm out}_w \rangle_{X^-} = \sum_{\Omega} \beta^{\rm out *}_{\Omega,\omega} \beta^{\rm out}_{\Omega,\omega} = \frac{1}{e^{8\pi G M w} - 1}$$

~ bosonic thermal radiation

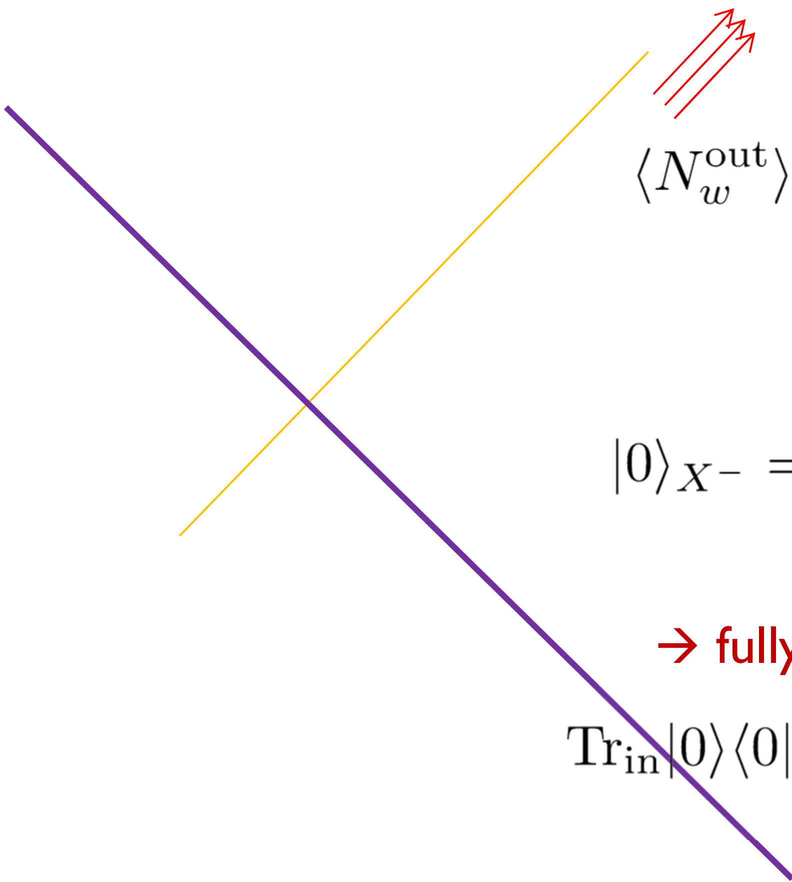
out of a pure quantum state

$$|0\rangle_{X^-} = \sum_{\vec{n}} \frac{1}{\sqrt{e^{8\pi G M w_{\vec{n}}} - 1}} \, |\vec{n}\rangle^{\rm in}_{x^-} \otimes |\vec{n}\rangle^{\rm out}_{x^-}$$

perfectly entangled pure quantum state

back to the Schwarzschild black hole

$$X^{\pm}_{\text{out}} = \pm r_H e^{\pm x^{\pm}_{\text{out}}/2r_H} \qquad\qquad x^{\pm} = t \pm r_*$$



$$\langle N^{\text{out}}_w \rangle_{X^-} = \sum_{\Omega} \beta^{\text{out}*}_{\Omega,\omega} \beta^{\text{out}}_{\Omega,\omega} = \frac{1}{e^{8\pi G M w} - 1}$$

