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SUSY

1. Motivation

□ SM – a consistent QFT, which can be valid M_{pl}.

- has intrinsic scales at low energy:

Λ_{QCD} (~200MeV),

there is no problem, Λ_{QCD}/M_{pl}.
dynamically generated (due to dimensional transmutation from dimensionless coupling g_s)

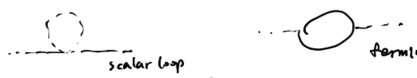
$$\Lambda_{QCD} \sim M \exp\left(-8\pi^2/b_0 g^2(M^2)\right)$$

EWSB (246 GeV) ← not dynamically generated
quantum correction

- e.g. consider Yukawa model with massless scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{\lambda}{4!}\varphi^4 - y\varphi\bar{\psi}\psi$$

Consider renormalization of scalar mass.



scalar loop　　　　fermion loop

$$-i\delta m^2 \sim -i\frac{\lambda}{32\pi^2}M^2_{cutoff} \qquad -i\delta m^2 \sim i\frac{4y^2}{16\pi^2}M^2$$

$\Rightarrow$ quadratically divergent unless $\lambda = \frac{y^2}{8}$

$$m^2 \sim \frac{\lambda/2 - 4y^2}{16\pi^2}M^2$$

hierarchy, gauge hierarchy, naturalness problem, etc.

$\Rightarrow$ fine-tuning problem for $m \ll M$

for $m \ll M$, low energy physics is sensitive to physics at high energy scale.

e.g.: $m^2_{higgs} \sim 10^4\,\mathrm{GeV}^2$

$M^2_{pl} \sim 10^{38}\,\mathrm{GeV}^2$

c.f. what about fermion mass?

→ theories of massless fermions

∃ chiral theory, forbiding mass term.

mass term is generated only if symmetry is broken.

eg. tree-level mass term explictly break it. (← explictly breaking)

even so, radiative correction to fermion mass is proportional to tree-level mass. → no fine-tuning problem.

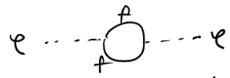
- same story for gauge boson.

⇒ We would like to find a solution to the fine-tuning problem associated with scalar sector.

i.e., find the origin and stability of electroweak symmetry breaking scale
EWSB

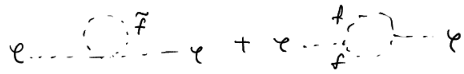
What is SUSY?

the symmetry relate particles with different spins, and provide a natural solution to the hierarchy problem.



e.g. for each chiral fermion (f_L, f_R)

∃ 2 scalar fields $(\tilde{f}_L, \tilde{f}_R)$　　$N(\tilde{f}) = N(f)$



$$\tilde{\lambda}_f = \lambda_f^2$$

→ the divergence ($\propto M^2$) is cancelled.

If $m_f = m_{\tilde{f}}$ → exact cancellation

But we haven't observed any single superpartners yet!

$m_f \neq m_{\tilde{f}}$ ⇒ ~~SUSY~~ ⇒ SUSY must be broken at some scale, and it is only an approximate symmetry.

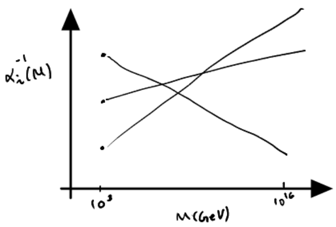
at low energy,

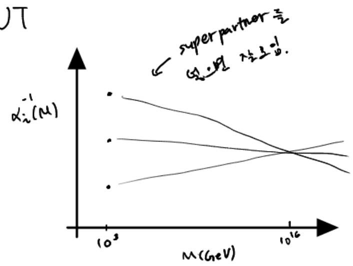
$\mathcal{L}_{eff} \supset \cancel{SUSY}$ ← SUSY breaking term

but, if it is broken explicitly, fine-tuning problem will be reintroduced.

⇒ it has to be broken spontaneously.

- Other motivation: GUT





- Dark matter

SUPERSYMMETRY

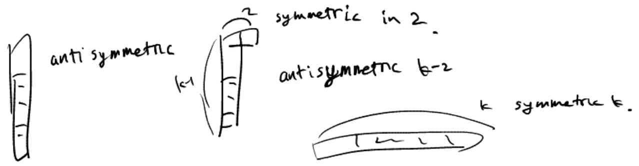
= Extension of Poincare algebra.

P_μ, $S_{\mu\nu} = M_{\mu\nu}$ + generator of internal symmetry

→ Lorentz scalar (∃ sym.)

ref. Weinberg Vol.1 Chap.2.

2. Representation of the Lorentz & Poincaré group



- symmetries always form groups!

why? assume 2 symmetries → can apply one after the other, can do backward.

· Representation of linear groups

take $GL(n,\mathbb{C})$: group of $n\times n$ complex matrices A, $\underline{A^i{}_k}$.

Defining representation

$(D(A)T)^i = A^i{}_k T^k$

as usual can take tensor products

$(D(A)T)^{i_1\cdots i_k} = A^{i_1}{}_{i_1'} A^{i_2}{}_{i_2'} \cdots A^{i_k}{}_{i_k'} T^{i_1'\cdots i_k'}$

A acts on every index!

$T^{i_1\cdots i_k}$ usually reducible representation.

To make it irreducible: "symmetrize" the indices

Well-defined based on representation of S_n symmetry group.

For k indices,

$P \in S_k$ $\begin{pmatrix} 1 \cdots k \\ p_1 \cdots p_k \end{pmatrix}$ k-dim permutation group

→ Young tableaux's

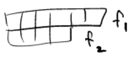
column permutation

$Y_f = \sum_q \delta_q\, q \sum_p P$ row permutation.

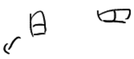
ex)

$S = f_1 + \cdots + f_s$: partition of S

$f_1 \geq f_2 \geq \cdots \geq f_k$



for example, 2 index



$\Sigma_p \; e(12)$

$\Sigma \delta_q q = e(12)$

$[Y_f, D(A)] = 0$ since Y_f made of permutation, each of which commute. $Y_f Y_{f'} = M_f Y_f \delta_{ff'}$

irreducible reps: tensors with Young symmetry 2 index case

⊟ $T^{ij} = (T^{ij} - T^{ji})$ anti-symmetric.

◫ $T^{ij} = (T^{ij} + T^{ji})$ symmetric

for more indices → could be more complicated.

$Y_f = \sum_q \delta_q q \sum P$

first symmetrize, then anti-symmetrize in the entry:

antisym in certain # of indice, for sure!

Important: can find 3 other basic reps!

upper index	$D(A) = A$	generically inequivalent for $GL(n,\mathbb{C})$
dotted lower	$D^*(A) = A^*$	
lower index	$\tilde{D}(A) = A^{T-1}$	denote 4 basic reps with 4 types of indices.
dotted upper	$\bar{D}(A) = (A^+)^{-1}$	

$(AT)^{i_1\cdots i_k} = A^{i_1}{}_{i_1'} \cdots A^{i_k}{}_{i_k'} T^{i_1'\cdots i_k'}$

$(A^*T)_{\dot{k}_1\cdots \dot{k}_k} = (A^*)^{\dot{k}_1'}{}_{\dot{k}_1} \cdots (A^*)^{\dot{k}_k'}{}_{\dot{k}_k} T_{\dot{k}_1'\cdots \dot{k}_k'}$

$= T_{\dot{k}_1'\cdots \dot{k}_k'} (A^+)^{\dot{k}_1'}{}_{\dot{k}_1} \cdots (A^+)^{\dot{k}_k'}{}_{\dot{k}_k}$

Just wired name for index to distinguish from A

Similarly

$(\tilde{A}T)_{i_1\cdots i_k} = (A^{T-1}T)_{i_1\cdots i_k}$

$= (A^{T-1})_{i_1}{}^{i_1'} \cdots (A^{T-1})_{i_k}{}^{i_k'} T_{i_1'\cdots i_k'}$

$= T_{i_1'\cdots i_k'} (A^{-1})^{i_1'}{}_{i_1} \cdots (A^{-1})^{i_k'}{}_{i_k}.$

$\overline{(\bar{D}(A)T)}^{\dot{k}_1\cdots \dot{k}_k}$

$= (A^{+-1})^{\dot{k}_1}{}_{\dot{k}_1'} \cdots (A^{+-1})^{\dot{k}_k}{}_{\dot{k}_k'} T^{\dot{k}_1'\cdots \dot{k}_k'}$

A → upper index. A acts from left.

A^* → lower dotted A^* acts from right

$\tilde{A}$ → lower index $\tilde{A}$ acts from right

$\bar{A}$ → upper dotted $\bar{A}$ acts from left.

