3. 인플레이션

최 기 영

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에너지 밀도의 변화

$$\rho \propto R^{-3(1+\omega)} \quad \left(\begin{array}{lll} \rho \propto R^{-4} & \text{Radiation} & (p = \frac{1}{3}\rho) \\ \rho \propto R^{-3} & \text{(Cold) Matter} & (p = 0) \\ \rho \propto \text{const} & \text{Vacuum energy} & (p = -\rho) \\ & \text{(cosmological constant)} & \end{array} \right.$$

For k=0 (flat), and $p = \omega\rho$ we obtain $R \propto t^{2/3(1+\omega)}$

Radiation $(p = \frac{1}{3}\rho)$	$R \propto t^{1/2}$
Matter $(p = 0)$	$R \propto t^{2/3}$
Vacuum energy $(p = -\rho)$	$R \propto \exp(Ht)$

입자 지평선(particle horizon)

$d_p(t) = 2t$ Radiation $(p = \frac{1}{3}\rho)$

$d_p(t) = 3t$ Matter $(p = 0)$

우주의 나이와 허블지평선

$$t_0 = \frac{2}{3(1+\omega)} H_0^{-1}$$

$t_0 = \frac{1}{2} H_0^{-1}$ Radiation $(p = \frac{1}{3}\rho)$

$t_0 = \frac{2}{3} H_0^{-1}$ Matter $(p = 0)$

Evolution in the big bang cosmology

	Inflation	RD	MD	DE
ρ	const	R^{-4}	R^{-3}	const
$R(t)$	$\exp(Ht)$	$t^{1/2}$	$t^{2/3}$	$\exp(Ht)$
H^{-1}	const	$2t$	$\frac{3}{2}t$	const

Scale

Comoving scale, Physical scale

$$\lambda \equiv \frac{2\pi}{k}, \quad \lambda_{\rm phys} = R(t)\lambda$$

Comoving mode, Physical mode

$$k = \frac{2\pi}{\lambda}, \qquad k_{\rm phys} = \frac{2\pi}{\lambda_{\rm phys}} = R^{-1}k$$

Hubble scale

Comoving scale, Physical scale

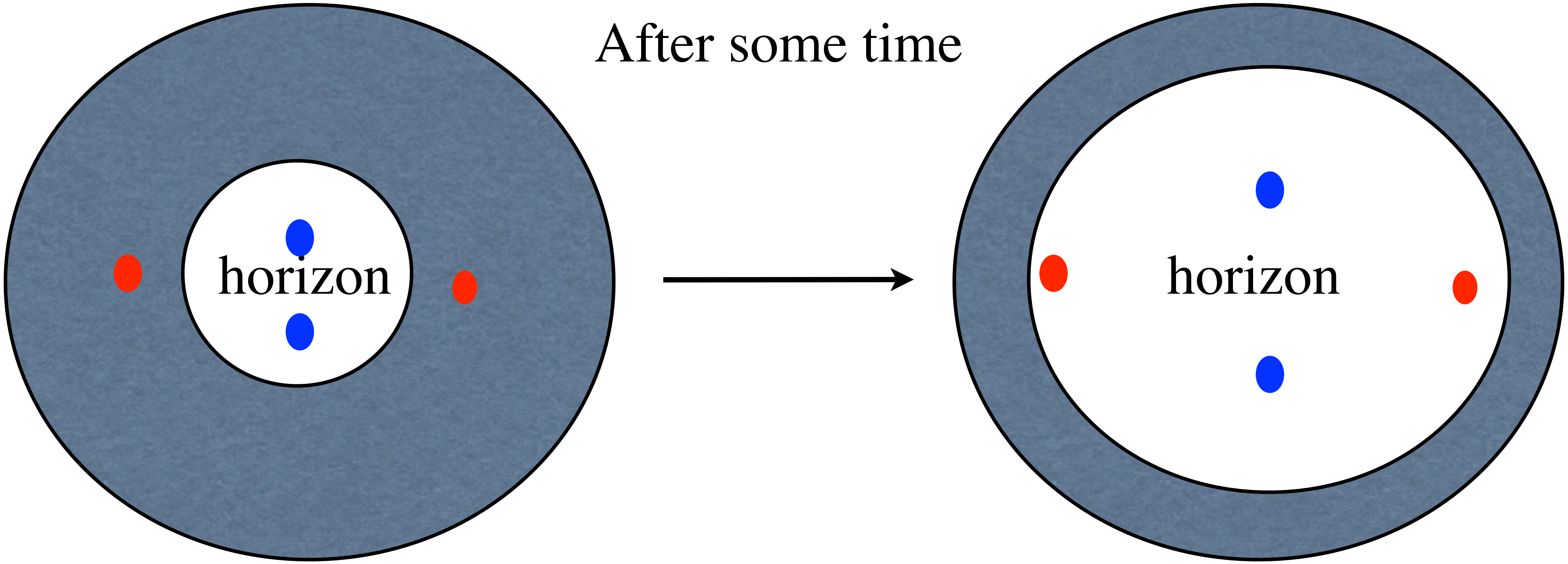
$$R^{-1}H^{-1} \qquad H^{-1}$$

Comoving mode, Physical mode

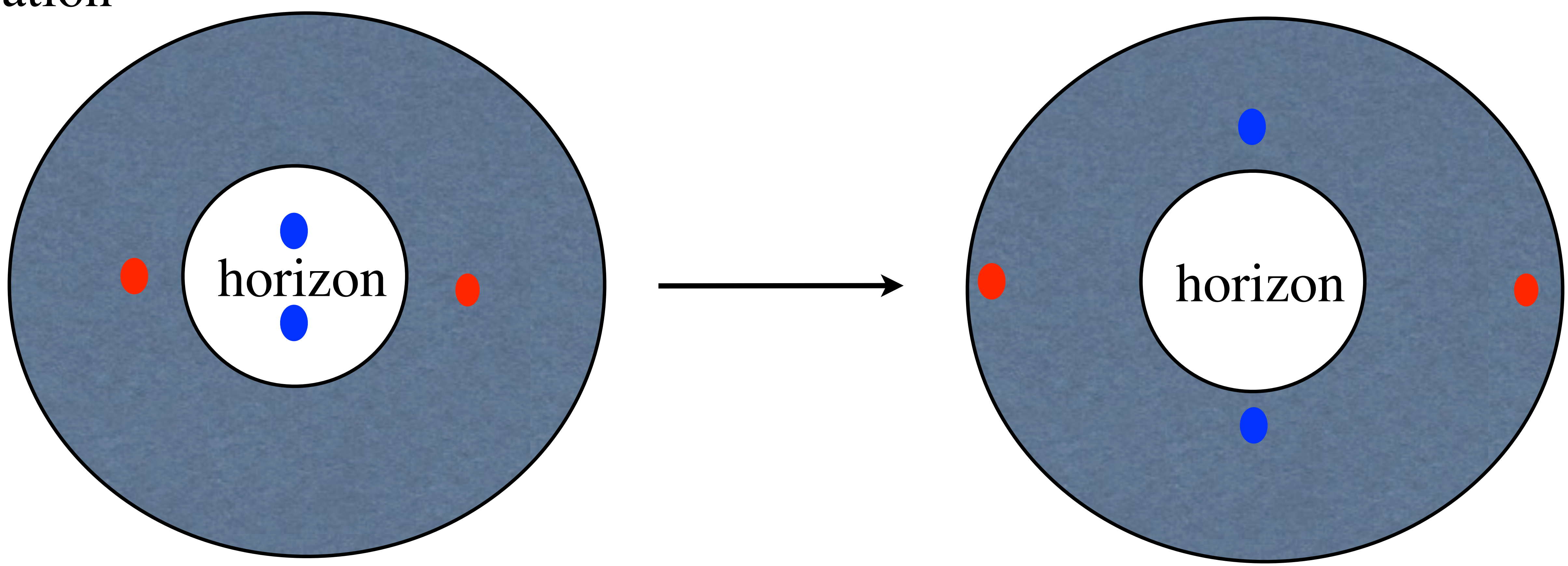
$$2\pi(RH) \qquad 2\pi H$$

$\lambda_{\text{phys}} = R(t)\lambda$ and H^{-1}

RD or MD

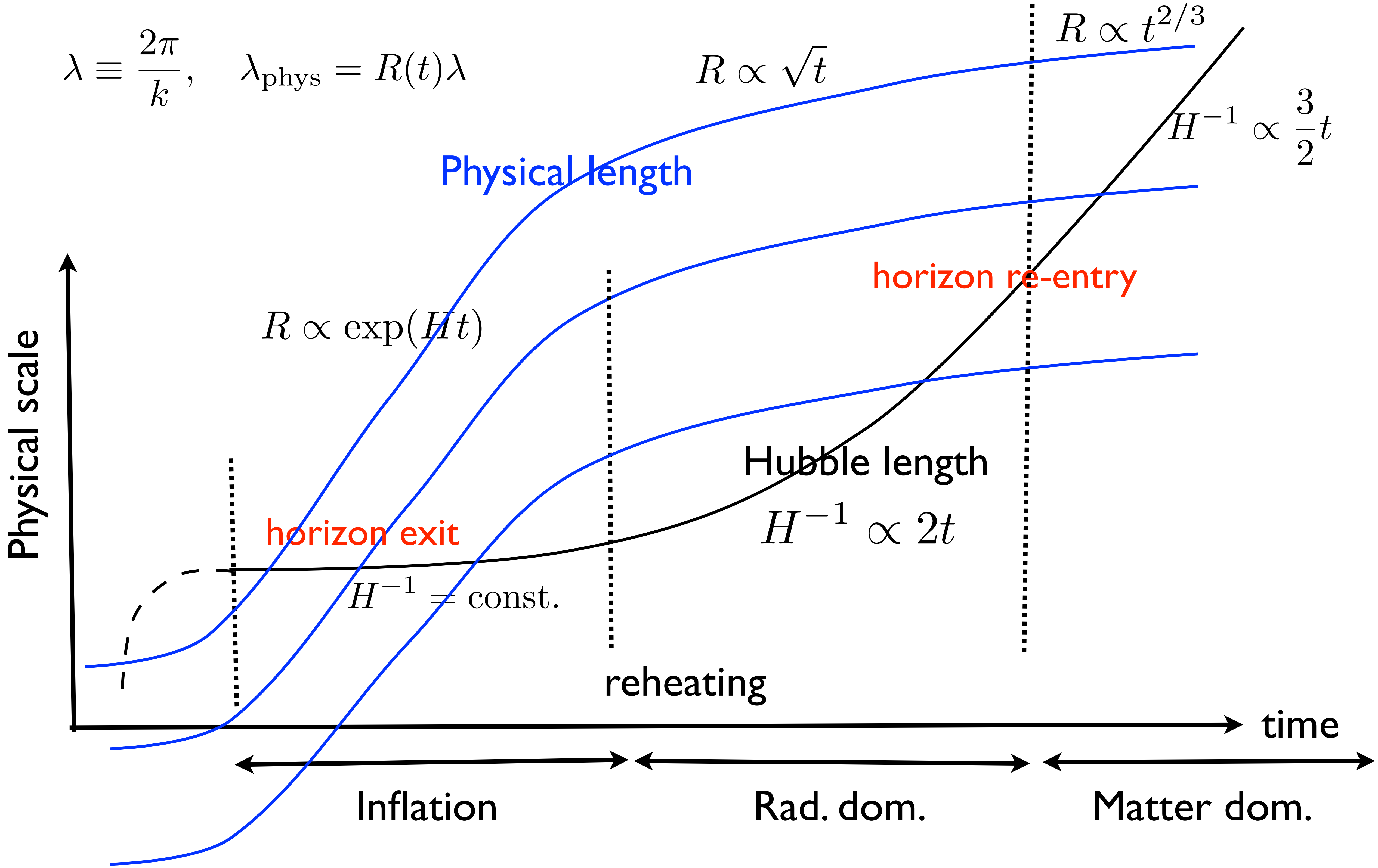


Inflation



Horizon exit and reentry: physical scale

$$\lambda \equiv \frac{2\pi}{k}, \quad \lambda_{\rm phys} = R(t)\lambda$$



For inflation, we need

$\frac{R}{H^{-1}}$ increases, or $R^{-1}H^{-1}$ decreases during inflation

↑ (comoving Hubble distance)

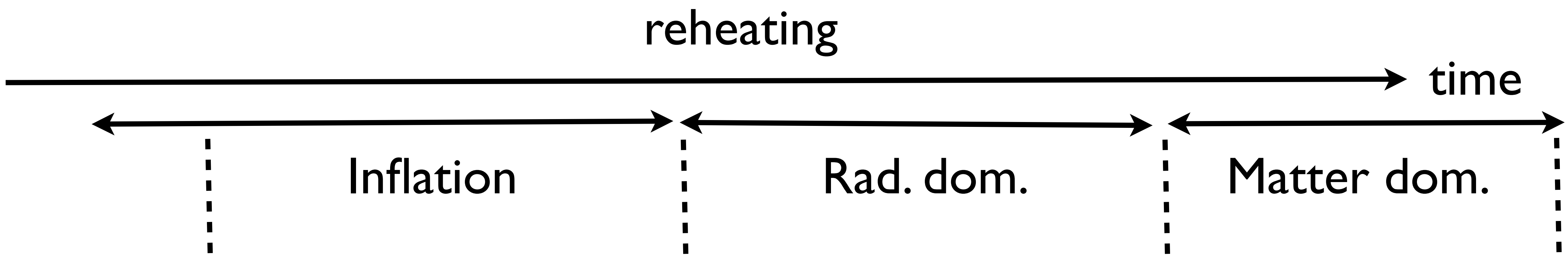
$(RH)^{-1}$ 가 감소하기위한 조건 : 가속팽창하는우주

$$\frac{\partial (RH)^{-1}}{\partial t} < 0 \Rightarrow \ddot{R} > 0$$

또는, 팽창계수의 가속공식에서

$$\ddot{a} = -\frac{4\pi}{3} G(\rho + 3p) a > 0 \quad \Rightarrow \quad \boxed{\omega = \frac{p}{\rho} < -\frac{1}{3}}$$

How much inflation do we need?