~ bosonic thermal radiation

out of a pure quantum state

$$|0\rangle_{X^-} = \sum_{\vec{n}} \frac{1}{\sqrt{e^{8\pi G M w_{\vec{n}}} - 1}} |\vec{n}\rangle^{\text{in}}_{x^-} \otimes |\vec{n}\rangle^{\text{out}}_{x^-}$$

perfectly entangled pure quantum state

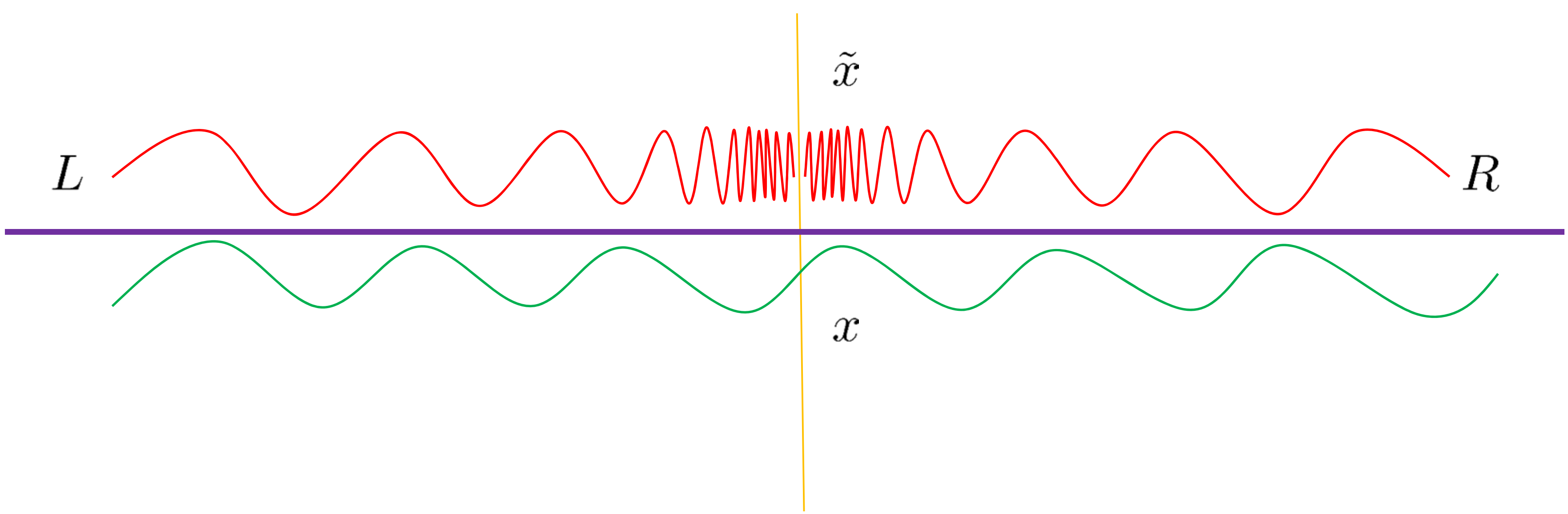
→ fully mixed, thermal state after partial trace

$$\text{Tr}_{\text{in}} |0\rangle\langle 0|_{X^-} = \sum_{\vec{n}} \frac{1}{e^{8\pi G M w_{\vec{n}}} - 1} |\vec{n}\rangle^{\text{out}}_{x^-} \langle \vec{n}|^{\text{out}}_{x^-}$$