→ we can make tensors of all these.

generic rep: $T^{i_1\cdots i_k\, \dot{k}_1\cdots \dot{k}_p}{}_{j_1\cdots j_m\, \dot{s}_1\cdots \dot{s}_n}$ + Young symmetry

But the traces invariant same with dotted indices

$T^i{}_i \to \delta^j{}_i\, T^i{}_j \to \delta^j{}_i A^i{}_{j'} T^{j'}{}_{i'} A^{-1\, i'}{}_j \to T^i{}_i$

For subgroups of $GL(n,\mathbb{C})$, not all are equivalent

Example: $U(n) \subset GL(n,\mathbb{C})$

$A^+ = A^{-1} \to \bar{A} = A.$

for $U(n)$, no distinction between dotted & undotted indices, just upper and lower.

$SL(n,\mathbb{C}) \to \det(A) = 1.$

$T_\beta{}^{a_1\cdots a_n} = T\,\varepsilon^{a_1\cdots a_n}$ invariant tensor due to $\det = 1$

$T_k \to T^{k_1\cdots k_{n-1}} = \varepsilon^{k_1\cdots k_{n-1} k_n} T_{k_n}$

1 lower index = $n-1$ anti-symmetric upper indices

All lower → upper

we can write in terms of

$T^{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_n}$ just dotted or un-dotted

$SU(n)$ may or may not introduce upper & lower index.

If n larger, may still introduce lower, but

$\bar{n}$ ← lower index ~ $n-1$ upper.

Most basic symmetry: spacetime symmetry.

<u>Poincaré group</u>

$x' = \underbrace{\Lambda x}_{\text{Lorentz transf.}} + \underbrace{a}_{\text{translation}}$

definition: $x \cdot y = x^0 y^0 - \vec{x}\cdot\vec{y}$

$:= g_{\mu\nu} x^\mu y^\nu = x_\mu y^\mu$ remains invariant

Poincaré: $\{(\Lambda, a)\} = P$

$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2)$

there are

<u>2 important subgroup</u>

homogeneous Lorentz $L = \{(\Lambda, 0)\}$

4 Translation $T_4 = \{(I, a)\}$

$(\Lambda, 0)(I, a) = (\Lambda, \Lambda a)$

$(I, a)(\Lambda, a) = (\Lambda, a)$ } do not commute! ∵ not $L \otimes T_4$. <u>semi direct group!</u>

P is not a direct product of L, T_4

$P \neq L \otimes T_4$, $P = \underbrace{L \wedge T_4}_{\text{semidirect product}}$

① <u>Homogeneous Lorentz group</u>

$x' = \Lambda x$ on 4-vectors

$x' = x^2$

$(\Lambda x, g\Lambda x) = (x, gx)$

$\boxed{\Lambda^T g \Lambda = g}$ to find generators, write as exponential

$\Lambda = e^{-\omega}$, then

$\boxed{g\omega g^{-1} = -\omega^T}$ Λ: one upper, one lower index $\omega^\mu{}_\nu$

$\omega^\mu{}_\rho g^{\rho\nu} = -g^{\mu\rho}(\omega^T)_\rho{}^\nu$

$\to \omega^{\mu\nu} = -\omega^{\nu\mu}$

<u>generators</u>: antisymmetric 4×4 matrices, 6 real parameters

<u>write them as</u>: $(-\omega)^{\lambda\rho} = \frac{i}{2}\omega^{\mu\nu}\underbrace{(S_{\mu\nu})^{\lambda\rho}}_{\mu\nu\text{ th generators}}$

$S_{\mu\nu}$: 6 indep. generators.

$(S_{\mu\nu})^{\lambda\rho} = i(\delta^\lambda_\mu \delta^\rho_\nu - \delta^\lambda_\nu \delta^\rho_\mu)$

To make it more physical

$\omega^{ik} = \varepsilon^{ikl}\omega^l$ rotation

$\omega^{i0} = -\omega^{0i} = u^i$ boost

$S^{ik} = \varepsilon^{ikl} J^l$

$S^{0i} = -S^{i0} = K^i$ } explicit form of Lorentz algebra.

$[J_i, J_k] = i\varepsilon_{ikl} J_l$

$[J_i, K_k] = i\varepsilon_{ikl} K_l$

$[K_i, K_k] = -i\varepsilon_{ikl} J_l$

$\Lambda = e^{i\omega^{\mu\nu}S_{\mu\nu}} = e^{i(\vec{\omega}\cdot\vec{J} + \vec{u}\cdot\vec{K})}$

$R(\vec{\omega}) = \Lambda(\vec{\omega}, 0)$: rotation

$L(\vec{v}) = \Lambda(0, \vec{u})$: Lorentz boost, $\vec{u} = \frac{\vec{v}}{v}\,\text{arc tanh}\, v$

$L(\vec{v})$ Not a subgroup!

4 distinct components of Lorentz group

$\Lambda^T g \Lambda = g$ $(\det\Lambda)^2 = 1$, $\det\Lambda = \pm 1$

$\Lambda^\rho{}_\mu g_{\rho\omega} \Lambda^\omega{}_\nu = g_{\mu\nu}$ $(\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1$

↳ $(\Lambda^0{}_0) \geq 1$ or $\Lambda^0{}_0 \leq -1$.

only $\Lambda^0{}_0 \geq 1$ $\det\Lambda = +1$ continuously connected to unit element

4 disconnected components:

$L = L^\uparrow_+ \oplus L^\uparrow_- \oplus L^\downarrow_+ \oplus L^\downarrow_-$

$L^\uparrow_+$	$L^\uparrow_-$	$L^\downarrow_+$	$L^\downarrow_-$
$\det\Lambda = +1$	$\det\Lambda = -1$	$\det\Lambda = 1$	$\det = -1$
$\Lambda^0{}_0 \geq 1$	$\Lambda^0{}_0 \geq 1$	$\Lambda^0{}_0 \leq 1$	$\Lambda^0{}_0 \leq -1$

$L^\uparrow_+$: real Lorentz group.

<u>space inversion</u>: $I_s x = (x_0, -\vec{x})$

$I_s = g$

$I_s L^\uparrow_+ = L^\uparrow_-$

<u>time reversal</u>: $I_T = -I_s = -g$

$I_T L^\uparrow_+ = L^\downarrow_+$

The previous parameterization is only valid for $L^\uparrow_+$

Representations of the Lorentz group

<u>1st method</u> $\Lambda = e^{i(\vec{\omega}\cdot\vec{J} + \vec{u}\cdot\vec{K})}$

trick: $\vec{M} = \frac{1}{2}(\vec{J} + i\vec{K})$, $\vec{N} = \frac{1}{2}(\vec{J} - i\vec{K})$ } for these generators,

$[M^i, M^j] = i\varepsilon^{ijk} M^k$, $[N^i, N^j] = i\varepsilon^{ijk} M^k$, $[N^i, M^j] = 0$

⇒ two seperated $SU(2)$'s $L^\uparrow_+ \sim SU(2) \times SU(2)$

what does that mean?

$$\Lambda = e^{i(\vec{\omega}\cdot(\vec{\mu}+\vec{\nu}) + \frac{\vec{u}\cdot(\vec{\mu}-\vec{\nu})}{2})}$$

$$= e^{i\vec{\mu}\cdot(\vec{\omega}-i\vec{u})} \cdot e^{i\vec{\nu}\cdot(\vec{\omega}+i\vec{u})}$$

we have $SU(2)$'s with complex parameters.

which means that every reps → product of 2 $SU(2)$ reps.

$$\underbrace{\underset{SU(2)_1}{M}, \underset{SU(2)_2}{N}}_{\text{quantum numbers}} \to D^{M,N} \text{ rep. matrix}$$

$$\hookrightarrow D^{M}_{mm'}(\vec{\omega}-i\vec{u})\, D^{N}_{nn'}(\vec{\omega}+i\vec{u})$$

2^{nd} method

$L^\uparrow_+$ equivalent to $SL(2,\mathbb{C})$

$$\underset{\text{only 4-vector}}{\{x^\mu\}} \longrightarrow \underset{2\times2 \text{ matrix}}{[x]} = x_0 \mathbb{1} - i\vec{x}\cdot\vec{\tau} \quad \text{pauli matrix}$$

$$= \begin{bmatrix} x^0 - x^3 & -x_1 + ix_2 \\ -x_1 - ix_2 & x^0 + x^3 \end{bmatrix} = [x]^\dagger$$

$$\det[x] = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x^2$$

Let us take $A \in SL(2,\mathbb{C})$ arbitrarly — 2×2 complex matrix with $\det A = 1$.