	$R(t_i)$	$R(t_f)$	$R(t_{eq})$	R_0
Uniform at present	$R(t_i)\lambda_h$	$R(t_f)\lambda_h$	$R(t_{eq})\lambda_h$	$R_0\lambda_h$
Hubble size	$H^{-1}(t_i)$	$H^{-1}(t_f)$	$H^{-1}(t_{eq})$	H_0^{-1}

$$\frac{R(t_i)\lambda_h}{H^{-1}(t_i)} = \left(\frac{R_i H_i}{R_f H_f}\right)\left(\frac{R_f H_f}{R_{eq} H_{eq}}\right)\left(\frac{R_{eq} H_{eq}}{R_0 H_0}\right)\left(\frac{R_0 \lambda_h}{H_0^{-1}}\right) < 1$$

	Inflation	RD	MD	DE
ρ	const	R^{-4}	R^{-3}	const
$R(t)$	$\exp(Ht)$	$t^{1/2}$	$t^{2/3}$	$\exp(Ht)$
H^{-1}	const	$2t$	$\frac{3}{2}t$	const

$$\frac{R(t_i)\lambda_h}{H^{-1}(t_i)} = \left(\frac{R_i H_i}{R_f H_f}\right)\left(\frac{R_f H_f}{R_{eq} H_{eq}}\right)\left(\frac{R_{eq} H_{eq}}{R_0 H_0}\right)\left(\frac{R_0 \lambda_h}{H_0^{-1}}\right) < 1$$

$$= e^{-H(t_f - t_i)}\left(\frac{R_{eq}}{R_f}\right)\left(\frac{R_0}{R_{eq}}\right)^{1/2}$$

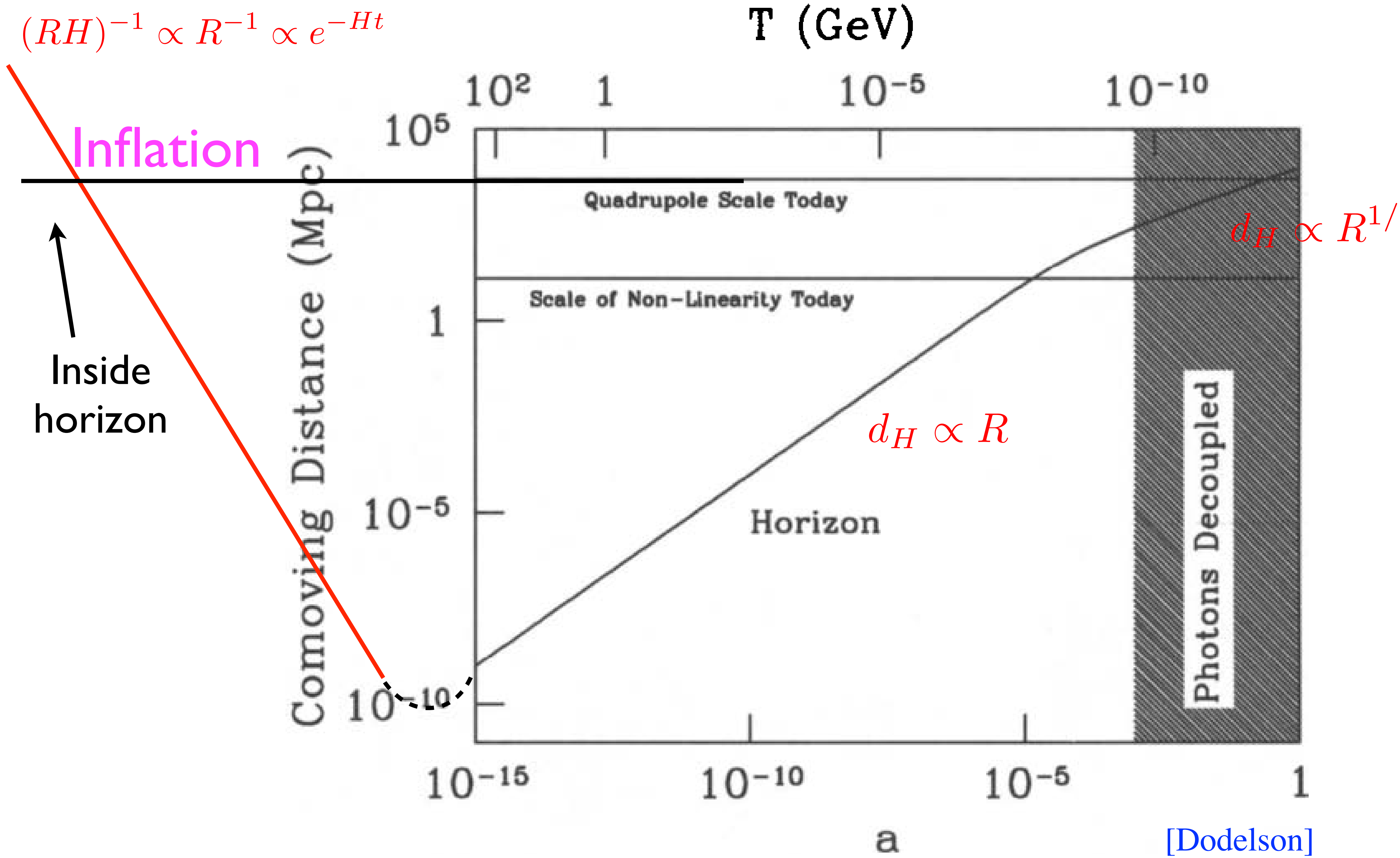
Therefore

$$e^N > \left(\frac{R_{eq}}{R_f}\right)\left(\frac{R_0}{R_{eq}}\right)^{1/2} = \left(\frac{T_f}{T_{eq}}\right)\left(\frac{T_{eq}}{T_0}\right)^{1/2} = 50\left(\frac{T_f}{1\,\mathrm{eV}}\right)$$

$50 \times 10^6 \simeq e^{18}$ $\qquad$ $50 \times 10^{25} \simeq e^{61}$

$$\boxed{N \gtrsim 50 - 60}$$

Horizon exit and reentry: co-moving scale: λ and $R^{-1}H^{-1}$



Inflation

스타로빈스키 모형(Starobinsky model) [Starobinsky, 1979, 1980]

- 아인슈타인 방정식에 양자수정항을 고려하면 드지터 메트릭이 나올 수 있고, 이것은 불안정하여 곧 붕괴하면서 뜨거운 프리드만 우주로 바뀌어진다.

- 특이점이 없는 드지터 중력에서 우주가 시작함으로 인하여 특이점 문제를 풀려고 하였다 (initial singularity).

$$S = \int d^4x \sqrt{-g}\frac{M_{\rm pl}^2}{2}\left(R + \frac{R^2}{6M^2}\right)$$

카자나스 [Kazanas, 1980]

- Temperature dependent vacuum energy (horizon problem)

사토 [Sato, 1981]

- First order phase transition similar to Guth's model (domain wall problem)

Old inflation [Guth, 1981]

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Inflationary universe: A possible solution to the horizon and flatness problems

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The standard model of hot big-bang cosmology requires initial conditions which are problematic in two ways: (1) The early universe is assumed to be highly homogeneous, in spite of the fact that separated regions were causally disconnected (horizon problem); and (2) the initial value of the Hubble constant must be fine tuned to extraordinary accuracy to produce a universe as flat (i.e., near critical mass density) as the one we see today (flatness problem). These problems would disappear if, in its early history, the universe supercooled to temperatures 28 or more orders of magnitude below the critical temperature for some phase transition. A huge expansion factor would then result from a period of exponential growth, and the entropy of the universe would be multiplied by a huge factor when the latent heat is released. Such a scenario is completely natural in the context of grand unified models of elementary-particle interactions. In such models, the supercooling is also relevant to the problem of monopole suppression. Unfortunately, the scenario seems to lead to some unacceptable consequences, so modifications must be sought.

V. PROBLEMS OF THE INFLATIONARY SCENARIO[37]

As I mentioned earlier, the inflationary scenario seems to lead to some unacceptable consequences. It is hoped that some variation can be found which avoids these undesirable features but maintains the desirable ones. The problems of the model will be discussed in more detail elsewhere,[37] but for completeness I will give a brief description here.

The central problem is the difficulty in finding a smooth ending to the period of exponential expansion. Let us assume that $\lambda(t)$ approaches a constant as $t \to \infty$ and $T \to 0$. To achieve the desired expansion factor $Z > 10^{28}$, one needs $\lambda_0/\chi^4 < 10^{-2}$ [see (3.18)], which means that the nucleation rate is slow compared to the expansion rate of the universe. (Explicit calculations show that λ_0/χ^4 is typically much smaller than this value.[18,19,36]) The randomness of the bubble formation process then leads to gross inhomogeneities.

To understand the effects of this randomness, the reader should bear in mind the following facts.

(i) All of the latent heat released as a bubble expands is transferred initially to the walls of the

(v) Each cluster will contain only a few of the largest bubbles. Thus, the collisions discussed in (iii) cannot occur.

The above statements do not quite prove that the scenario is impossible, but these consequences are at best very unattractive. Thus, it seems that the scenario will become viable only if some modification can be found which avoids these inhomogeneities. Some vague possibilities will be mentioned in the next section.

Note that the above arguments seem to rule out the possibility that the universe was ever trapped in a false vacuum state, unless $\lambda_0/\chi^4 \gtrsim 1$. Such a large value of λ_0/χ^4 does not seem likely, but it is possible.[19]

VI. CONCLUSION

I have tried to convince the reader that the standard model of the very early universe requires the assumption of initial conditions which are very implausible for two reasons:

(i) *The horizon problem.* Causally disconnected regions are assumed to be nearly identical; in par-

스칼라장을 이용한 인플레이션

스칼라장과 중력의 액션

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

첫 째항은 아인슈타인 텐서 항,

둘, 셋째 항은 스카라장의 텐서를 준다.