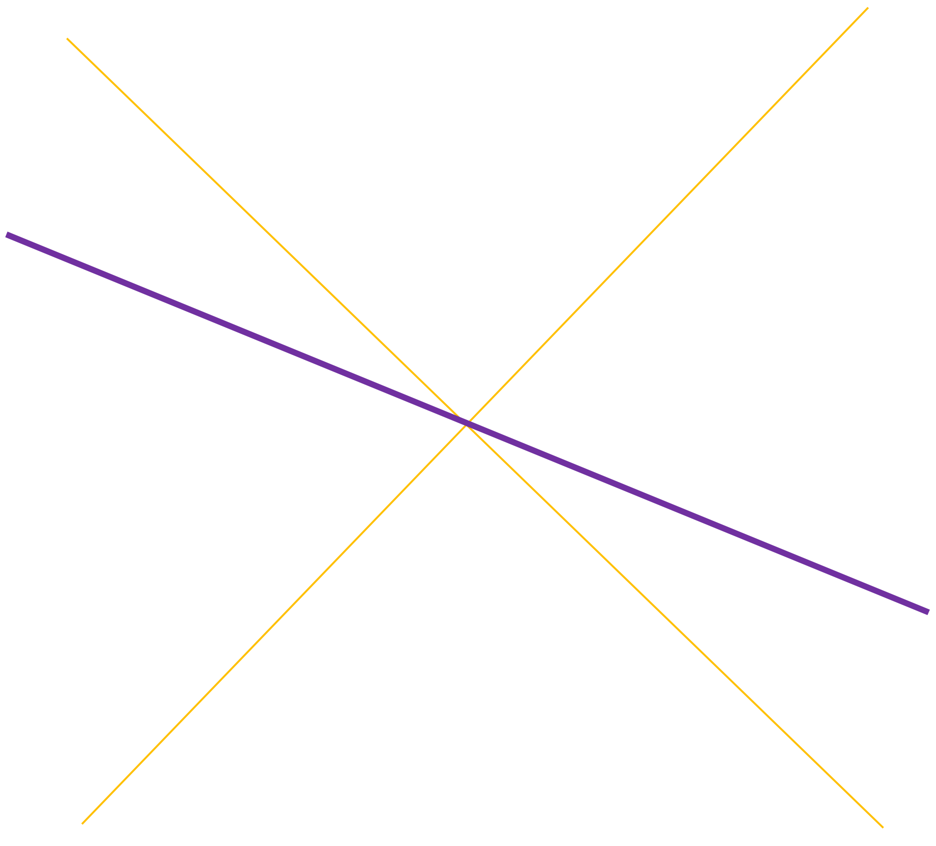
fully mixed state from a perfect quantum entanglement

$$|0\rangle_{\text{Total}} = \sum_{\vec{n}} a(\vec{n})\ |\vec{n}\rangle_{\text{L}} \otimes |\vec{n}\rangle_{\text{R}}$$

$$\rho\big|_{\text{R}} = \text{Tr}_{\text{L}} |0\rangle\langle 0|_{\text{Total}} = \sum_{\vec{n}} |a(\vec{n})|^2\ |\vec{n}\rangle_{\text{R}}\langle\vec{n}|_{\text{R}}$$