Define $[x]_A = A[x]A^\dagger$

this mean that $[x]_A^\dagger = [x]_A$

$$\det[x]_A = \det A \det[x] \det A^\dagger = \det[x] = x^2.$$

$[x]_A \to [x_A]$: a transformed 4-vector

$x_A = \Lambda_A x$ with $x_A^2 = x^2$, Λ is a Lorentz transformation

$SL(2,\mathbb{C}) \sim L^\uparrow_+$

$\Lambda \to \Lambda_A$, but $A, -A \to \Lambda_A$ (not isomorphic)

we already know all reps of $SL(2,\mathbb{C})$:

upper index dotted or undotted tensor

$D^{m,n}$ $\varphi^{a_1 \dots a_m \dot{b}_1 \dots \dot{b}_n}$ with Young symmetry of traceless

But $SL(2,\mathbb{C})$: Young symmetries only 1 row

→ need to be symmetric in dotted & undotted indices.

dim. of rep. $D^{M,N}$: $(2M+1)(2N+1)$

<u>simplest</u> : single undotted index.

$D^{M,N} \to M = \frac{1}{2}, N = 0$, $(\frac{1}{2}, 0)$ rep $\leftarrow D$.

<u>Weyl spinor</u> (2 component LH spinor)

$M = 0, N = \frac{1}{2} \to (0, \frac{1}{2})$ rep → one dotted upper index

$\sim$ RH 2 comp. spinor $\leftarrow \bar{D}$.

4-vector : $D^{\frac{1}{2},\frac{1}{2}}$: 4-dimensional $\{x^\mu\} \to [x]$ transformed as $A[x]A^\dagger$ (transform upper index; transform lower dotted index.)

$[x]^a{}_{\dot{b}}$ to get $\frac{1}{2}, \frac{1}{2}$ → raise with ε

$$[\tilde{x}]^{a\dot{b}} = [x]^a{}_{\dot{b}'}\, \varepsilon^{\dot{b}'\dot{b}}$$

clearly, both are equivalent to the transformation of 4-vector.

Also leaves :

$\sigma^\mu := (1, \vec{\sigma})^a{}_{\dot{\beta}}$ — indices from $A, A^\dagger$

How about adding parity!

$$L^\uparrow = L^\uparrow_+ \oplus I_s L^\uparrow_+$$

$I_s = g$ know $g\Lambda g = (\Lambda^T)^{-1}$

$$I_s \Lambda I_s^{-1} = (\Lambda^T)^{-1}$$

$$(\Lambda^T)^{-1} = e^{-i(\vec{\omega}\cdot\vec{J}^T + \vec{u}\cdot\vec{K}^T)}$$

generators known explicitly:

$\vec{J}^T = -\vec{J}$ (real vector)

$\vec{K}^T = \vec{K}$ (pseudo vector)

$$(\Lambda^T)^{-1} = e^{i(\vec{\omega}\cdot\vec{J} - \vec{u}\cdot\vec{K})}$$

$$I_s \Lambda(\vec{\omega}, \vec{u}) I_s^{-1} = \Lambda(\vec{\omega}, -\vec{u})$$

Remember that action of $D^{M,N}(\vec{\omega}, \vec{u})$ is

$$D^M(\vec{\omega}+i\vec{u}) \oplus D^N(\vec{\omega}-i\vec{u})$$

Action of parity : $\vec{u} \to -\vec{u}$

$\sim$ like $M \leftrightarrow N$

so irrep's including parity:

① if $M = N$ already good rep.

② if $M \neq N$ we need to double the space

$$D^{M,N} \oplus D^{N,M}$$

<u>Example:</u>

$$D^{(\frac{1}{2},0)} \to D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$$

$\psi = \begin{pmatrix} \varphi^a \\ \bar{\chi}^{\dot{b}} \end{pmatrix}$ Dirac bispior

$D(\Lambda) = \begin{pmatrix} A & \\ & \bar{A} \end{pmatrix} = S(\Lambda)$, $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by definition

$A = e^{\frac{1}{2}\vec{\tau}\cdot\vec{z}}$, $S = e^{\frac{i}{2}(\vec{\tau}\cdot(\vec{\omega}-i\vec{u}) \quad \vec{\tau}\cdot(\vec{\omega}+i\vec{u}))}$

$\to \vec{\sigma} = \begin{pmatrix} \vec{\tau} & \\ & \vec{\tau} \end{pmatrix}$ $\vec{\alpha} = \begin{pmatrix} -\vec{\tau} & \\ & \vec{\tau} \end{pmatrix}$, $\beta = P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

→ get the entire Dirac matrix algebra, just out of group theory.

$$\gamma^i = \beta\alpha^i, \quad \gamma^0 = \beta, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Representations of the Poincaré's group

(aka: what are particles)

$P = L \wedge T_4$

$P_+^\uparrow = L_+^\uparrow \wedge T_4$

want $(\Lambda, a) \to U(\Lambda, a)$ unitary reps. of Poincaré group

$|p,\alpha\rangle$: a state of momentum p & some quantum # α

$P_\mu |p,\alpha\rangle = p_\mu |p,\alpha\rangle$

$\underset{\text{translation}}{U(a)}|p,\alpha\rangle = e^{iap}|p,\alpha\rangle$

For a fixed p, $\mathcal{H}_p = \{|p,\alpha\rangle\}$

$U(\Lambda)|p,\alpha\rangle = |p,\alpha\rangle_\Lambda$ ← transformed state

$U(a)|p,\alpha\rangle_\Lambda = U(\Lambda,a)|p,\alpha\rangle$

$= U(\Lambda)U(\Lambda^{-1}a)|p,\alpha\rangle$

$= e^{i(\Lambda^{-1}a\cdot p)}U(\Lambda)|p,\alpha\rangle = e^{ia\cdot\Lambda p}|p,\alpha\rangle_\Lambda$

This means $|p,\alpha\rangle_\Lambda$ is $\mathcal{H}_{\Lambda p}$

Define $O_q = \{\Lambda q\}$ track contain q.
(act with all possible Λ's)

$\mathcal{H}_{O_p} = \{|q,\alpha\rangle, q = \Lambda p\}$ is an invariant space

→ space of irreps. will correspond to these tracks

track determined by $\hat{p}$ characteristic momentum!

$\mathcal{H}_{\hat{p}} = (\Lambda \mid \Lambda\hat{p} = \hat{p})$ LITTLE GROUP of $\hat{p}$
(invariant group of $\hat{p}$)

Track	$\hat{p}$	$\mathcal{H}_p$
$p^2=m^2>0$, $p^0>0$	$(m,0)$	$SO(3)$
$p^2=m^2>0$, $p^0<0$	$(-m,0)$	$SO(3)$
$p^2=0$ $p^0>0$	$(1,\hat{e}_3)$	$\tilde{E}_2$ ← Euclidean 2 dim rot.
$p^2=0$ $p^0<0$	$(-1,\hat{e}_3)$	
$p=0$		$L_+^\uparrow \sim SL(2,C)$
$p^2=-m^2<0$	$(0, m\hat{e}_3)$	$SU(1,1)$ $\lvert z_1\rvert^2-\lvert z_2\rvert^2 = \text{inv.}$

Want to characterize arbitrary rep.

Define: $L(p)\hat{p} = p$
matrix that takes chosen $\hat{p}$ to arbitrary p

if $L'\hat{p} = p$ $L^{-1}L'\hat{p} = \hat{p}$

$\underbrace{L^{-1}L'}_{\text{part of little group}} \in \mathcal{H}$ $L' = L\Lambda$ $\Lambda \in \mathcal{H}_{\hat{p}}$

pick one choice, pick $\hat{p}$ $p \to L(p)$

$U(\Lambda)|\hat{p},\alpha\rangle = ?$

Assume $\Lambda \in \mathcal{H}_{\hat{p}}$ leaves $\hat{p}$ invariant

must just give a rep of the little group

$U(\Lambda)|\hat{p},\alpha\rangle = D_{\alpha'\alpha}|\hat{p},\alpha'\rangle$
← rep. of little group

define $|p,\alpha\rangle = U(L(p))|\hat{p},\alpha\rangle$

Then acting on ground state,

$U(\Lambda)|p,\alpha\rangle = U(\Lambda)U(L(p))|\hat{p},\alpha\rangle$

$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p))\,U(\Lambda)U(L(p))|\hat{p},\alpha\rangle$

$= U(L(\Lambda p))\underbrace{U(L^{-1}(\Lambda p)\Lambda L(p))}_{\text{this is within the little group}}|\hat{p},\alpha\rangle$

$L^{-1}(\Lambda p)\Lambda \underbrace{L(p)\hat{p}}_{p} = \hat{p}$

$U(\Lambda)|p,\alpha\rangle = D_{\alpha'\alpha}(L^{-1}(\Lambda p)\Lambda L(p))\,|\Lambda p,\alpha'\rangle$

: Wigner rotation

Meaning: irrep of $P_+^\uparrow$ = track + rep. of little group

the possible tracks:

① $p = \hat{p} = (0,0)$ correspond to vacuum

Little group $SL(2,\mathbb{C}) \sim L_+^\uparrow$

All unitary irreps. infinite dimensional, except trivial unitary rep. $1 : \Lambda \to 1$

This will describe the vacuum of the theory,

not just 1 vacuum. $|0\rangle \to 1$

$P_\mu|0\rangle = 0$, $\vec{J},\vec{K}|0\rangle = 0$. $U(a)|0\rangle = U(\Lambda)|0\rangle = 0$

② **Massive particles** $\hat{p} = (m,0)$, $p^2 = m^2$

particles with mass m.