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi + V(\phi)\right)$$

균일한 스칼라장에 대하여,

에너지 밀도 $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

압력 밀도 $p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

$$\ddot{a} = -\frac{4\pi}{3}G(\rho + 3p)a > 0$$

스칼라장을 이용한 인플레이션

우주의 에너지가 스칼라장에 의하여 주로 결정된다면

에너지 밀도 $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

압력 밀도 $p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

$\longrightarrow \rho + 3p = \dot{\phi}^2 - V(\phi) < 0$

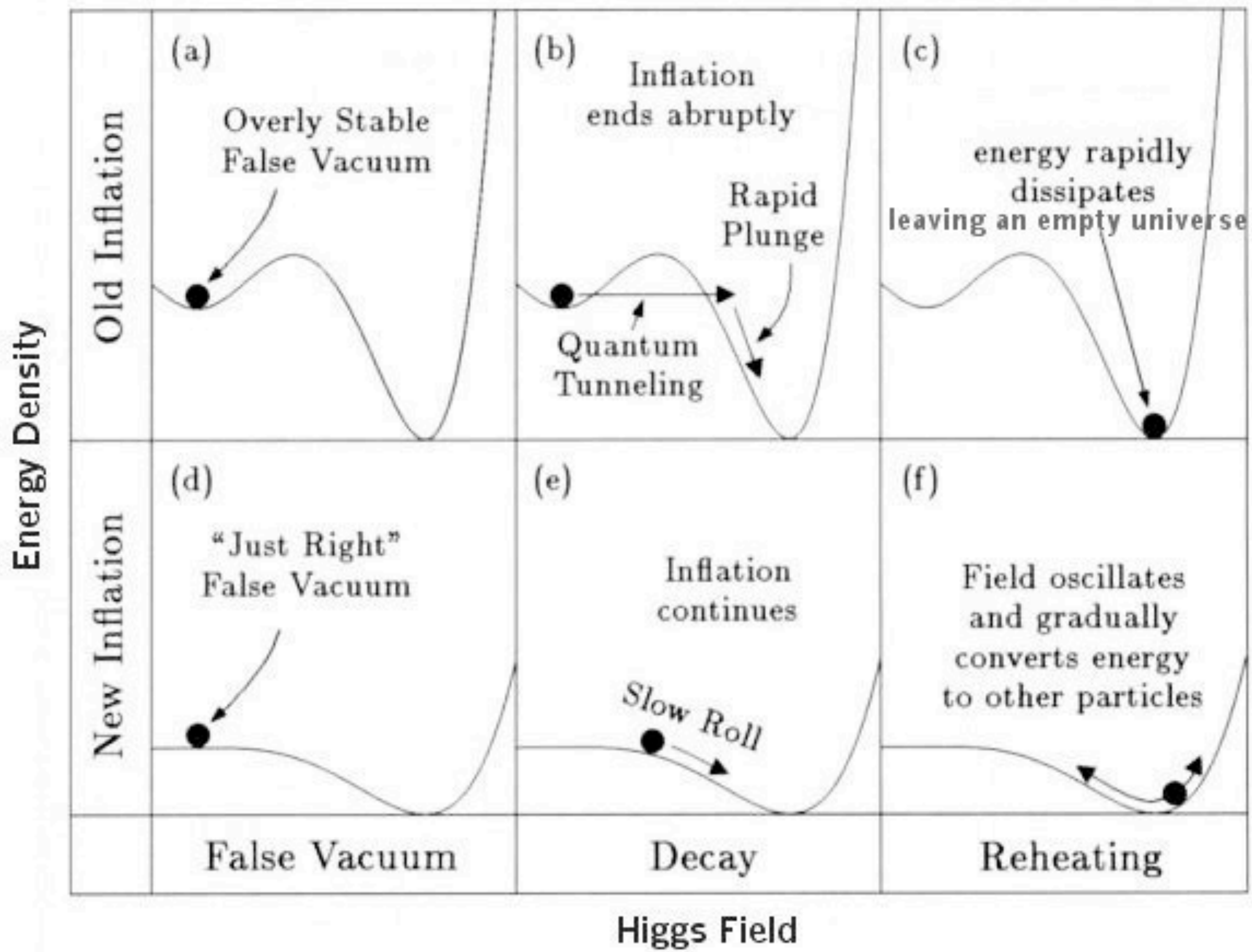
위치에너지가 운동에너지보다 훨씬 클 때, $\dot{\phi}^2 \ll V(\phi)$

압력 밀도 = - 에너지 밀도, :상수의 허블팽창

:스칼라 장이 자신의 위치에너지에서 아주 천천히 움직이면서 인플레이션을 일으킨다.

Slow-roll inflation

Old inflation and New inflation



Old inflation [Guth, 1981]

1. 초기에 우주는 아주 높은 온도에서 팽창하고 있었다. 열적항에 의하여 스칼라장은 대칭성을 회복하고 $\phi = 0$ 인 지점에 위치하고 있다.
2. 시간이 지나 온도가 낮아졌지만, 스칼라장은 여전히 local minimum 에서 과냉각된 상태로 위치하여 있고, 우주의 에너지를 지배하게된다. 이 때 인플레이션이 일어난다. (delayed first order phase transition)
3. 스칼라장이 상전이를 통하여 0 이 아닌 지점으로 옮겨가면서 인플레이션이 끝이 난다. 상전이는 우주의 여기 저기에서 상전이의 거품이 만들어지고, 그 거품들이 충돌을 하여 합쳐지면서 우주의 온도가 높아지게 되고, 뜨거운 우주가 다시 시작된다.

문제점:거품의 충돌로 인하여 우주는 더욱 불균일한 상태로 이르게 된다.

Graceful Exit problem

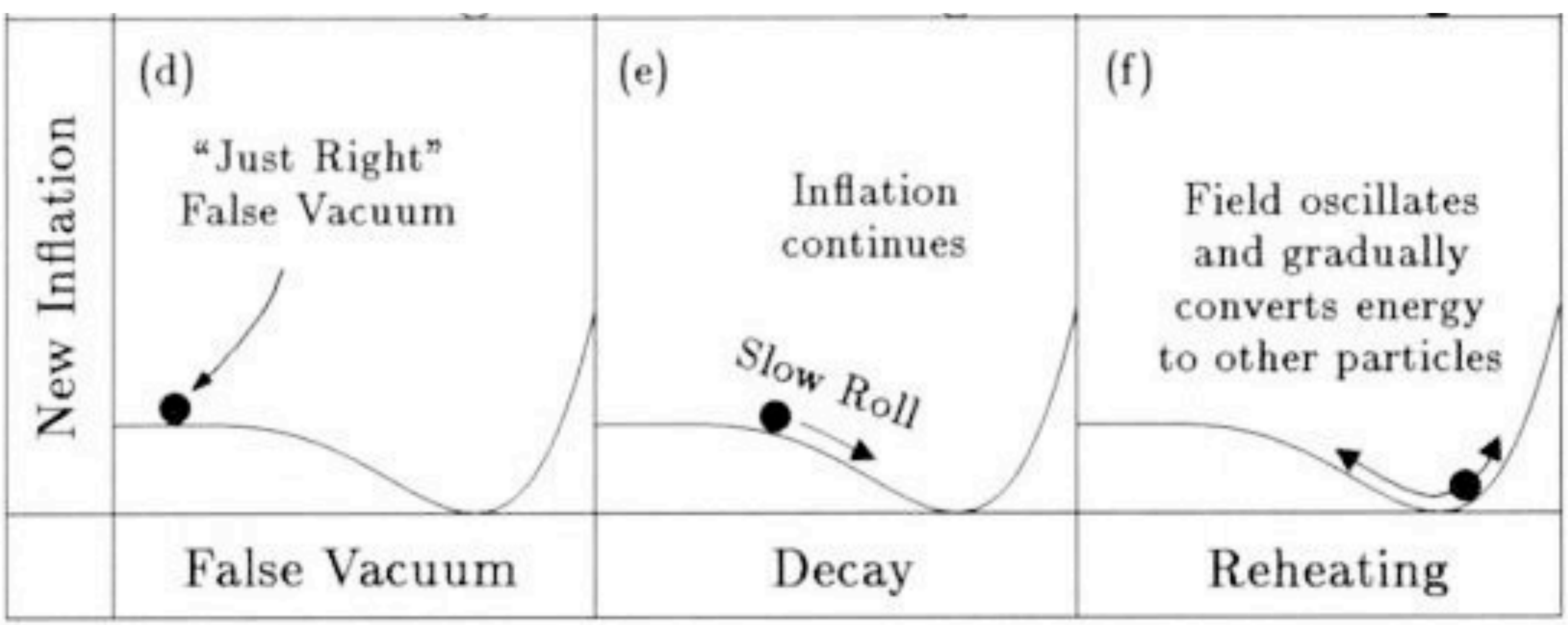
New inflation [Linde, 1982, Albrecht, Steinhardt, 1982]

1. 초기에 우주는 아주 높은 온도에서 팽창하고 있었다. 열적항에 의하여 스칼라장은 대칭성을 회복하고 $\phi = 0$ 인 지점에 위치하고 있다.

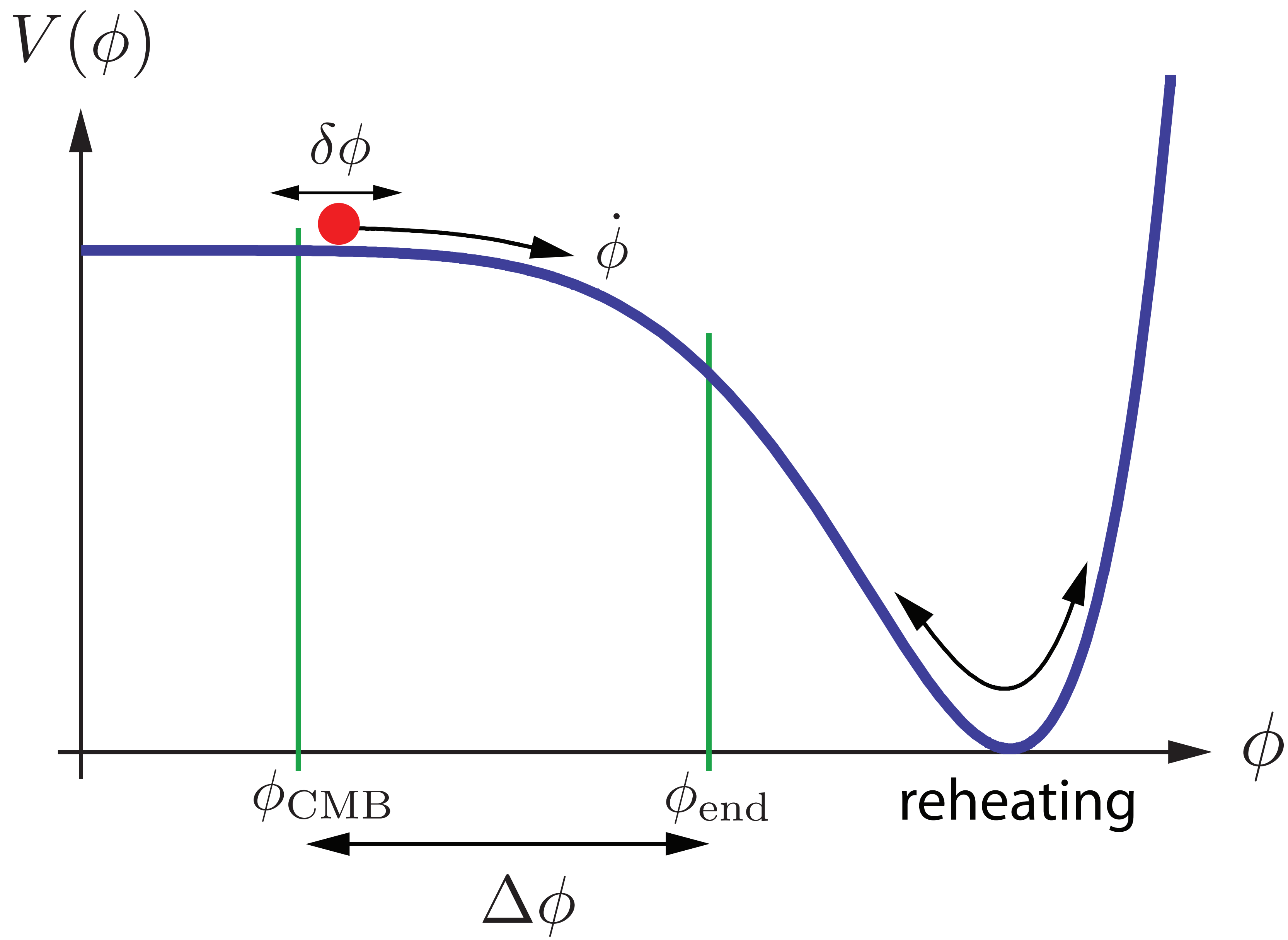
2. 스칼라장이 minimum 으로 이동하면서 인플레이션을 일으킬 수 있다. 이 때 스칼라는 아주 천천히 움직일 수 있도록 위치에너지가 $\phi = 0$ 근처에서 아주 평탄하여야한다.

3. 스칼라장이 minimum 그처에 이르면 인플레이션은 끝이 나고, 그 근처에서 스칼라가 damped 진동을 하며 입자들을 만들어낸다. 이 과정에서 우주는 다시 뜨거워지고 팽창하는 우주가 시작된다.