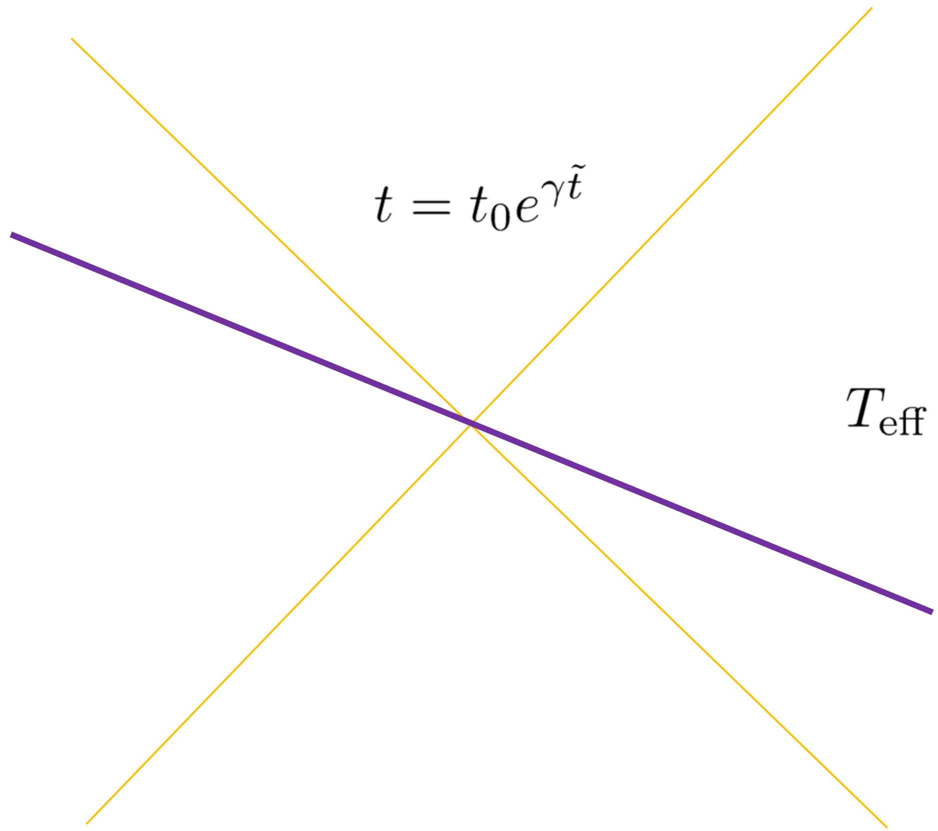


which happens whenever there is an event horizon



with thermal spectra

whenever two coordinates are related via exponential

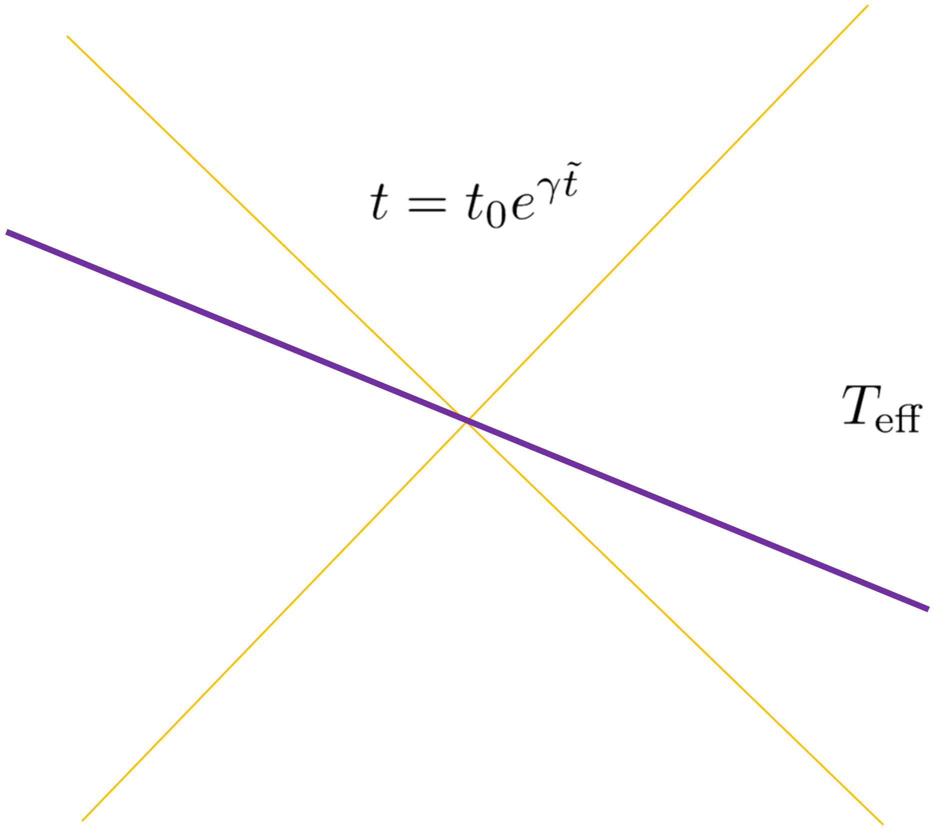


$$t = t_0 e^{\gamma \tilde{t}}$$

$$T_{\rm eff} = \frac{\gamma}{2\pi}$$

with thermal spectra

whenever the time coordinates are related via exponential