Little group : $SO(3) \sim SU(2)$

irreps D^j ← spin j rep of $SU(2)$

A state is given by $\boxed{m, j}$

$|\hat{p},\alpha\rangle \to |\hat{p},j\sigma\rangle$

little group $\mathcal{H}_{\hat{p}} = SU(2)$

$L(p) = L(-v) = e^{i\vec{\omega}\cdot\vec{K}}$, $\vec{v} = \frac{\vec{p}}{p^0}$

where $u = \frac{1}{\sqrt{1-v^2}} = \frac{p^0}{m}$

so generic state, $|p,\alpha\rangle \to |p,j\alpha\rangle$ an we know

$U(\Lambda)|p,j\sigma\rangle = D^j_{\sigma'\sigma}(L(\Lambda p)^{-1}\Lambda L(p))\,|\Lambda p, j\sigma\rangle$

For particle at rest:

$$U(R)\,|\hat p, j\sigma\rangle = D^{j}_{\sigma'\sigma}(R)\,|\hat p, j\sigma'\rangle$$

$\downarrow$

$$J_3\,|\hat p, j\sigma\rangle = \sigma\,|\hat p, j\sigma\rangle$$

J_{op} : spin of particle at rest.

For moving particles, better to define the Helicity

$$\Sigma = \frac{\vec J\cdot\vec p}{|\vec p|} \qquad \left.\begin{matrix}[\vec J, \Sigma] = 0\\ [p^i, \Sigma] = 0\end{matrix}\right\} \text{ can measure simultaneously}$$

$$|p, j\sigma\rangle \longrightarrow |p, j\,\lambda\rangle \quad \text{(helicity states)}$$

③ massless particles $p^2 = 0$, $p^0 > 0$

$\hat p = (1, \vec e_3) = (1, 0, 0, 1)$

First find out what is $L(p)$

(by def, makes p out of $\hat p$)

First a Lorentz boost $L(-v)$

$$\begin{pmatrix}\cosh u & \sinh u\\ \sinh u & \cosh u\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = (\cosh u + \sinh u)\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \underbrace{(\cosh u + \sinh u)}_{\frac{1}{\sqrt{1-v^2}} + \frac{v}{\sqrt{1-v^2}}}\,\hat p = |p|\,\hat p$$

Lorentz boost : increases Length of the momentum

then do rotation $R(\vec p)$

$R(\vec p)\,\vec e_3 = \frac{\vec p}{p}$

so $L(p) = R(\vec p)\,L(-v)$

Important: what is a little group

$\Lambda p = p$ look for generators of group

$\to$ do infinitesimal transformation!

$\Lambda e^{-\omega} \approx (1-\omega)$

$(1-\omega)p = p \to \omega p = 0$. ← antisymmetric matrix.

$\omega^{\mu\nu}p_\nu = 0 \longrightarrow$ for $\mu = i$, $\underbrace{\omega^{ik}}_{\epsilon^{ikl}\omega_l}p_k + \omega^{i0}p_0 = 0$.

$(\vec\omega\times\vec p)^i + p_0 u^i = 0$ for $\mu = i$

$\omega^{\mu\nu}p_\nu = 0 \xrightarrow[\text{for } \mu=i]{} \omega^{0i}p_i = \omega^{i0}p^i = \vec u\cdot\vec p = 0$

So, $\omega^{\mu\nu}p_\nu = (\vec u\cdot\vec p,\ \vec\omega\times\vec p + p_0\vec u)$ for us $p = \hat p = (1, \vec e_3)$

so $\omega\hat p = (u^3,\ \vec\omega\times\vec e_3 + \vec u)$

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = 0$$

$\omega\hat p = 0 \longrightarrow u^3 = 0$, $\vec\omega\times\vec e_3 + \vec u = 0$ $\quad \begin{pmatrix}\omega_2 + u_1\\ -\omega_2 + u_2\\ 0\end{pmatrix} = 0.$

$u_1 = -\omega_2$, $u_2 = \omega_1$, ω_3 arbitrary, $(u_3 = 0)$

so transformation is

$$\omega^3 J^3 + \omega^1\underbrace{(J^1 + K^2)}_{=:M^1} + \omega^2\underbrace{(J^2 - K^1)}_{=:M^2}$$

↑ the generators of the little group of the massless particles.

we want calculate: $[J^3, M^{1,2}] = \pm i M^{1,2}$

$[M^1, M^2] = 0$.

this is an Euclidean group of plane, E_2.

$M^{1,2}$: 2dim translation

J^3 : rotation

E_2 : $SO(2) \ltimes T_2$ ← another semi direct product.

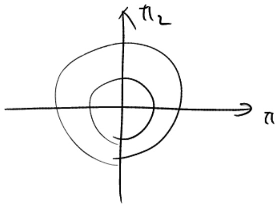
same structure as P itself!

need to use the same method

$|\vec\pi, \alpha\rangle$ states ← 2-dim momentum

$\vec M\,|\vec\pi, \alpha\rangle = \vec\pi\,|\vec\pi, \alpha\rangle$

need to have track again to find irrep



track	$\hat\pi$	$H_{\hat p}$
$\pi^2 = c^2$	$c(1,0)$	I
$\pi^2 = 0$	$(0,0)$	$SO(2)$

1) irreps of $\pi^2 = c^2$ track are all trivial

2) irreps of $\pi^2 = 0$ track $\to$ $SO(2) \sim U(1)$ irrep

$R(\omega) = e^{i\omega\lambda}$ $\quad \lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$

has to be integers so that field is single valued (bosons) or half-integer (fermions)

$U(\vec h)\,|\lambda\rangle = |\lambda\rangle$

$U(R)\,|\lambda\rangle = e^{i\omega\lambda}\,|\lambda\rangle$

pick non-trivial ones for irrep. of $H_{\hat p}$

$|p\,\alpha\rangle \to |p\,\lambda\rangle$

How does it actually act?

$L^{-1}(\Lambda p)\,\Lambda L(p) = e^{i\omega J_3}\,e^{i\vec h\vec M}$ ← find ω, h for given Λ

$\to \omega(L^{-1}(\Lambda p)\,\Lambda L(p))$

$U(\Lambda)|p\,\lambda\rangle = e^{i\lambda w}\,|\hat{p},\lambda\rangle$

λ : eigenvalue of J_3 for generic p's

$\Sigma = \frac{\vec{J}\cdot\vec{p}}{p}$ helicity again,

$\Sigma\,|p,\lambda\rangle = \Sigma\, U(R(\vec{p}))\,||p|\,\hat{p},\lambda\rangle$

$= U(R(\hat{p}))\, J^3\,||p|\,\hat{p},\lambda\rangle = \lambda\,|p,\lambda\rangle$

λ : helicity of zero mass particle

0 mass particle : has only one helicity state!

But : if you also want $P^\uparrow = P_+^\uparrow + I_s P_T^\uparrow$

including space inversion (parity) we need to double the space

If include the parity : will get 2 helicity states

$\left(\begin{array}{ll} \text{photon: } 2 \text{ helicity state}, & \mathcal{L}=1 \\ \text{graviton: } \quad " & \mathcal{L}=2 \end{array}\right)$

generically for QFT :

particles $\to$ Hilbert space

$\mathcal{H} = \{|0\rangle,\ |p;\sigma\rangle,\ |\widetilde{p\ j\sigma}\rangle\}$ $\leftarrow$ conjugate

$|p_1\ j_1\ \sigma_1,\ \cdots, p_r\ j_r\ \sigma_r\rangle$ - - -

$\uparrow$ multiparticle state

$|p\,j\,\sigma\rangle = N_p\, a^\dagger(p\,j\,\sigma)\,|0\rangle$ $\qquad a|0\rangle = b|0\rangle = 0$

$|\widetilde{p\,j\,\sigma}\rangle = N_p\, b^\dagger(p\,j\,\sigma)|0\rangle$ $\qquad [a(p\sigma), a^\dagger(p'\sigma')]$

$\qquad = \delta_{\sigma\sigma'}\,\delta(\vec{p}-\vec{p}\,')$

For every particle : also a field acting on the Hilbert space,

$$\varphi_r(x) = \int \frac{d^3p}{N_p}\left[U_r(p\sigma)\,a(p\sigma)\,e^{-ip\cdot x} + V_r(p\sigma)\,b^\dagger(p\sigma)\,e^{ip\cdot x}\right]$$

Field is a tensor operator according to the finite dimensional rep of Lorentz group

SUPERSYMMETRY

= Extension of the Poincaré algebra

Poincaré group:

P_μ ; $S_{\mu\nu} = M_{\mu\nu}$ + generators of internal symmetry

$\to$ Lorentz scalar

Can we enlarge the Poincaré algebras include a nontrivial commutation relation between internal choices & space time symmetry ?