4. 인플레이션 중에 밀도의 불균일성이 만들어지게 되는데, 그것이 우주거대구조의 형성을 설명할 수 가 있었다.



Slow-roll inflation



Slow-roll 인플레이션

$$\frac{\delta S}{\delta \phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu \phi) + V_{,\phi} = 0$$

$$M_{\rm pl} = (8\pi G)^{-1/2}$$

운동방정식 $$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + V_\phi = 0,$$

프리드만방정식 $$H^2 = \frac{1}{3{M_{\rm pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right).$$

$\dot{\phi}^2 \ll V(\phi)$

$\ddot{\phi} \ll H\dot{\phi}$

$$3H(t)\dot{\phi}(t) \approx -V_\phi\,,$$

$$H^2 \approx \frac{V(\phi)}{3{M_{\rm pl}}^2}\,.$$

인플레이션의 끝

Slow-roll parameters 가 O(1) 보다 커지면서 끝남.

Slow-roll parameters

$$\epsilon_V = \frac{M_{\rm pl}^2 V_\phi^2}{2V^2}\,, \quad << 1$$

$$\eta_V = \frac{M_{\rm pl}^2 V_{\phi\phi}}{V}\,. \quad << 1$$

E-folding 수 $N \gtrsim 50$

$$N = \log\frac{a_f}{a_i} = \int_{t_i}^{t_f} H dt = \int \frac{H}{\dot\phi} d\phi \simeq -\int \frac{3H^2}{V_\phi} d\phi \simeq -\frac{1}{M_P^2}\int \frac{V}{V_\phi} d\phi$$

$$\simeq -\frac{1}{M_P}\int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon}} d\phi$$

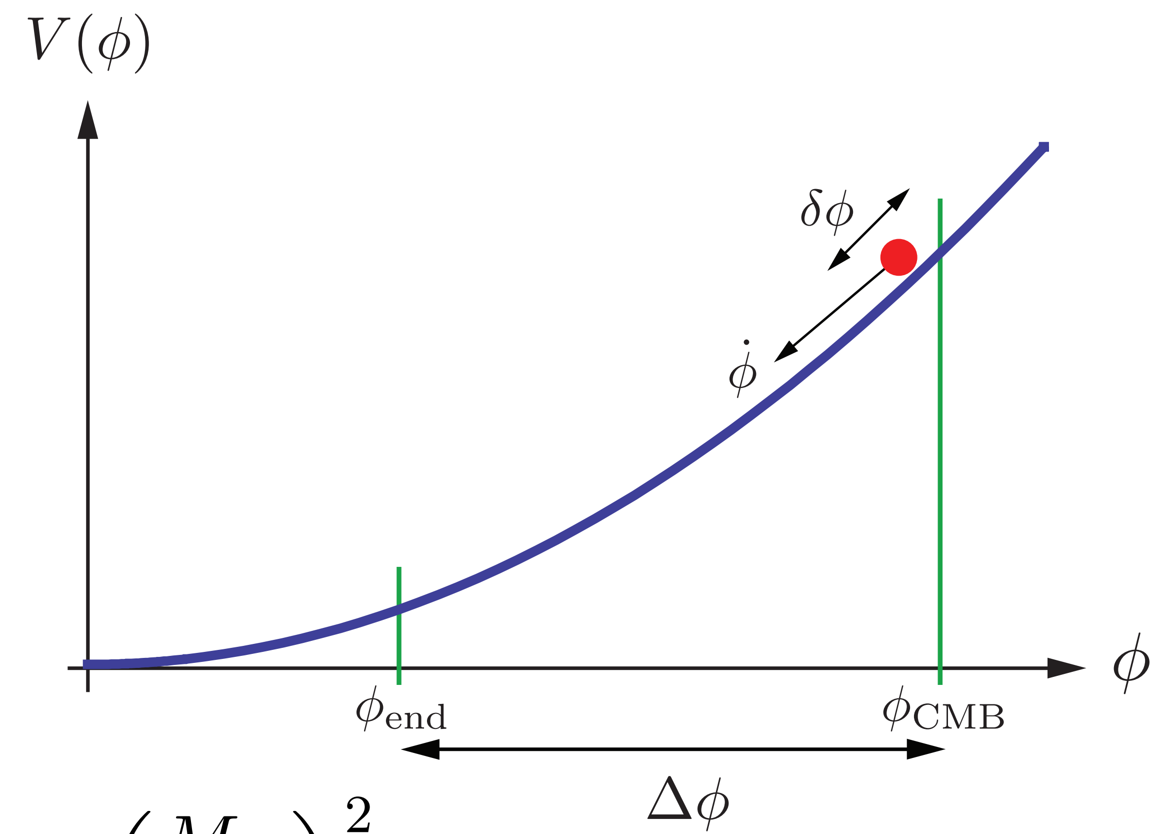
여기서 위치에너지는 양수이고 감소한다.

작은 값의 slow-roll parameters 와 때문에 충분한 팽창을 얻을 수 있다.

간단한 예: large field inflation

$m^2\phi^2$ 인플레이션:

$$V(\phi) = \frac{1}{2}m^2\phi^2$$



slow-roll 변수들: $$\epsilon_{\rm v}(\phi) = \eta_{\rm v}(\phi) = 2\left(\frac{M_{\rm pl}}{\phi}\right)^2$$

slow-roll 조건 $\epsilon_{\rm v}, |\eta_{\rm v}| < 1,$ 을 만족하기 위하여: $\phi > \sqrt{2}M_{\rm pl} \equiv \phi_{\rm end}$

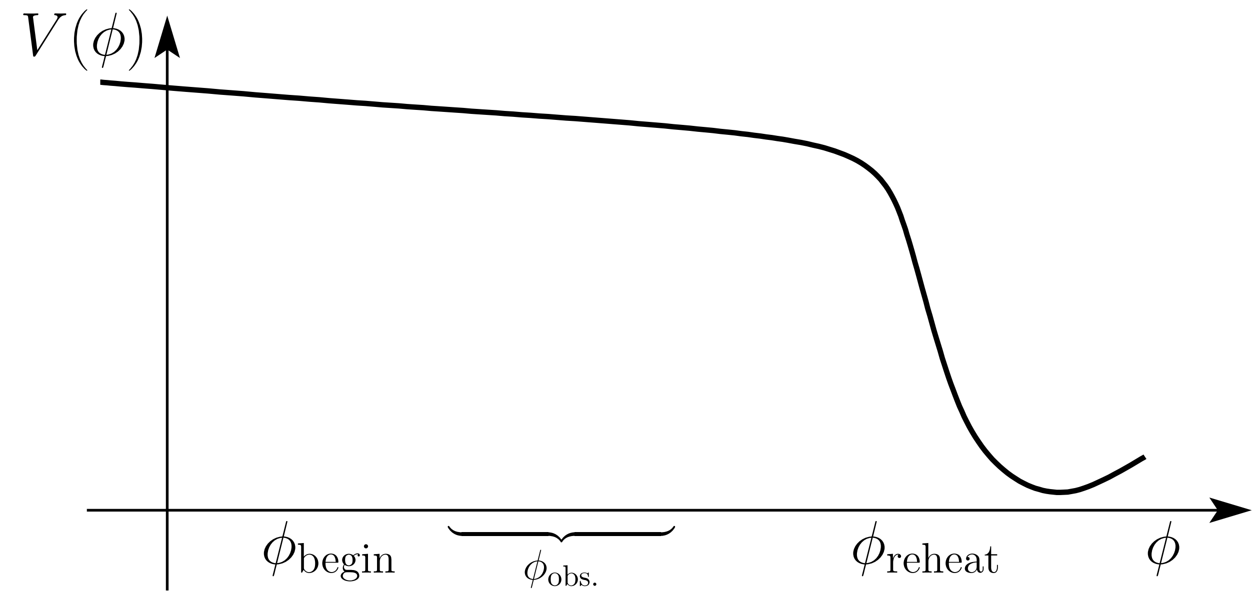
인플레이션 끝으로부터 e-folds: $$N(\phi) = \frac{\phi^2}{4M_{\rm pl}^2} - \frac{1}{2}$$

CMB 관측이 만들어지는 지점 $$\phi_{\rm cmb} = 2\sqrt{N_{\rm cmb}}\,M_{\rm pl} \sim 15M_{\rm pl}$$

$$N_{\rm CMB} \sim 60$$

간단한 예: small field inflation

$$V(\phi) = V_0 \left(1 - \left(\frac{\phi}{M}\right)^2\right)$$



slow-roll 변수들: $\epsilon \simeq \frac{M_{\rm Pl}^2 \phi^2}{M^4}$ $\eta \simeq \frac{M_P^2}{M^2}$

인플레이션의 끝 $\phi_{\rm end} \simeq \frac{M^2}{M_P}$

인플레이션 끝으로부터 e-folds: $N = \frac{M^2}{2M_P^2} \ln\left(\frac{\phi_i}{\phi_e}\right)$

$N_{\rm CMB} \sim 60$

인플레이션의 끝과 우주의 재가열 (Reheating)

인플라톤이 움직이다가 slow-roll 조건이 만족되지 못하면,
인플라톤은 재빨리 움직여 포텐셜의 최소점으로 움직이며,
그 지점을 중심으로 진동을 하게된다.

$$V(\phi) \simeq \frac{1}{2} m^2 (\phi - \phi_0)^2$$

이 때, 인플라톤의 에너지는 물질지배 때처럼 비슷하게 감소한다.

$$\frac{d\bar{\rho}_\phi}{dt} + 3H\bar{\rho}_\phi = 0 \quad \longrightarrow \quad \bar{\rho}_\phi \propto R^{-3}$$

재가열 온도 (Reheating temperature)

인플라톤이 다른 가벼운 입자들과 상호작용이 있어 붕괴를 할 수 있다면,

$$\frac{d\bar{\rho}_\phi}{dt} + (3H + \Gamma_\phi)\bar{\rho}_\phi = 0 \quad \xrightarrow[\Gamma_\phi \gg H]{} \quad \bar{\rho}_\phi(t) = \bar{\rho}_\phi(t_i)\left(\frac{R(t)}{R(t_i)}\right)^{-3} e^{-\Gamma_\phi t}$$

인플라톤의 붕괴 후 뜨거운 복사 시대가 시작되며, 그 때의 온도는

reheating temperature $\quad T_{\rm reh} \simeq \left(\frac{90}{4\pi^2 g_*}\right)^{1/4} \sqrt{\Gamma_\phi M_{\rm P}}$