$$t = t_0 e^{\gamma \tilde{t}}$$

$$T_{\rm eff} = \frac{\gamma}{2\pi}$$

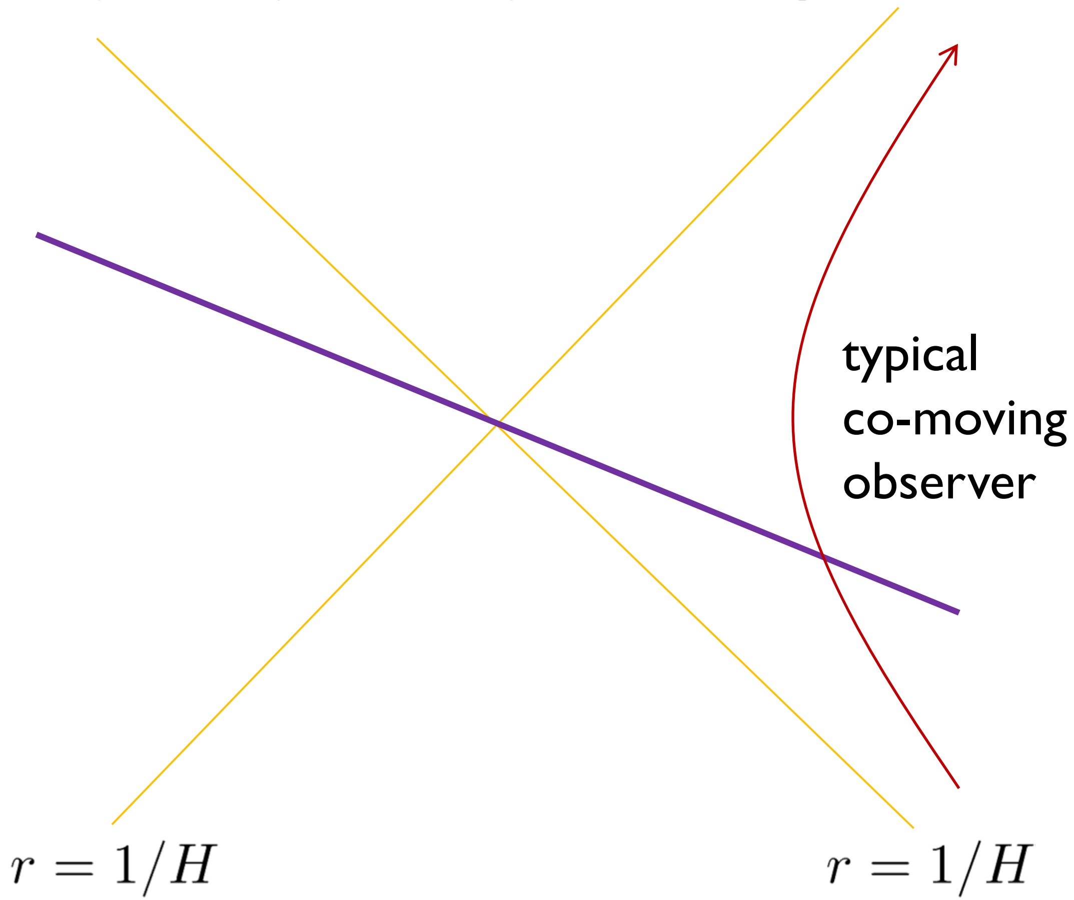
$$\frac{1}{8\pi GM}$$

$$\frac{H}{2\pi}$$

$$\frac{a}{2\pi}$$

cosmological horizon in de Sitter universe

$$-(1-H^2r^2)dt^2 + (1-H^2r^2)^{-1}dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2]$$