1960's : SU(3) internal of Gell-Mann
SU(2)

can you use an approximate SU(6) symmetry

Coleman–Mandula theorem

No possible conserved charge, that is non-trivial under Lorentz

8/30 ref. Wess-Bagger Chap 1~3

Coleman-Mandula no go theorem

Example: assume non-trivial Lorentz structure exists.

e.g. $Q_{\mu\nu}$ with $Q_\mu{}^\mu = 0$ conserved.
symmetric under $\mu \leftrightarrow \nu$

Consider a field carrying this charge $|p\rangle$

$\langle p | Q_{\alpha\beta} | p \rangle = C(p_\alpha p_\beta - \frac{1}{d} p^2 \eta_{\alpha\beta})$

if $C \neq 0$, then this should be conserved even during scattering.

Assume $P_1 + P_2 \to P_1' + P_2'$

$$C[P_{1\alpha} P_{2\beta} + P_{2\alpha} P_{2\beta} - \frac{1}{d}\eta_{\alpha\beta}(m^2+m^2)]$$
$$= C[P_{1\alpha}' P_{1\beta}' + P_{2\alpha}' P_{2\beta}' - \frac{1}{d}\eta_{\alpha\beta}(m^2+m^2)]$$

Since you already have momentum conservation, $P_1+P_2 = P_1'+P_2'$

this gives an additional kinematic constraint on scattering.

$P_i' = \pm P_i$

Scattering can only happen in forward or backward direction! but this is possible only in $d=2$, unless interaction is trivial.

General form of C-M no go theorem

(1) S-matrix based on local, relativistic QFT in $d=4$

(2) finite # of different particles without particle state of given mass.

(3) energy gap (no massless particles)

$\Rightarrow$ Lie algebra of symmetries $= P +$ Lorentz scalar
internal scalar

Supersymmetry: get around Coleman-Mandula theorem by enlarging the notion of symmetry.

Till now, we assumed implicitly: all symmetry are bosonic.

(e.g. a symmetry thakes a boson $\to$ boson
fermion $\to$ fermion)

Other possibility: fermion $\longleftrightarrow$ boson

In this case, some generators fermionic

$\Rightarrow$ Need to extend notion of symmetry algebra & symmetry group.

Usually, symmetry

$[x, x'] = x''$

For SUSY: "graded Lie algebra"

:= 2-types of generators (= bosonic & fermionic)

$\to \{Q, Q'\}_+ = X$ Q: odd element which is anti-commuting

$[Q, X]_- = Q''$ X: even " " symmetric

super charges

Haag, Lopuszanski-Sohnius:

most general graded Lie algebra consistent with relativistic interacting QFT's: SUSY algebra

The SUSY algebra

X: even $\leftarrow$ by Coleman-Mandula $= P +$ internal symmetry

Q: odd

Q: fermionic, with the same Lorentz invariant structure.

$Q = \oplus \Sigma\, Q_{\underbrace{\alpha_1 \cdots \alpha_a}_{\text{symmetric}} \underbrace{\dot\alpha_1 \cdots \dot\alpha_b}_{\text{anti-symmetric}}}$ generically

Consider

$\{Q_{\cdots i \cdots}, \bar{Q}_{\cdots i, \cdots}\} = ?$

under ordinary spin, $Q, \bar{Q}$: $\frac{1}{2}$

so anti-commute

$\to$ symmetrize

$\to$ spin $(a+b)$

Must close into P_μ or $M_{\mu\nu}$

but this is antisymmetric in $\mu \leftrightarrow \nu$

$\to$ not possible

so we have to $a+b=1$, so only possibility:

$\{Q_\alpha^L, \bar{Q}_{\dot\alpha M}\} = 2 P_m \sigma^m_{\alpha\dot\alpha} \delta^L_M$

L, M: # of supercharge

convention. $\to$ normalization of Q's

$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2P_m \sigma^m_{\alpha\dot\beta}$ just for 1 supercharge

Show $[P_m, Q_\alpha] = 0 \to$ homework.

Note: May need to use Jacobi identity generalized for graded Lie algebra.

$$\{Q_\alpha^L, Q_\beta^M\} = \varepsilon_{\alpha\beta} Z^{LM} + M_{\alpha\beta} Y^{LM}$$

antisymmetric ($\varepsilon_{\alpha\beta}$); symmetric $\to$ spin 1 ($M_{\alpha\beta}$) $\to$ must be proportional to $M_{\mu\nu}$

HW

use $[Q^M_\alpha, P_\nu] = 0$, Jacobi $[P_\xi, \{Q^L_\alpha, Q^M_\beta\}] = 0$.

but $[P, M] \neq 0$, so $Y^{LM} = 0$.

$\{Q^L_\alpha, Q^M_\beta\} = 0$.

Z^{LM} : central charge

For $N=1$, $\{Q_\alpha, Q_\beta\} = 0$ only possibility.

For general case, can show Z^{AB} commutes with everything.

Assume no central charge : then there is an internal symmetry by rotating the supercharges among themselves.

$$Q^{A'}_\alpha = U^{AB} Q^B_\alpha \qquad \bar{Q}^{A'}_{\dot\alpha} = (U^{AB})^* \bar{Q}^B_{\dot\alpha}$$

$U(N)$ unitary matrix $\to$ R-symmetry

For $N=1$, there is no central charge. $U(1)$ symmetry, $U(1)_R$!

Since Q^A_α has definite Lorentz-transform properties :

$$(Q^A_\alpha)' = (1 + \tfrac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu})_\alpha{}^\beta Q^A_\beta$$

$$= U(\Lambda)^\dagger Q_\alpha U(\Lambda) = Q_\alpha + \tfrac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}, Q_\alpha]$$

$$\to [M^{\mu\nu}, Q_\alpha] = -i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$

Let us look at the internal symmetry

$[B^r, B^s] = i C^{rst} B^t$, Q_α^A : some irrep. of B (part of $U(N)_R \cdots$)

$$[B^r, Q_\alpha^A] = -(b^r)^A{}_C Q_\alpha^C$$

so full SUSY algebra

$$\{Q_\alpha^A, \bar{Q}_{\dot\beta B}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta_B{}^A$$

$$\{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta} Z^{AB}$$

$$[Q_\alpha^A, P_\mu] = 0$$

$$[Q_\alpha^A, M_{\mu\nu}] = -i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^A$$

$+$ Lorentz, $[M^{\mu\nu}, P^\lambda] = i(\eta^{\nu\lambda}P^\mu - \eta^{\mu\lambda}P^\nu)$

$[M^{\mu\nu}, M^{\rho\sigma}] = \cdots$ — usual Poincaré algebra.

$[Z, \cdots] = 0$.

Consequence of the SUSY algebra

Q_α^A : fermionic operator

boson & fermions are contained in same reps.

symmetry connects bosons & fermions $\to$ supersymmetry.

① # of fermions = # of bosons

takes any reps R:

define fermion # operator :

$(-1)^{N_F}|q\rangle = +|q\rangle$ if $|q\rangle$ is bosonic

$-|q\rangle$ if $|q\rangle$ is fermionic

Q_α^A makes boson from fermion, fermion from boson.

$$(-1)^{N_F} Q_\alpha^A |q\rangle = -Q_\alpha^A (-1)^{N_F} |q\rangle$$

$$\to \{(-1)^{N_F}, Q_\alpha\} = 0.$$

Now consider an irrep of SUSY

$$\mathrm{Tr}[(-1)^{N_F}\{Q_\alpha^A, \bar{Q}^B_{\dot\beta}\}]$$

$$= \mathrm{Tr}[(-1)^{N_F}(Q_\alpha^A \bar{Q}^B_{\dot\beta} + \bar{Q}^B_{\dot\beta} Q_\alpha^A)]$$

$$= \mathrm{Tr}[-Q_\alpha^A (-1)^{N_F} \bar{Q}^B_{\dot\beta} + (-1)^{N_F} \bar{Q}^B_{\dot\beta} Q_\alpha^A] = 0.$$

But : $\mathrm{Tr}[(-1)^{N_F}\{Q_\alpha^A, \bar{Q}^B_{\dot\beta}\}]$

$= \mathrm{Tr}[(-1)^{N_F}(2\sigma^\mu_{\alpha\dot\beta} P_\mu)]$ in an irrep for fixed P_μ

$= \underbrace{\mathrm{Tr}[(-1)^{N_F}]}_{=0} (2\sigma^\mu_{\alpha\dot\beta} P_\mu)$

$$\mathrm{Tr}(-1)^{N_F} = 0$$

$\to$ in SUSY rep, equal # of fermions & bosons.