인플라톤 붕괴에 의하여 생긴 물질과 복사

$$\dot{\rho}_M + 3H(\rho_M + p_M) = \Gamma \rho_\varphi$$

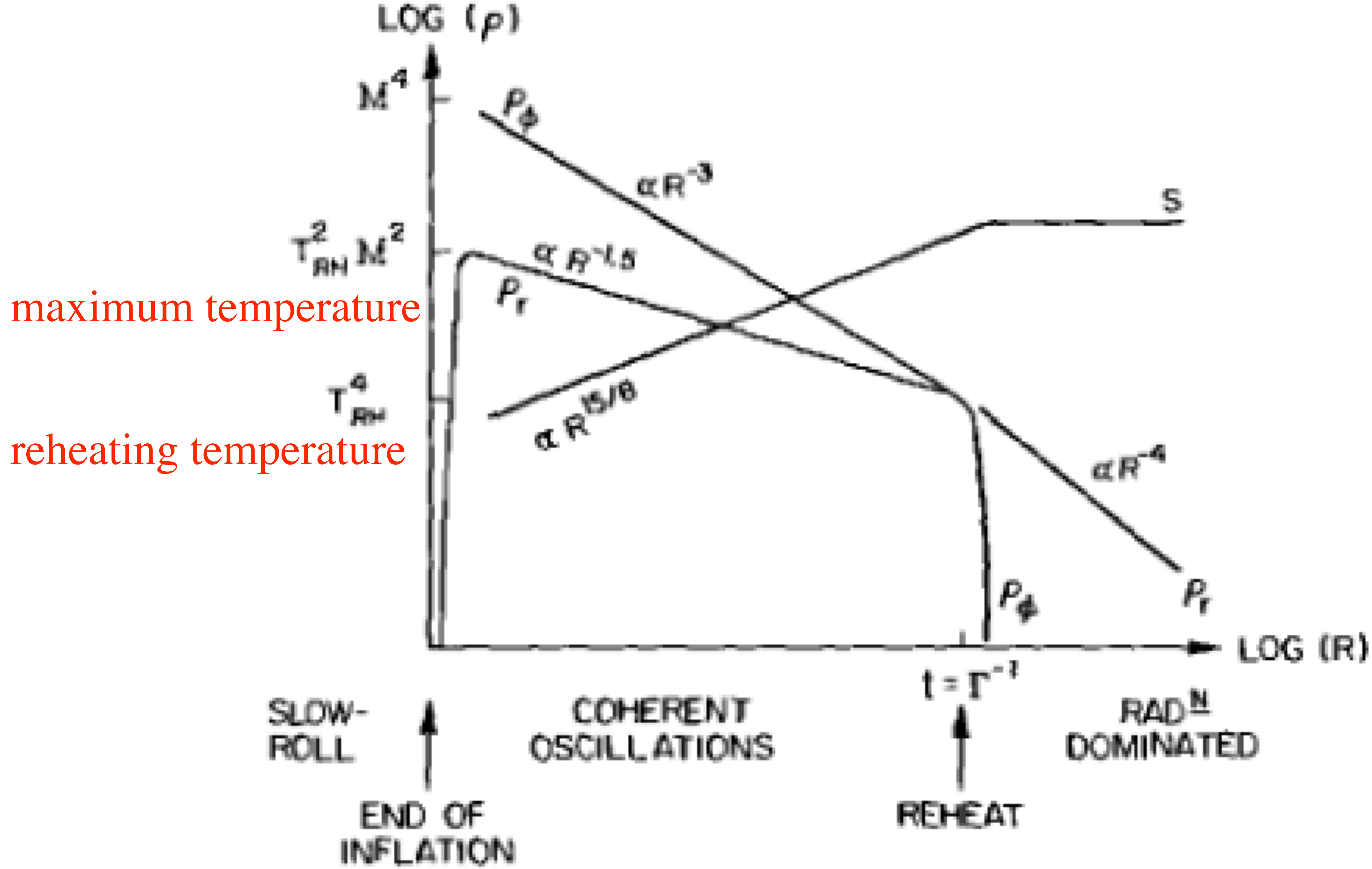
인플라톤의 붕괴에 의하여 생긴 물질들이 아주 상대론적이라고 하면,

$$p_M = \rho_M / 3$$

물질밀도의 해를 구할 수 있다.

$$\rho_M(t) = \frac{\rho_\varphi(t_I)\, \Gamma\, a^3(t_I)}{a^4(t)} \int_{t_I}^{t} a(t')\, e^{-\Gamma(t'-t_I)}\, dt'$$

그리고 표준 빅뱅 팽창이 시작된다.



LOG (ρ)
M⁴
ρφ
α R⁻³
S
T²RH M²
α R⁻¹·⁵
ρr
maximum temperature
T⁴RH
α R¹⁵/⁸
reheating temperature
α R⁻⁴
ρφ
ρr
LOG (R)
t = Γ⁻¹
SLOW-ROLL
COHERENT OSCILLATIONS
RADN DOMINATED
END OF INFLATION
REHEAT

New inflation and eternal inflation

The classical movement of inflaton field on 1/H time scale,

$$\Delta\phi_{cl} \simeq -\frac{V_\phi}{3H} H^{-1} \simeq M_P\sqrt{2\epsilon}$$

The quantum movement of inflaton field on 1/H time scale,

$$\Delta\phi_{qu} \simeq \frac{H}{2\pi}$$

For new inflation,

$$\frac{\Delta\phi_{qu}}{\Delta\phi_{cl}} \simeq \frac{H}{2\pi M_P\sqrt{2\epsilon}} \sim \frac{M^2}{M_P^2} \gg 1$$

$$\epsilon \simeq \frac{M_{\rm Pl}^2\phi^2}{M^4} \qquad \phi \sim \frac{H}{2\pi}$$

Chaotic inflation [Linde, 1983]

우주초기에 스칼라장은 공간의 위치에 따라 무작위적인 값을 가지고 카오스적으로 분포하고 있는 것으로 고려한다.

그 중에 인플레이션 조건을 만족하는 영역이 기하급수적 팽창을 하여 균일한 공간의 우주 영역을 만들게 된다.

초기우주에서의 상전이나 열적평형상태 같은 가정은 필요하지 않다.

Chaotic Inflation [Linde 1983, 1405.0270]

=/ monomial potential

The name of this broad class of inflationary theories is related to the observation that inflation can be realized even in the theories where the inflaton potential does not have any special features such as local minima or maxima with extraordinary small curvature, and even if the universe was not born in the hot Big Bang. To put it to a proper context, one should compare it to other approaches to inflation.

여러가지 인플레이션 모형들

Power law potential $V(\phi) = \lambda M_{\rm pl}^4 \left(\frac{\phi}{M_{\rm pl}}\right)^n$

Exponential potential and power law inflation $V(\phi) = \Lambda^4 \exp\left(-\lambda \frac{\phi}{M_{\rm pl}}\right)$

$$a(t) \propto t^{2/\lambda^2}.$$

Inverse power law potential $V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{\rm pl}}\right)^{-\beta}$

Hilltop models $V(\phi) \approx \Lambda^4 \left(1 - \frac{\phi^p}{\mu^p} + \ldots\right)$ $\mu \gtrsim 9\, M_{\rm pl}.$

Natural inflation $V(\phi) = \Lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right)\right]$

Hybrid inflation $V(\phi,\chi) = \Lambda^4 \left(1 - \frac{\chi^2}{\mu^2}\right)^2 + U(\phi) + \frac{g^2}{2}\phi^2\chi^2$

여러가지 인플레이션 모형들

Non-minimal coupling to gravity

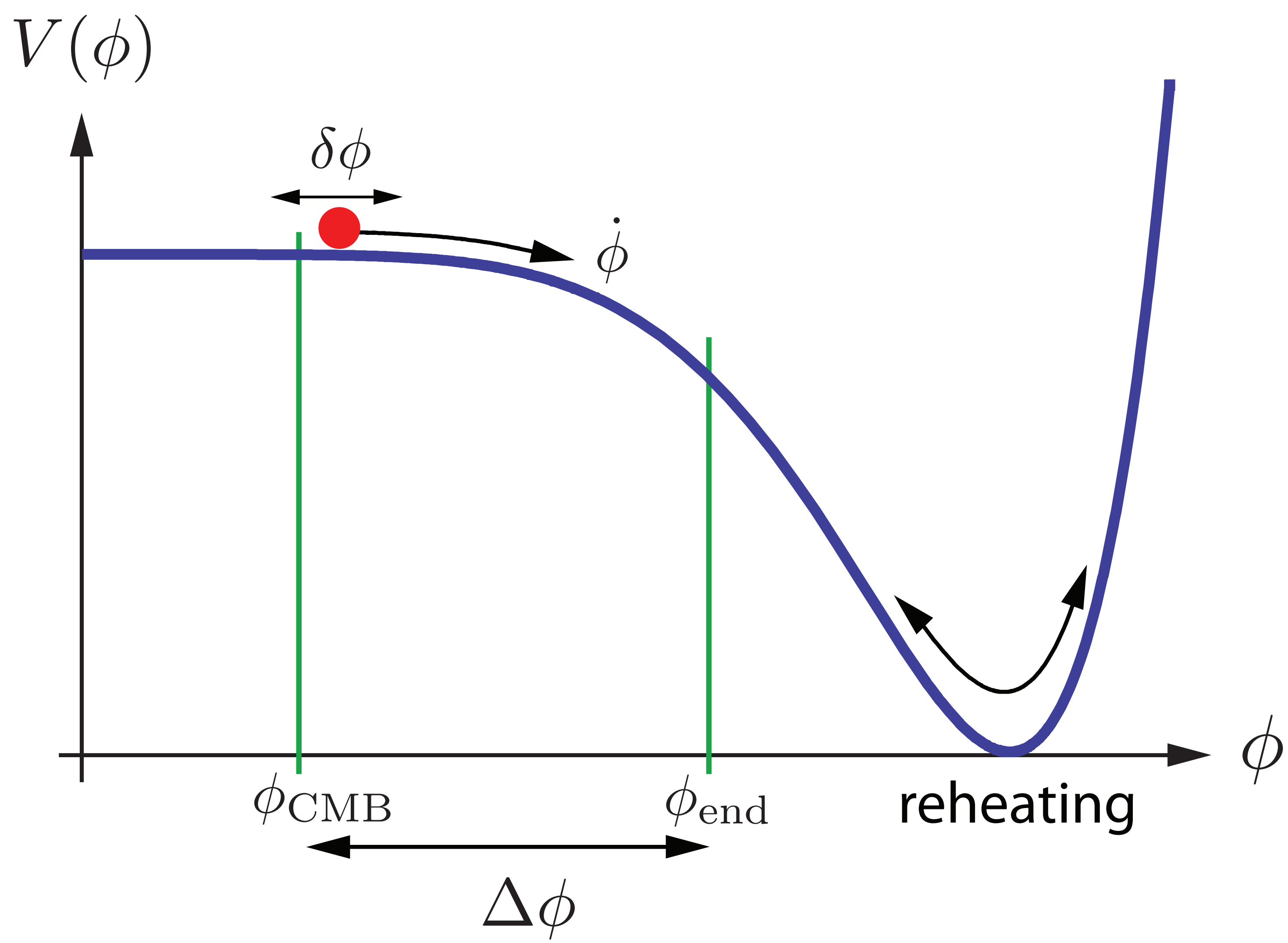
Modified gravity $\qquad f(R)$ theories

Non-canonical kinetic terms $\qquad \mathcal{L}_\phi = X - V(\phi)\,, \qquad X \equiv \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

Multi-inflaton fields

1. Small-field inflaiton

- Field moves over a small distance: $\Delta\phi < M_{\rm pl}$
- predict very small gravitational waves produced during inflation
- related to the spontaneous symmetry breaking

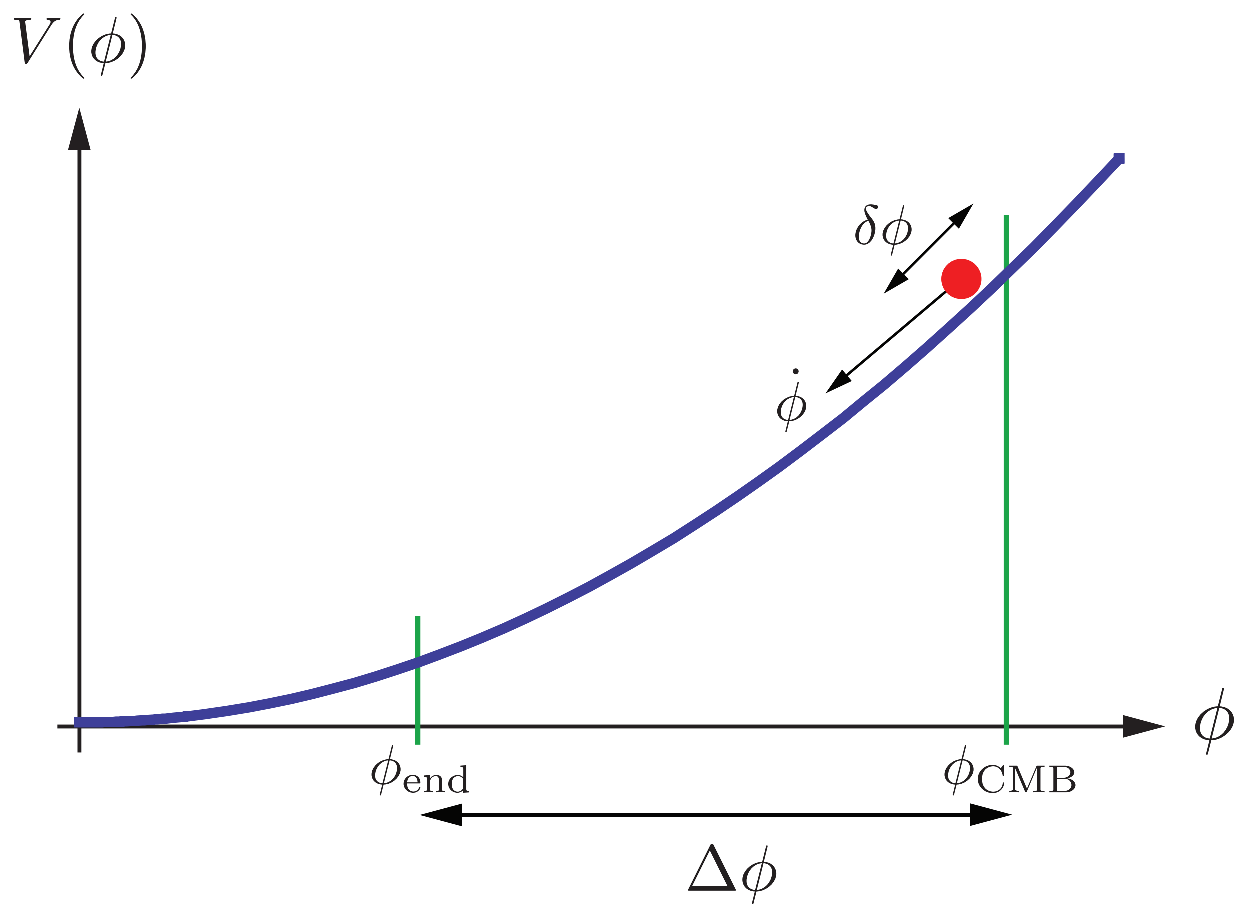


2. Large-field inflation

$$\Delta\phi > M_{\rm pl}$$

- Field moves over a large distance and moves to the minimum: $\phi = 0$
- predict observable gravitational waves produced during inflation