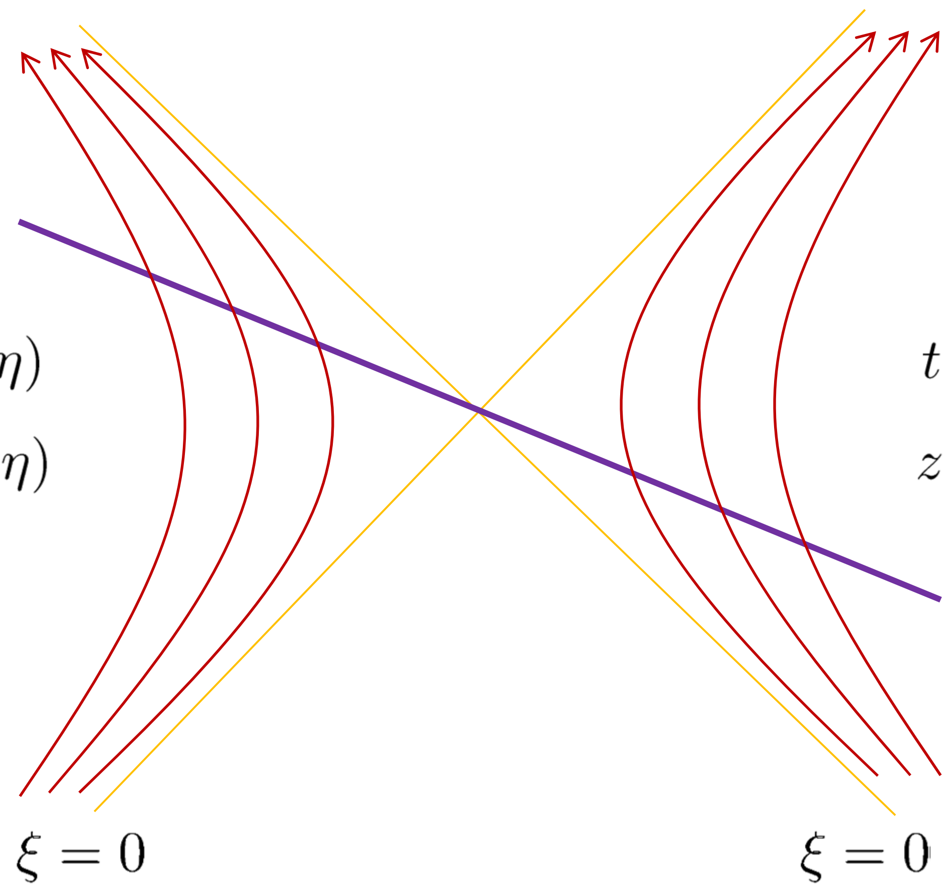


Rindler horizon for uniformly accelerating observers

$$-dt^2 + dz^2 + dx^2 + dy^2 = -(a\xi)^2 d\eta^2 + d\xi^2 + dx^2 + dy^2$$

$$t = -\xi \sinh(a\eta)$$

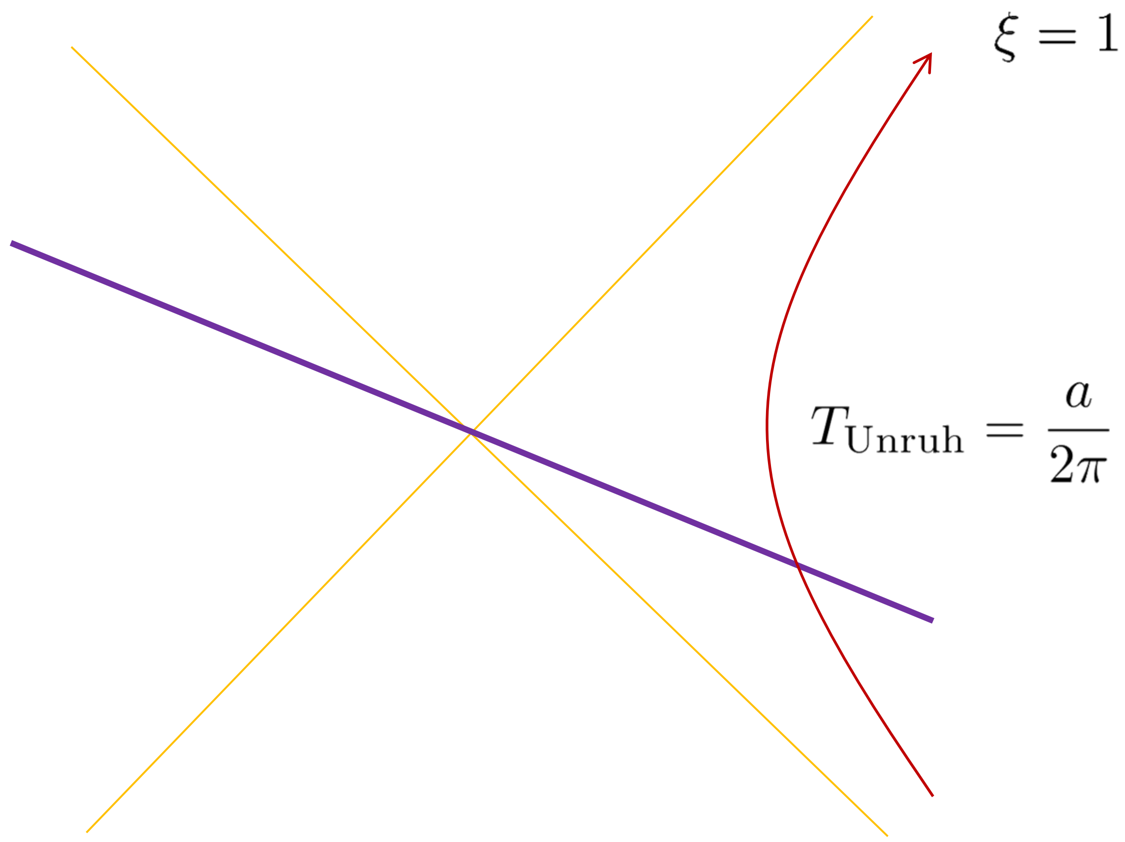
$$z = -\xi \cosh(a\eta)$$



$$t = \xi \sinh(a\eta)$$

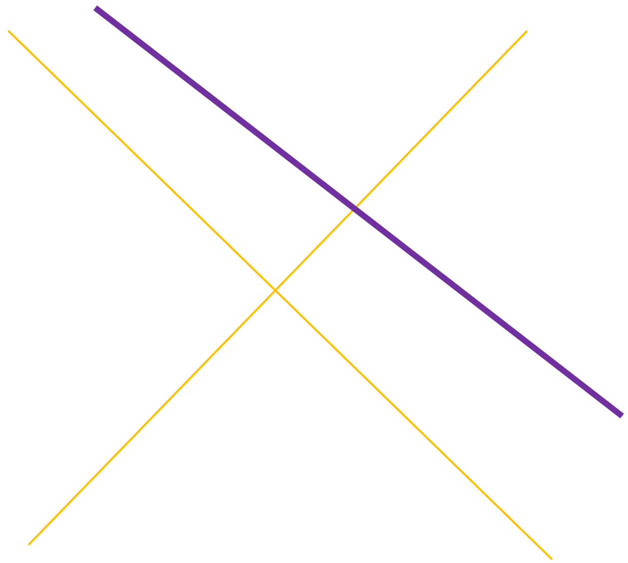
$$z = \xi \cosh(a\eta)$$

the Unruh temperature is a little more subtle