② $[P_\mu, Q_\alpha^A] = 0 \to [P^2, Q_\alpha^A] = 0$

P^2 : casimir operator,

$\to$ every particle in SUSY irrep has the same mass.

Equal # of fermions and bosons, and equal mass for them.

③ Positivity of energy

$H = P_0$. $\{Q_\alpha^A, \bar{Q}^B_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta_B{}^A$

multiply by $\bar\sigma^{\nu\dot\beta\alpha}$ & take trace, $\sigma^\mu = (1, \vec\sigma)$, $\bar\sigma^\mu = (1, -\vec\sigma)$

$$\mathrm{Tr}\{Q_\alpha^A, \bar{Q}_{\dot\beta}^B\}\bar\sigma^{\nu\dot\beta\alpha} = 2\,\mathrm{Tr}(\underbrace{\sigma^\mu_{\alpha\dot\beta}\bar\sigma^{\nu\dot\beta\alpha}}_{2\eta^{\mu\nu}}) P_\mu \delta^A{}_B$$

take $\nu = 0$, $P_0 = +1$, $\bar\sigma^0 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$\underline{4H = Q_1^A \bar{Q}_{\dot 1}^A + \bar{Q}_{\dot 1}^A Q_1^A + Q_2^A \bar{Q}_{\dot 2}^A + \bar{Q}_{\dot 2}^A Q_2^A} \text{ positive definite}$$

$$\Rightarrow \langle\psi|H|\psi\rangle \geq 0$$

$\Rightarrow$ energy is positive definite in SUSY theories.

if $H|\Psi\rangle = 0$

$\rightarrow$ all terms varies

$$Q_\alpha^A |\Psi\rangle = \bar{Q}_{\dot\alpha}^A |\Psi\rangle = 0$$

$\rightarrow$ SUSY vacuum energy $= 0$ if SUSY unbroken

$\rightarrow H = 0$ is vacuum.

Ireps of SUSY

Assume for now no central charge.

Use the same method as for Poincaré group:

find track (orbits), little group, ect, ...

Massless particles

$p = (E, 0, 0, E)$, Little group E_2 as before which has irreps defined by helicity $J = J_3$

$$J|\lambda\rangle = \lambda|\lambda\rangle$$

But, now we have additional generators, $Q, \bar{Q}$.

How are they represented?

$$\{Q_\alpha^A, \bar{Q}_{\dot\beta}^B\} = 2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu$$

$$\{Q_\alpha^A, \bar{Q}_{\dot\beta}^B\} = 2(\sigma^0_{\alpha\dot\beta} - \sigma^3_{\alpha\dot\beta})E\delta^{AB}$$

$$= 2E\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right] = 4E\begin{pmatrix}0&0\\0&1\end{pmatrix}\delta^{AB}$$

$\{Q_1^A, Q_1^{*B}\} = 0$ + positivity

$\rightarrow$ only trivial representation

$$\{Q_2^A, Q_2^{*B}\} = 4E\delta^{AB}$$

Let us $\underline{\text{define}}$: $a^A = \frac{1}{2\sqrt{E}} Q_2^A$, $Q^{+A} = \frac{1}{2\sqrt{E}}\bar{Q}_2^A = (a^A)^+$

N fermionic creation & annihilation operators.

$$\{a^A, a^{+B}\} = \delta^A{}_B$$

$$\{a, a\} = \{a^+, a^+\} = 0.$$

a^+, a also raise & lower helicity!

since $\underset{\substack{\uparrow \\ \text{spinor index}}}{a^A \sim Q_\alpha^A}$ $\rightarrow$ changes helicity by $\pm\frac{1}{2}$

take lowest helicity state $|\lambda\rangle$

$\rightarrow$ annihilated by all a's, $a^A|\lambda\rangle = 0 \quad \forall A$.

$$\underset{\substack{\uparrow \\ \text{helicity } \lambda + \frac{1}{2}n}}{|\lambda + \tfrac{1}{2}n, A_1, \cdots, A_n\rangle} = \frac{a^{+A_n}\cdots a^{+A_1}}{\sqrt{n!}}|\lambda\rangle$$

antisymmetric in $A_1, \cdots, A_n$

$\binom{N}{n}$ possibility $\frac{N!}{(N-n)!\,n!}$

The helicity state $\lambda + \frac{1}{2}n$ appears $\binom{N}{n}$

$\underline{\text{the highest helicity mode}}$: every a^+ appears once.

$\lambda + \frac{1}{2}N$. dim of irrep: $\sum_{n=0}^{N}\binom{N}{n} = 2^N$

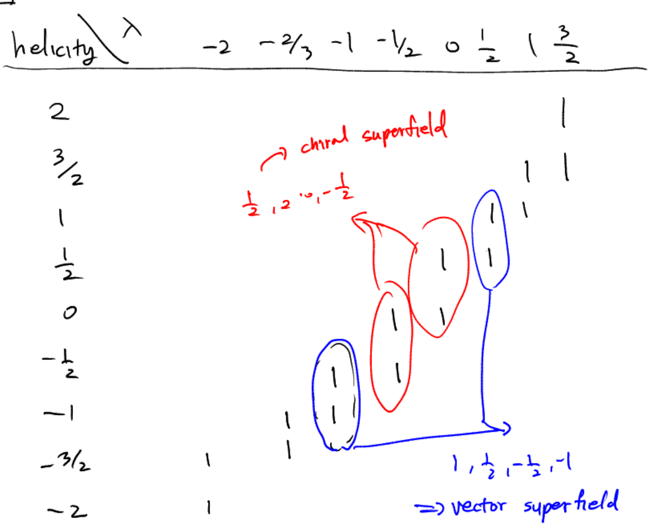
$\underline{\text{CPT}}$: reverse sign of all helicity.

Generally needed to double the rep:

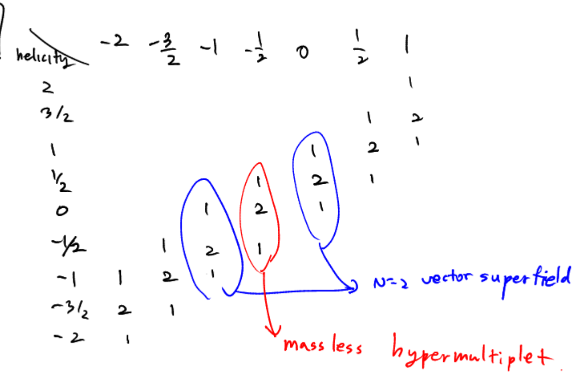
(unless it is automatically CPT complete)

states	helicity	# of state
$\lvert\Omega\rangle$	λ_{min}	1
$a^{+A}\lvert\Omega\rangle$	$\lambda_{min} + \frac{1}{2}$	N
$a^{+A}a^{+B}\lvert\Omega\rangle$	$\lambda_{min} + 1$	$\binom{N}{2}$
$\vdots$	$\vdots$	$\vdots$
$a^{+N}a^{+N-1}\cdots a^{+1}\lvert\Omega\rangle$	$\lambda_{min} + \frac{N}{2}$	1

$\boxed{N=1}$



$\boxed{N=2}$



$\boxed{N=4}$ interesting example $\lambda = 1$.

1	1
$\frac{1}{2}$	4
0	6
$-\frac{1}{2}$	4
-1	1

$N=4$ vector multiplet.

Massive representation

$\hat{p} = (M, 0, 0, 0)$

SUSY algebra in rest frame is

$\{ Q^A_\alpha, \bar{Q}^B_{\dot\beta} \} = 2\sigma^0_{\alpha\dot\beta} P_0 \delta^{AB} = 2\delta_{\alpha\beta} M \delta^{AB}$

now both $\{Q^A_1, Q^{+B}_1\} = 2M\delta^{AB}$

$\{Q^A_2, Q^{+B}_2\} = 2M\delta^{AB}$

define $a^A_\alpha = \frac{1}{\sqrt{2M}} Q^A_\alpha$, $(a^A_\alpha)^+ = \frac{1}{\sqrt{2M}} \bar{Q}^A_{\dot\alpha}$

$\{ a^A_\alpha, (a^B_\beta)^+ \} = \delta^\beta_\alpha \delta^B_A$

$\{a, a\} = \{a^+, a^+\} = 0$

Now we have $2N$ fermionic creation & annihilation operators.