- e.g., monimial inflation, natural inflation etc

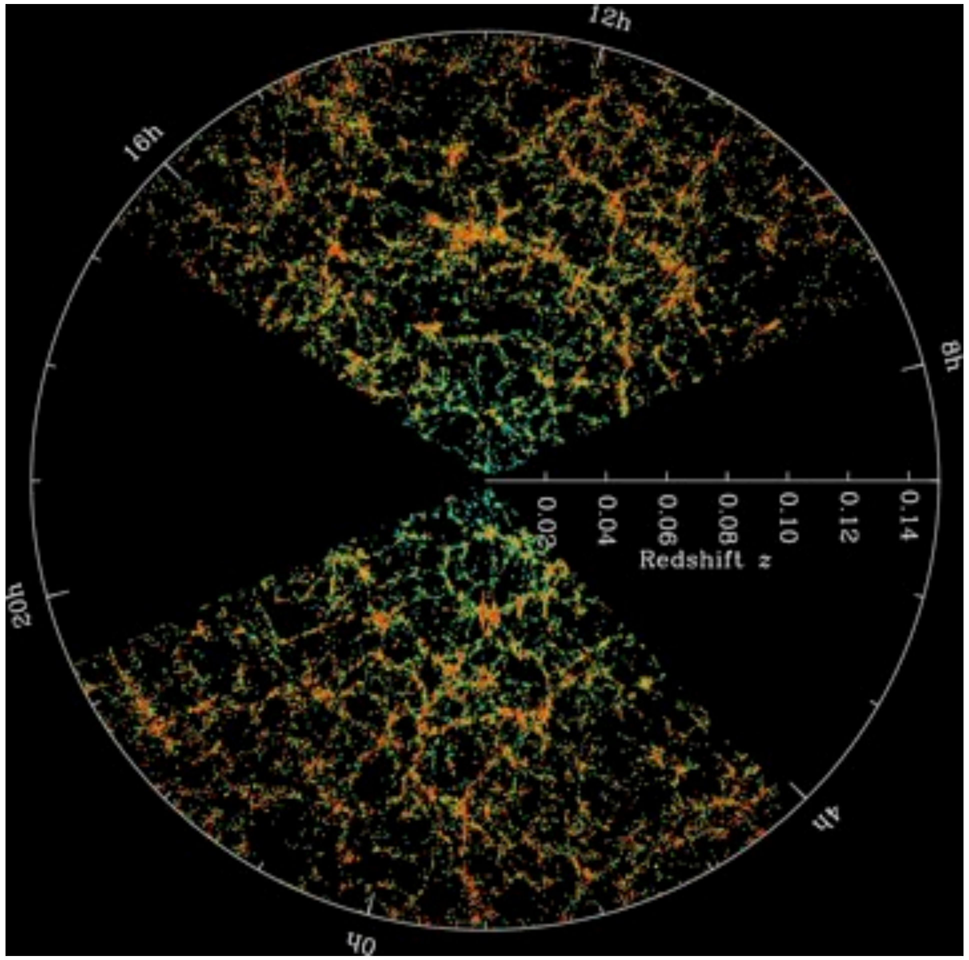


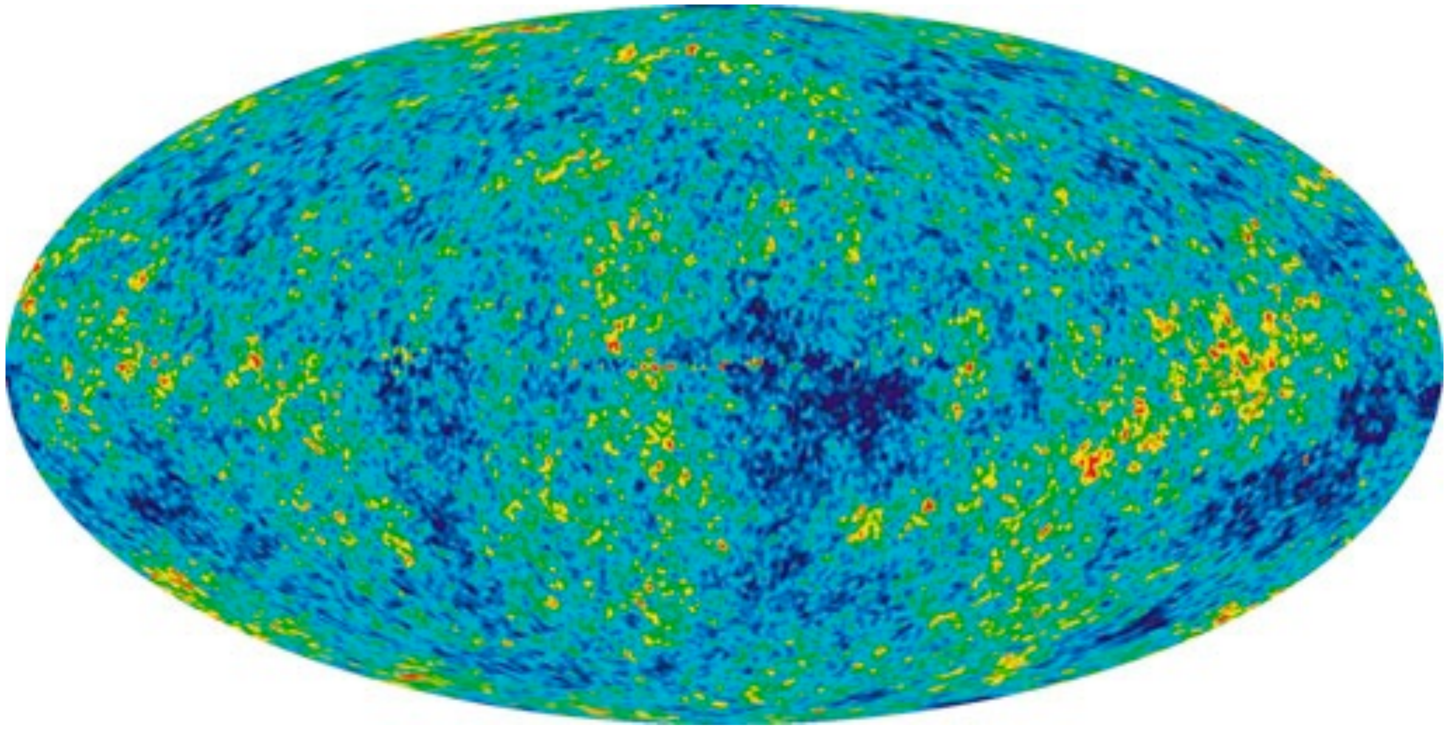
Up to now

$$\rho(t, \mathbf{x}) = \rho(t), T(t, \mathbf{x}) = T(t), \cdots$$

Cosmological principle:

Homogeneous and Isotropic Universe





- Large Scale Structure formation

- CMB temperature anisotropy

실제 우리 우주는 불균일하고 비등방적이다.

$$\frac{\delta\rho(t,\mathbf{x})}{\rho_0(t)} \sim \frac{\delta T(t,\mathbf{x})}{T_0(t)} \sim 10^{-5}$$

We need

$$\rho(t,\mathbf{x}) = \rho_0(t) + \delta\rho(t,\mathbf{x}), T(t,\mathbf{x}) = T_0(t) + \delta T(t,\mathbf{x}), \cdots$$

Initial Small fluctuations to the background values

$$\frac{\delta\rho(t,\mathbf{x})}{\rho_0(t)} \sim \frac{\delta T(t,\mathbf{x})}{T_0(t)} \sim 10^{-5}$$

Metric perturbation and gauge

We can solve Einstein equation

Einstein equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

with linear perturbation with scalar and tensor

$$ds^2 = (1+2\phi)dt^2 - a^2[(1-2\psi)\delta_{ij} - h_{ij}]dx^i dx^j$$

$\delta\rho(t, \mathbf{x})$ and $\delta p(t, \mathbf{x})$

we can solve the perturbed equations

$$G^{\mu\nu} = G_0^{\mu\nu}(t) + \delta G^{\mu\nu}(t, \mathbf{x})$$

→

$$T^{\mu\nu} = T_0^{\mu\nu}(t) + \delta T^{\mu\nu}(t, \mathbf{x})$$

In the 0-th order, background equation,

$$G_0^{\mu\nu}(t) = \frac{8\pi G}{c^4} T_0^{\mu\nu}(t)$$

In the 1st order (linear), perturbation equation,

$$\delta G^{\mu\nu}(t, \mathbf{x}) = \frac{8\pi G}{c^4} \delta T^{\mu\nu}(t, \mathbf{x})$$

Looks complicated!

Metric perturbations

$$ds^2 = \left[{}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}(x^\gamma)\right] dx^\alpha dx^\beta ,$$

0-th order with conformal time $dt = a(t) d\eta$

$${}^{(0)}g_{\alpha\beta} dx^\alpha dx^\beta = a^2(\eta)\left(d\eta^2 - \delta_{ij} dx^i dx^j\right) .$$

1-st order metric tensor is symmetric 4x4 matrix : 10 components

classify them based on the symmetry of rotation

S	V	T	
1			$\delta g_{00} = 2a^2\phi ,$
1	2		$\delta g_{0i} = a^2\left(B_{,i} + S_i\right) .$ with constraint $S^i_{,i} = 0$
2	2	2	$\delta g_{ij} = a^2\left(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}\right) .$
4 +	4 +	2	=10

traceless, transverse $h^i_i = 0, \quad h^i_{j,i} = 0,$

zero divergence $F^i_{,i} = 0$

Scalar perturbations ϕ, B, ψ, E

They are induced by the energy density inhomogeneities.
Related to the gravitational instability.

Vector perturbations S_i and F_i

They are related to the rotational motion of the fluid. They decay very quickly and are not interesting in the linear perturbation.

Tensor perturbations h_{ij}

They describe gravitational waves.

They are decoupled in the Einstein equations and thus can be studied separately.