as each Rindler wedge has no obvious asymptotic frame;
the Unruh temperature refers to the observer at

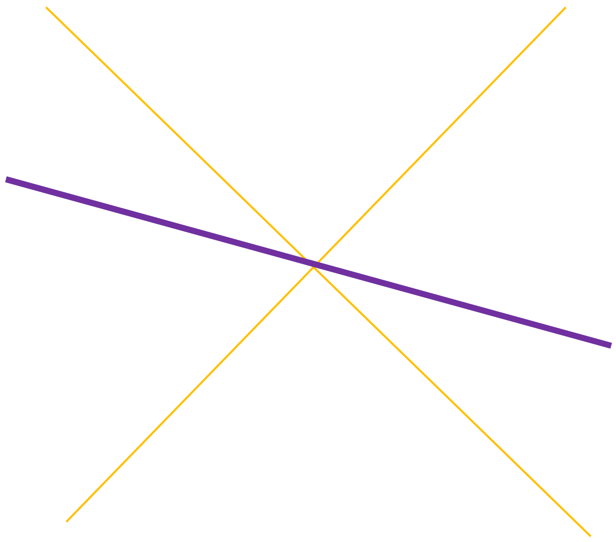


more generally, the same formula works for other accelerated observers if a is replaced by her own acceleration

radiative vs. thermal equilibrium



radiation vacuum



Hartle-Hawking vacuum for BH;
Bunch-Davies vacuum for de Sitter;
Unruh vacuum for Rindler wedges