$\rightarrow$ Clifford algebra with $2N$ fermionic generators

Remember: basic states is characterized by spin j, representation of little group $SO(3)$.

Clifford vacuum of fermionic creation & annihilation ops.

$|\Omega\rangle$ has spin j : $(2j+1)$ degeneracy

$a^A_\alpha |\Omega\rangle = 0$, $P^2|\Omega\rangle = M^2|\Omega\rangle$

apply creation operators,

$\frac{1}{\sqrt{n!}} (a^{A_1}_{\alpha_1})^+ \cdots (a^{A_n}_{\alpha_n})^+ |\Omega\rangle$

a^+'s are anti-commute : anti-symmetric in exchange of

$(\alpha_1, A_1) \leftrightarrow (\alpha_k, A_k)$

There are $2N$ pairs of indices possible for given n

$\binom{2N}{n}$ different state,

dim of irrep : $\sum_{n=1}^{N} \binom{2N}{n} = 2^{2N}$ ← twice of fermionic rep.

multiply degeneracy of $|\Omega\rangle$ $(2j+1)$

$\rightarrow (2j+1) 2^{2N}$

if $|\Omega\rangle$ not degenerated : fundamental massive multiplet

$d = 2^{2N} \rightarrow$ 2^{2N-1} fermionic, 2^{2N-1} bosonic $\}$ states

Highest spin : symmetrize in as many spinor indices possible

But antisymmetric in exchange of pair of indice, if symmetric in all spinor,

$\rightarrow$ max # : N maximum spin you can get is $\frac{N}{2}$

(assuming original vacuum non-degenerated)

For $N=1$ $|\Omega\rangle$ spin 0, non-degenerated ground state $(j=0)$

$a^+_\alpha |\Omega\rangle$ spin $\frac{1}{2}$

$\frac{\epsilon^{\alpha\beta}}{2\sqrt{2}} a^+_\alpha a^+_\beta |\Omega\rangle$ spin 0

1 complex scalar + 1 Weyl spinor

if ground state spin j, for $N=1$

$\underbrace{j, j+\frac{1}{2}, j-\frac{1}{2}}, j$

$a^+_\alpha |\Omega\rangle$

spin $\frac{1}{2} \otimes j$

$\rightarrow j+\frac{1}{2} \oplus j-\frac{1}{2}$

$2(2j+1) + (2j+2) + 2j$

$= 4(2j+1) = (2j+1) 2^{2\cdot 1}$

$\boxed{N=1}$

# of rep \ j	0	$\frac{1}{2}$	1	$3/2$
$j=0$	2	1		
$j=\frac{1}{2}$	1	2	1	
$j=1$		1	2	1
$j=3/2$			1	2
$j=2$				1

$\boxed{N=2}$ look at $j=0$

# of rep \ j	0	$\frac{1}{2}$	1
0	5	4	1
$\frac{1}{2}$	4	6	4
1	1	4	6
$3/2$		1	4
2			1

Notation we use :

2-component, $(+,-,-,-)$ metric

$\underset{\text{dirac}}{\psi_D} = \begin{pmatrix} \xi_\alpha \\ \chi^{+\dot\alpha} \end{pmatrix}$ ← LH spinor, RH spinor

$\gamma_\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$, $\sigma^\mu = (1, \vec\sigma)$, $\bar\sigma^\mu = (1, -\vec\sigma)$

$P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$P_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

ξ, χ will be Grossman fields that carry spinor index

- complex conjugate changes undotted to dotted & raise/low index
- can raise/low index with $\epsilon_{ij} = \epsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\xi\chi$ spinor sum defined as

$\xi\chi = \xi^\alpha \chi_\alpha = \xi^\alpha \epsilon_{\alpha\beta} \chi^\beta = -\xi_\alpha \chi^\alpha = -(\epsilon_{\alpha\beta} \xi^\beta \chi^\alpha)$

$\xi\chi = \chi\xi$ in this notion $\chi^+ \xi^+ = \chi^+_{\dot\alpha} (\xi^+)^{\dot\alpha}$

N = 1 SUSY Field theory

Why is N=1 the most important?

N > 1 will be $SU(N)_R$ symmetry will imply that theory is non-chiral! Very different from SM.

N=1 can be chiral.

If I only want fields with spin $0, \frac{1}{2}, 1$:

from N=1 table:

only chiral superfield $2\times(j=0)+1\times(j=1/2)$

vector superfield $1\times(j=0)+2\times(j=\frac{1}{2})+1\times(j=1)$

in massive case, we want to write down field theories realizing these symmetries.

First: Wess-Zumino models. (N=1 chiral superfield)

simplest rep. of SUSY

chiral superfield 1 spin $\frac{1}{2}$ Weyl spinor

2 spin 0 scalar

1 complex scalar

+ 1 chiral spinor

Free theory:

$$S=\int d^4x\,(\partial^m\phi^*\partial_m\phi+i\psi^\dagger\bar\sigma^m\partial_m\psi)$$

How to do SUSY transformation?

continuous transformation, so

$\phi\to\phi+\delta\phi$ \
$\psi\to\psi+\delta\psi$ } but fermionic, so define SUSY transformation parameter, ϵ^α

a continuous Grassman #

but GLOBAL

(e.g. ϵ is fixed everywhere in space)

$\delta\phi\propto\psi$

$\delta\psi\propto\phi$

$$\Rightarrow\quad \delta\phi=\underset{\substack{\uparrow\\ \dim[\phi]=1}}{\epsilon^\alpha\psi_\alpha}=\epsilon\underset{\substack{\uparrow\\ \dim[\psi]=\frac{3}{2}}}{\psi}$$

$\to$ ϵ mass dim $-\frac{1}{2}$ (like $\frac{1}{\sqrt{x}}$)

$$\delta\phi^*=(\epsilon\psi)^\dagger=(\epsilon^\alpha\psi_\alpha)^\dagger$$
$$=\psi_\alpha^\dagger(\epsilon^\alpha)^\dagger=(\psi^\dagger)^{\dot\alpha}(\epsilon^\dagger)_{\dot\alpha}$$
$$=(\epsilon^\dagger\psi^\dagger)$$

σ matrix carries $\sigma^m_{\alpha\dot\beta}$, $\bar\sigma^{m\,\dot\alpha\beta}$ } indices

Then, $\delta\mathcal{L}_s$

$$=\epsilon^\dagger\partial_m\psi^\dagger\partial^m\phi+\partial^m\phi^*\epsilon\partial_m\psi$$

what could $\delta\psi$ be?

$\mathcal{L}_f$ linear in derivative.

To have a chance of working MUST contain another derivative. Also dimensional argument.

$$\underset{\substack{\uparrow\\ 3/2}}{\delta\psi}\propto\underset{\substack{\uparrow\\ -1/2}}{\epsilon}\underset{\substack{\uparrow\\ 1}}{\phi}\quad\to\text{need }\partial_m$$

But, these also need to worry about indices.

$$\delta\psi_\alpha\sim\sigma^\nu_{\alpha\dot\beta}\epsilon^{\dagger\dot\beta}\partial_\nu\phi$$

The actual transformation will be

$$\delta\psi_\alpha=-i(\sigma^\nu\epsilon^\dagger)_\alpha\partial_\nu\phi$$
$$\delta\psi^\dagger_{\dot\alpha}=i(\epsilon\sigma^\nu)_{\dot\alpha}\partial_\nu\phi^*$$

Check that this really leaves $\mathcal{L}$ invariant.

$$\delta\mathcal{L}_f=i\delta\psi^\dagger\bar\sigma^m\partial_m\psi+i\psi^\dagger\bar\sigma^m\partial_m\delta\psi$$
$$=-\epsilon\sigma^\nu\bar\sigma^m\partial_\nu\phi^*\partial_m\psi+\psi^\dagger\bar\sigma^m\partial_m(\sigma^\nu\epsilon^\dagger)\partial_\nu\phi$$
$$=-\epsilon\sigma^\nu\bar\sigma^m\partial_m\psi\partial_\nu\phi^*+\psi^\dagger\underbrace{\bar\sigma^m\sigma^\nu}_{\text{can be symmetrized}}\epsilon^\dagger\underbrace{\partial_m\partial_\nu}_{\text{symmetric in } m\leftrightarrow\nu}\phi$$

use Fierz identity.

$$\{\bar\sigma^m,\sigma^\nu\}=\eta^{m\nu}$$

$$=(\psi^\dagger\epsilon^\dagger)\partial_m\partial^m\phi-\epsilon\sigma^\nu\bar\sigma^m\partial_m\partial_\nu\phi^*$$

integration by parts

$$=-(\epsilon^\dagger\partial_m\psi^\dagger)\partial^m\phi+\partial_m[(\epsilon^\dagger\psi^\dagger)\partial^m\phi]$$
$$-\partial_m(\epsilon\sigma^\nu\bar\sigma^m\psi\partial_\nu\phi^*)+\epsilon\underbrace{\sigma^\nu\bar\sigma^m}_{\text{symmetrize}\to\eta^{m\nu}}\psi\partial_m\partial_\nu\phi^*$$

$$=-(\epsilon^\dagger\partial_m\psi^\dagger)\partial^m\psi+\underbrace{\epsilon\psi\partial_m\partial^m\phi^*}_{-\epsilon\partial_m\psi\partial^m\phi^*+\text{ total derivative}}+\text{total derivative}$$

$$\delta\mathcal{L}_f=-(\epsilon^\dagger\partial_m\psi^\dagger)\partial^m\phi-(\epsilon\partial_m\psi)(\partial^m\phi^*)$$

+ total derivative.