General coordinate transformation

$$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha,$$

$\xi^\alpha = \xi^\alpha(\mathbf{x}, t)$

infinitely small functions of space and time

Scalar $\quad \tilde{\phi}(\tilde{x}) = \phi(x)$

Vector $\quad \tilde{V}_\alpha(\tilde{x}) = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} V_\gamma(x)$

$$\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} = \frac{\partial(\tilde{x}^\gamma - \xi^\gamma)}{\partial \tilde{x}^\alpha} = \delta^\gamma_\alpha - \xi^\gamma_{,\alpha}$$

Tensor $\quad \tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \frac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta}(x^\rho)$

$$\approx {}^{(0)}g_{a\beta}(x^\rho) + \delta g_{\alpha\beta} - {}^{(0)}g_{\alpha\delta}\xi^\delta_{,\beta} - {}^{(0)}g_{\gamma\beta}\xi^\gamma_{,\alpha},$$

Using $\quad \tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = {}^{(0)}g_{a\beta}(\tilde{x}^\rho) + \delta\tilde{g}_{a\beta}, \quad$ and $\quad {}^{(0)}g_{a\beta}(x^\rho) \approx {}^{(0)}g_{a\beta}(\tilde{x}^\rho) - {}^{(0)}g_{a\beta,\gamma}\xi^\gamma,$

we infer the gauge transformation law for tensor perturbation

$$\delta g_{\alpha\beta} \to \delta\tilde{g}_{a\beta} = \delta g_{\alpha\beta} - {}^{(0)}g_{a\beta,\gamma}\xi^\gamma - {}^{(0)}g_{\gamma\beta}\xi^\gamma_{,\alpha} - {}^{(0)}g_{\alpha\delta}\xi^\delta_{,\beta}.$$

Gauge invariant variables

$$ds^2 = (1+2\phi)dt^2 - a^2[(1-2\psi)\delta_{ij} - h_{ij}]dx^i dx^j$$

Curvature perturbation on uniform-density hypersurface

$$\zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}_0}$$

It remains constant outside horizon for adiabatic matter perturbations, $k \ll aH$

$$\delta p_{\rm en} \equiv \delta p - \frac{\dot{p}_0}{\dot{\rho}_0}\delta\rho = 0$$

In the spatially flat hypersurface, $\zeta = \dfrac{\delta\rho}{3(\rho_0 + p_0)}$

During slow-roll inflation, $\rho = V(\phi),\ \delta\rho = \dfrac{\partial V}{\partial\phi}\delta\phi$ $\qquad \dot{\rho} = \dfrac{\partial V}{\partial\phi}\dot{\phi}$

$$\zeta \simeq -\psi - H\frac{\delta\phi}{\dot{\phi}}$$

Coordinate transformation

$$t \rightarrow t + \alpha$$
$$x^i \rightarrow x^i + \delta^{ij}\beta_{,j}$$

Linear order:

$$\psi \rightarrow \psi + H\alpha \qquad \text{and} \qquad \delta\rho \rightarrow \delta\rho - \dot{\rho}_0\alpha$$

The curvature perturbation

$$\zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}_0} \qquad \text{is invariant}$$

Gauge invariant variables

Comoving curvature perturbation

$$\mathcal{R} \equiv \psi - \frac{H}{\rho_0 + p_0}\delta q$$

δq is the scalar part of the 3-momentum density $T^0_i = \partial_i \delta q$.

During inflation, $$\mathcal{R} \simeq \psi + H\frac{\delta\phi}{\dot{\phi}} \simeq -\zeta$$

In general in the linearized Einstein equation,

$$-\zeta = \mathcal{R} + \frac{k^2}{(aH)^2}\frac{2\rho_0}{3(\rho_0 + p_0)}\Psi_B,$$

with $\Psi_{\rm B} \equiv \psi + a^2 H(\dot{E} - B/a)\,,$

Conservation outside horizon,

$$\dot{\mathcal{R}} = -\frac{H}{\bar{\rho} + \bar{p}}\,\delta p_{en} + \frac{k^2}{(aH)^2}\Big(\dots\Big)\,.$$

Gauge invariant variables

Inflaton perturbation on spatially flat slices

$$\boxed{Q \equiv \delta\phi + \frac{\dot{\phi}_0}{H}\psi}$$

Generation of perturbations

from inflation

Quantum fluctuation of scalar field generates

scalar perturbation ⟶ density perturbation

⟶ CMB

tensor perturbation ⟶ gravity wave

During inflation, the background is uniform. The quantum fluctuation occurs in this uniform background. The average fluctuation is zero, since somewhere the fluctuation is larger than background and somewhere smaller. However the average of the square of the fluctuation is not zero, and it is called the variance.

The variance in the harmonic oscillator [Dodelson]

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Consider x as operator and expand as creation and annihilation operator,

$$\hat{x} = v(\omega, t)\hat{a} + v^*(\omega, t)\hat{a}^\dagger \qquad \text{with solution} \qquad v(\omega, t) = \frac{e^{-i\omega t}}{\sqrt{2\omega}}$$

Quantization $[\hat{a}, \hat{a}^\dagger] \equiv \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$ with vacuum $\hat{a}|0\rangle = 0$

The variance of x is then

$$\begin{aligned} \langle |\hat{x}|^2 \rangle &= |v(\omega, t)|^2 \langle 0|\hat{a}\hat{a}^\dagger|0\rangle \\ &= |v(\omega, t)|^2 \langle 0|[\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|0\rangle \\ &= |v(\omega, t)|^2 \end{aligned}$$

Tensor perturbation $h \equiv h^{+}, h^{\times}$ $dt = a d\eta$ $\eta \equiv \int_{a_e}^{a} \frac{da}{Ha^2} \simeq -\frac{1}{aH}$

Equation $$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + k^2 h = 0$$

With new variable, $\tilde{h} \equiv \frac{ah}{\sqrt{16\pi G}}$ it becomes $\left[\ddot{\tilde{h}} + \left(k^2 - \frac{\ddot{a}}{a}\right)\tilde{h}\right] = 0$

* conformal time derivative

As a quantum operator for each mode,

$$\hat{\tilde{h}}(\vec{k}, \eta) = v(k, \eta)\hat{a}_{\vec{k}} + v^*(k, \eta)a^{\dagger}_{\vec{k}}$$

the variance is

$$\langle \hat{\tilde{h}}^{\dagger}(\vec{k}, \eta)\hat{\tilde{h}}(\vec{k}', \eta)\rangle = |v(\vec{k}, \eta)|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

and therefore

$$\langle \hat{h}^{\dagger}(\vec{k}, \eta)\hat{h}(\vec{k}', \eta)\rangle = \frac{16\pi G}{a^2} |v(k, \eta)|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$\equiv (2\pi)^3 P_h(k)\delta^3(\vec{k} - \vec{k}')$$ tensor power spectrum

To solve $v(k,\eta)$ $\quad$ $\ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right) v = 0$ $\quad$ $\eta \simeq -\frac{1}{aH}$

During inflation $\ddot{v} + \left(k^2 - \frac{2}{\eta^2}\right) v = 0$ with $v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right]$

Superhorizon $\lim_{-k\eta \to 0} v(k,\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \frac{-i}{k\eta}$ or $h \propto \frac{v}{a} \sim \frac{e^{-ik\eta} H}{k^{3/2}}$

it becomes constant and the power spectrum is

$$P_h(k) = \frac{16\pi G}{a^2} \frac{1}{2k^3\eta^2} = \frac{8\pi G H^2}{k^3}.$$ at around horizon exit.

$$h \equiv h^+, h^\times$$

Tensor spectrum : the power spectrum for the two polarization modes

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = (2\pi)^3 \, \delta(\mathbf{k} + \mathbf{k}') \, P_h(k) \,, \qquad \mathcal{P}_h = \frac{k^3}{2\pi^2} P_h(k)$$

We define the tensor power spectrum as the sum of the power spectra for the two polarizations

$$\mathcal{P}_t = 2\mathcal{P}_h \qquad \mathcal{P}_t = \frac{H^2}{M_P^2 \pi^2}$$

Tensor spectral index

$$n_t \equiv \frac{d \ln \mathcal{P}_t}{d \ln k}$$

The tensor power spectrum can be written as

$$\mathcal{P}_t(k) = A_t(k_*) \left(\frac{k}{k_*} \right)^{n_t(k_*)}$$

Scalar perturbation

From action, $$S = \int \left(\tfrac{1}{2} g^{\gamma\delta} \varphi_{,\gamma}\varphi_{,\delta} - V\right)\sqrt{-g}\, d^4x$$

Klein-Gordon eq. in the curved spacetime,

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\, g^{\alpha\beta} \frac{\partial \varphi}{\partial x^\beta}\right) + \frac{\partial V}{\partial \varphi} = 0,$$

For the perturbed scalar field, $\varphi = \varphi_0(\eta) + \delta\varphi(\mathbf{x}, \eta)$

Background eq. $$\varphi_0'' + 2\mathcal{H}\varphi_0' + a^2 V_{,\varphi} = 0$$

Linear order eq. in the gauge invariant form,

$$\overline{\delta\varphi}'' + 2\mathcal{H}\overline{\delta\varphi}' - \Delta\overline{\delta\varphi} + a^2 V_{,\varphi\varphi}\overline{\delta\varphi} - \varphi_0'(3\Psi + \Phi)' + 2a^2 V_{,\varphi}\Phi = 0.$$

$$\Psi' + \mathcal{H}\Phi = 4\pi\varphi_0'\overline{\delta\varphi},$$

$$\Psi = \Phi$$

Equation during slow-roll

$$\ddot{\delta\phi} + 2aH\dot{\delta\phi} + k^2\delta\phi = 0$$

same as the equation for the tensor perturbation.

The power spectrum is

$$P_{\delta\phi} = \frac{H^2}{2k^3}$$

at around horizon exit.

The curvature perturbation is then

$$\zeta = -H\frac{\delta\phi}{\dot{\phi}_0} \simeq \frac{1}{M_P^2}\frac{V}{V_\phi}\delta\phi$$

Therefore the power spectrum of the curvature perturbation is

$$P_\zeta = \left(\frac{V}{V_\phi}\right)^2 \frac{H^2}{2k^3} = \frac{1}{2\epsilon M_P^2}\frac{H^2}{2k^3}$$

at around horizon exit.

Curvature perturbation: $\zeta(\mathbf{x}) = \int \zeta_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{d^3k}{(2\pi)^{3/2}}$

$\langle \cdots \rangle$

: ensemble average of fluctuations

Power spectrum : size of perturbation

$$\langle \zeta(\mathbf{k_1})\zeta(\mathbf{k_2})\rangle = (2\pi)^3 P(k_1)\delta^3(\mathbf{k_1}+\mathbf{k_2})$$

$$= (2\pi)^3 \frac{2\pi^2}{k_1^3}\mathcal{P}(k_1)\delta^3(\mathbf{k_1}+\mathbf{k_2})$$

$$\int \frac{\mathrm{d^3k}}{(2\pi)^3} P_\zeta(k) = \int \mathcal{P}_\zeta(k)\,\mathrm{d\,logk}$$

$$\mathcal{P}(k) = \frac{k^3}{2\pi^2}P(k)$$

Scalar spectral index

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}}{d\ln k}$$

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2\epsilon M_P^2}$$

Running of spectral index

$$\alpha_s \equiv \frac{dn_\zeta}{d\ln k}$$

Approximation by a power law around pivot scale k_*

$$\mathcal{P}(k) = A_s\left(\frac{k}{k_*}\right)^{n_s(k_*)-1+\frac{1}{2}\alpha_s(k_*)\ln(k/k_*)}$$

Power spectrum

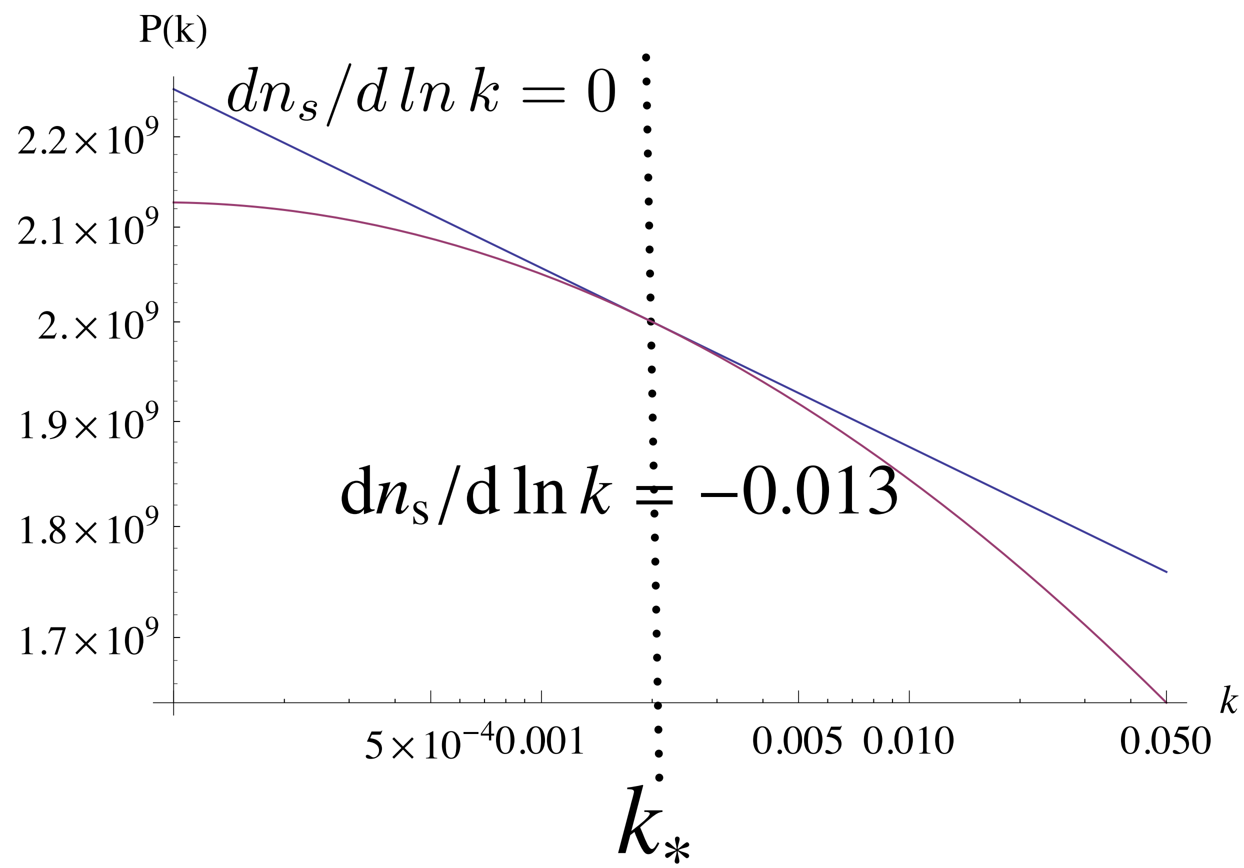
$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{1}{2}\, dn_s/d\ln k\, \ln(k/k_*) + \frac{1}{6}\, d^2 n_s/d\ln k^2 (\ln(k/k_*))^2 + \ldots}$$