$\delta\mathcal{L}_f+\delta\mathcal{L}_s=0$ up to total derivative!

$\to$ SUSY transformation leaves $\mathcal{L}$ invariant!

Is this really representing SUSY algebra?

$$\{Q_\alpha,\bar Q_{\dot\beta}\}=2\sigma^m_{\alpha\dot\beta}P_m$$
$$\epsilon^\alpha(Q_\alpha\bar Q_{\dot\beta}+\bar Q_{\dot\beta}Q_\alpha)\epsilon^{\dagger\dot\beta}=\epsilon^\alpha(2\sigma^m_{\alpha\dot\beta}P_m)\epsilon^{\dagger\dot\beta}$$
$$\epsilon Q\epsilon^\dagger\bar Q-\epsilon^\dagger\bar Q\epsilon Q=2\epsilon\sigma^m\epsilon^\dagger P_m$$
$$[\epsilon Q,\epsilon^\dagger Q^\dagger]=2(\epsilon\sigma^m\epsilon^\dagger)^\dagger P_m$$

A SUSY transformation on a field:

$$\delta_\epsilon\phi=(\epsilon Q+\epsilon^\dagger Q^\dagger)\phi$$
$$\delta_\epsilon\psi=(\epsilon Q+\epsilon^\dagger Q^\dagger)\psi$$

If SUSY algebra, then

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\Phi$$

$$= [(\epsilon_2 Q + \epsilon_2^+ Q^+), (\epsilon_1 Q + \epsilon_1^+ Q^+)]\Phi$$

$$= [\epsilon Q, \epsilon_1^+ Q^+]\Phi - [\epsilon_1 Q_1, \epsilon_2^+ Q^+]\Phi$$

$$= 2\epsilon_2 \sigma^m \epsilon_1^+ P_m \Phi - 2\epsilon_1 \sigma^m \epsilon_2^+ P_m \Phi$$

$$\Rightarrow (\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\Phi = \delta_{\epsilon_2}\epsilon_1\psi - \delta_{\epsilon_1}\epsilon_2\psi$$

$$= -\epsilon_1(-i\sigma^\nu\epsilon_2^+)\partial_\nu\phi + \epsilon_2(-i\sigma^\nu\epsilon_1^+)\partial_\nu\phi$$

$$= i(\epsilon_1\sigma^\nu\epsilon_2^+ - \epsilon_2\sigma^\nu\epsilon_1^+)\partial_\nu\Phi$$

The right answer of SUSY algebra up to a factor 2

→ fixes normalization of $\delta\Phi$, so really

$$\delta\phi = \sqrt{2}\epsilon\psi$$

$$\delta\psi_\alpha = -i\sqrt{2}(\sigma^\nu\epsilon^+)_\alpha\partial_\nu\Phi$$

Do the same for fermions,

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha$$

$$= -2i(\epsilon_1\sigma^m\epsilon_2^+ - \epsilon_2\sigma^m\epsilon_1^+)\partial_m\psi_\alpha$$

using Fierz identity

$$+ \underbrace{i(\epsilon_{1\alpha}\epsilon_2^+\bar\sigma^m\partial_m\psi - \epsilon_{2\alpha}\epsilon_1^+\bar\sigma^m\partial_m\psi)}_{\text{extra piece → vanish on-shell due to Dirac equation!}}$$

Is that problem? Does it hold in QM?

Trick: add auxiliary field (scalar) F

no kinetic term

→ not propagating degrees of freedom

$$\mathcal{L}_{aux} = F^*F \qquad \dim[F] = 2$$

$$\left.\begin{aligned}\delta F &= -i\sqrt{2}\,\epsilon^+\bar\sigma^m\partial_m\psi \\ \delta F^* &= i\sqrt{2}\,\partial_m\psi^+\bar\sigma^m\epsilon\end{aligned}\right\}\ \delta F \propto \text{EOM of }\psi$$

We choose such that it exactly cancels out additional piece in $\delta\psi$.

$$\delta\psi_\alpha = -i\sqrt{2}(\sigma^\nu\epsilon^+)_\alpha\partial_\nu\phi + \sqrt{2}\epsilon F$$

$$\delta\psi_\alpha^+ = i\sqrt{2}(\epsilon\sigma^\nu)_\alpha\partial_\nu\phi^* + \sqrt{2}\epsilon_\alpha^+ F^*$$

extract piece to cancel $(\delta\delta - \delta\delta)\psi$

& $\delta S_{new} = 0$

(new piece in $\delta\psi$ cancels $F\delta F^*$ ..)

SUSY still OK. why?

on-shell $\phi \to 2$ DOF

$\psi \to 2$ DOF

off-shell $\phi \to 2$ DOF

$\psi \to 4$ DOF, $F \to 2$ DOF

For SUSY to hold off-shell as well,

we need another DOF, F, F^* that vanishes on-shell!

With this choice, SUSY algebra close for all

$$X = \phi, \phi^*, \psi, \psi^*, F, F^*$$

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = -2i(\epsilon_1\sigma^m\epsilon_2^+ - \epsilon_2\sigma^m\epsilon_1^+)\partial_m X$$

SUSY transformation

$$\delta\phi = \sqrt{2}\epsilon\psi$$

$$\delta\psi = -i\sqrt{2}(\sigma^\nu\epsilon^+)\partial_\nu\phi + 2\epsilon F$$

$$\delta F = -i\sqrt{2}\,\epsilon^+\bar\sigma^m\partial_m\psi$$

By Noether theorem, $\delta S = 0$ implies a conserved current → supercurrent.

What is it?

$$\mathcal{L}(x+\delta x) - \mathcal{L}(x) = \partial_m V^m \quad \leftarrow \text{total derivative}$$

$$= \frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial\partial_m x}\delta(\partial_m x)$$

$$\text{by EOM,}\quad \partial_m\left(\frac{\partial\mathcal{L}}{\partial\partial_m x}\right) = \frac{\partial\mathcal{L}}{\partial x}$$

$$\partial_m V^m = \partial_m\left(\frac{\partial\mathcal{L}}{\partial\partial_m x}\delta x\right)$$

$$\partial_m\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\partial_m x}\delta x - V^m\right)}_{\text{conserved current}} = 0$$

$$\epsilon\,\partial_m J^m = 0.$$

can read off from $\mathcal{L}$

fixed from explicit form of δS

$$\epsilon J^m + \epsilon^+ J^{m+} := \frac{\partial\mathcal{L}}{\partial\partial_m x}\delta x - V^m$$

$$\epsilon J^m + \epsilon^+ J^{m+} = \sqrt{2}\epsilon\psi\partial^m\phi^* + \sqrt{2}\epsilon^+\psi^+\partial^m\phi$$

$$+ i\psi^+\bar\sigma^m(-\sqrt{2}i\sigma^\nu\epsilon^+\partial_\nu\phi) - \epsilon\sigma^m\bar\sigma^\nu\psi\partial_\nu\phi^*\sqrt{2}$$

$$+ \epsilon\sqrt{2}\psi\partial^m\phi^* - \sqrt{2}\epsilon^+\psi^+\partial^m\phi$$

use the identity, $\sigma^m\bar\sigma^\nu + \sigma^\nu\bar\sigma^m = 2\eta^{m\nu}$

$$= \sqrt{2}\epsilon\sigma^\nu\bar\sigma^m\psi\partial_\nu\phi^* + \text{h.c.}$$

$$J_\alpha^m = \sqrt{2}(\sigma^\nu\bar\sigma^m\psi)\partial_\nu\phi^*$$

$$\Rightarrow Q_\alpha = \int d^3x\, J_\alpha^0 \quad \text{supercharge of SUSY algebra.}$$

"sample" references:

Balin & Love.

1. Stephan Martin, A SUSY Primer
2. Quevado, Supersymmetry and Extra Dimension.
3. Wess and Bagger, Supersymmetry and supergravity.
4. Dree, Godbole, Roy, Sparticles.
5. Terning, Modern SUSY
6. Agyres, Lectures on supersymmetry.

many more reference.