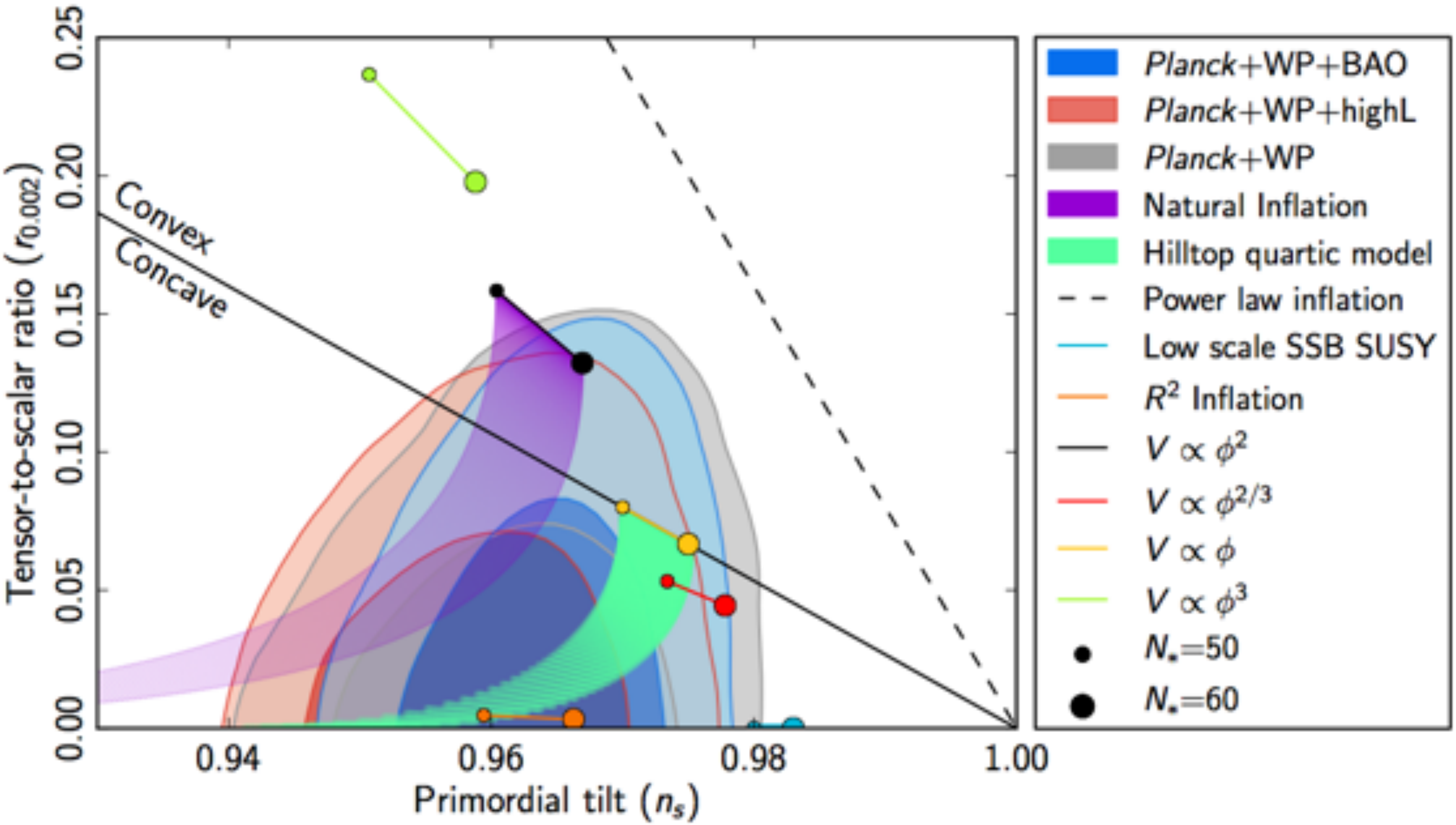
As : amplitude $A_s = 2 \times 10^{-9}$

k_* : pivot scale $k_* = 0.002\,\mathrm{Mpc}^{-1}$

n_s : spectral index $n_s = 0.96$

$dn_s/d\ln k$: running of spectral index





Tensor-to-scalar ratio ($r_{0.002}$)
Primordial tilt (n_s)
0.00
0.05
0.10
0.15
0.20
0.25
0.94
0.96
0.98
1.00
Convex
Concave
Planck+WP+BAO
Planck+WP+highL
Planck+WP
Natural Inflation
Hilltop quartic model
Power law inflation
Low scale SSB SUSY
R^2 Inflation
$V \propto \phi^2$
$V \propto \phi^{2/3}$
$V \propto \phi$
$V \propto \phi^3$
N_*=50
N_*=60

- Origin of the primordial curvature perturbation : Inflation

Radiation dominated era

Inflation → CMB

$$\delta\phi$$

quantum fluctuation
of a light scalar field

$$\zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}}$$

$$\delta\rho \simeq \frac{dV}{d\phi}\delta\phi$$

$$\zeta$$

curvature perturbation
: conserved outside horizon
if $p = p(\rho)$

[Lyth, Malik, Sasaki, 2005]

If not, it can be changed
eg, multi-field

$$\frac{\Delta T}{T} \sim 10^{-5}$$

anisotropy in CMB

$$\frac{\Delta T}{T} = -\frac{1}{5}\zeta$$ (SW limit)

- The properties of the primordial curvature perturbation [Planck, 2013]

  - Power spectrum $\mathcal{P}_\zeta = (2.198 \pm 0.056) \times 10^{-9}$: small perturbation

  - Spectral index $n_\zeta = 0.9603 \pm 0.0073$: almost scale-invariant

  - Running spectral index $dn_s/d\ln k = -0.013 \pm 0.009$ (68%; *Planck*+WP);

  - Tensor-to-scalar ratio $r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta}$

    $r_{0.002} < 0.11$ (95%; no running),

    $r_{0.002} < 0.26$ (95%; including running).

    : unobserved yet

  - Non-Gaussianity

f_{NL}		
Local	Equilateral	Orthogonal
2.7 ± 5.8	−42 ± 75	−25 ± 39

: No evidence of non-G yet

Tensor to scalar ratio

$$r \equiv \frac{\mathcal{P}_t}{\mathcal{P}_\zeta} \simeq 8\epsilon \qquad\qquad \mathcal{P}_t = \frac{H^2}{M_P^2 \pi^2}$$

The determination of r gives the energy scale of inflation

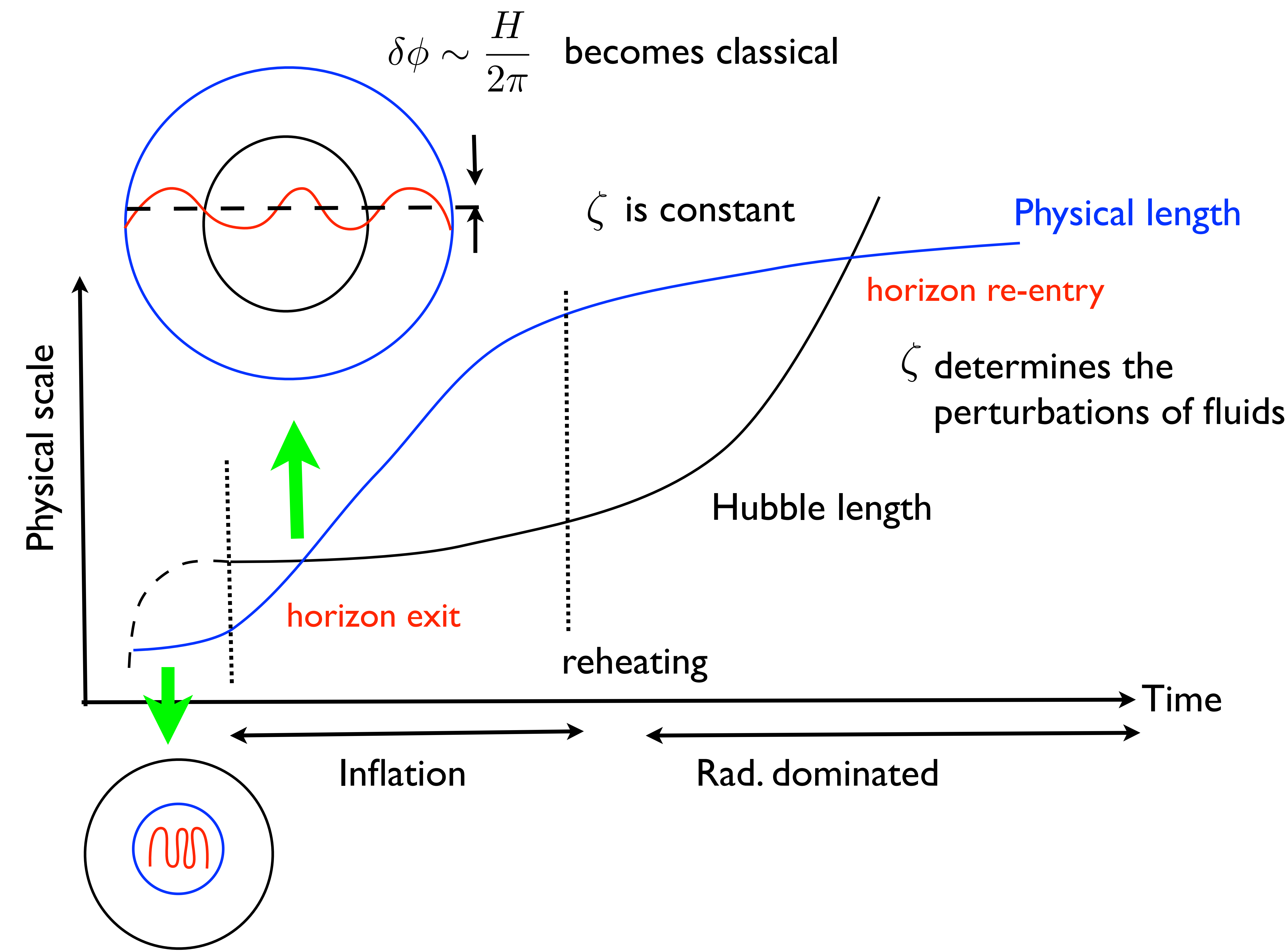
$$\boxed{V^{1/4} \sim \left(\frac{r}{0.01}\right)^{1/4} 10^{16}\,\mathrm{GeV}}\,.$$

Lyth bound

$$r = \frac{8}{M_{\rm pl}^2}\left(\frac{d\phi}{dN}\right)^2 . \qquad\qquad \frac{\Delta\phi}{M_{\rm pl}} = \int_{N_{\rm end}}^{N_{\rm cmb}} \mathrm{d}N\,\sqrt{\frac{r}{8}}\,.$$

$$\boxed{\frac{\Delta\phi}{M_{\rm pl}} = \mathcal{O}(1) \times \left(\frac{r}{0.01}\right)^{1/2}}\,,$$

The large value of tensor-to-scalar ratio is related to the large field inflation.


$\delta\phi \sim \frac{H}{2\pi}$ becomes classical
ζ is constant
Physical length
horizon re-entry
ζ determines the perturbations of fluids
Physical scale
Hubble length
horizon exit
reheating
Time
Inflation
Rad. dominated