Exactly Solvable Many-Body Stochastic Processes

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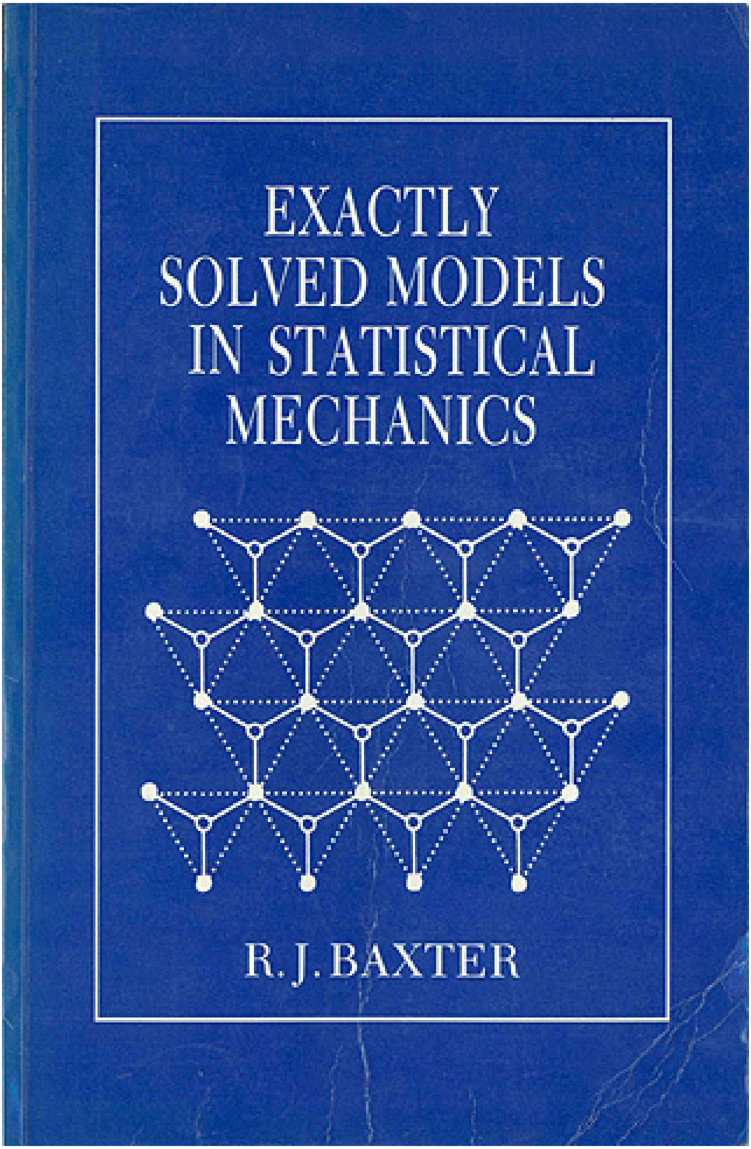
September 13, 2014

Preface

This is the lecture note for the Open KIAS PSI Summer Institute 2014 which is held at Alpensia Resort in August 24–30, 2014. In this lecture, I introduce a few stochastic systems that appear frequently in the study of interacting particle systems. The special feature of the selected models is that they are exactly solvable. By studying the models, the students can learn the important analytic techniques of nonequilibrium statistical mechanics. Hopefully, they can serve as useful tools for their future research.

“they are relevant and they can be solved, so why not do so and see what they tell us?”

in the preface of “Exactly solved models in statistical mechanics" by R.J. Baxter


EXACTLY
SOLVED MODELS
IN STATISTICAL
MECHANICS
R. J. BAXTER

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Introduction

A stochastic system may be represented by the Langevin equation of the form

$$m\ddot{q}_i = -\gamma_{ij}\dot{q}_j + f_i(q) + \xi_i(t) \tag{1.1}$$

with the stochastic noise forces ξ resulting from interactions with the thermal or non-thermal reservoir. Due to the stochastic noises, the dynamics can be described by the probability distribution $P(q,t)$. It satisfies the Fokker-Planck equation [Ris84, VK92, Gar85]

$$\frac{\partial}{\partial t}P(q,t) = -\frac{\partial}{\partial q_i}D_i(q)P(q,t) + \frac{1}{2}\frac{\partial^2}{\partial q_i \partial q_j}D_{ij}(q)\ .P(q,t) \tag{1.2}$$

A stochastic system may be defined by a dynamic rule or the transition rate $W(q|q')$. Then, the probability distribution satisfies the master equation

$$\frac{\partial}{\partial t}P(q,t) = \sum_{q'} W(q|q')P(q',t) - \sum_{q} W(q'|q)P(q,t) \ . \tag{1.3}$$

Once the probability distribution is known, then any physical quantities are obtained by the ensemble average

$$\langle \cdots B(q'(t'))A(q(t))\rangle = \int dq dq' \cdots B(q')P(q',t'|q,t)A(q)P(q,t) \ . \tag{1.4}$$

In general, it is hard to solve for the probability distribution and the ensemble average in the transient and stationary regimes. So, one usually relies on numerical means such as the Monte Carlo simulations or exact enumerations. Fortunately, there are a class of models that are exactly solvable. We will learn the important analytic methods that are powerful in studying interacting stochastic particle systems.

2 Random walks in 1D

A random walker is hopping around in 1D space. We will cover the probability distribution, the first passage time, and the record statistics. Extensive reviews on these topics are found in [Red01, Hug96].

2.1 Model

Let $x(t)$ be the position of a random walker at time t. A random walk motion is specified with the time interval τ between two consecutive jumps and the displacement d of each jump. The time interval may be chosen from a probability distribution

$$f(\tau) = \begin{cases} \delta(\tau - \tau_0) & \text{[discrete time interval]} \\ \mu e^{-\mu\tau} & \text{[Poisson process with a rate } \mu \text{]} \end{cases} . \tag{2.1}$$

For the Poisson case, the probability that the RWer makes a jump within an infinitesimal time interval Δt is given by $\mu \Delta t$ independent of all the previous jumps (Markov process). The displacement may also be discrete (e.g., $d = \pm a$ with a lattice constant $a = 1$) or distributed continuously according to a given distribution $w(d)$.

2.2 Probability distribution

Consider the continuous-time random walks with rate $\mu = 1$ on the 1D lattice. The displacement is $d = \pm 1$ with equal probability (no bias).

Let $P_t(y|x)$ be the probability that the RWer is found at position y at time t given that it was at position x at time $t = 0$. In the time interval dt, it evolves as

$$P_{t+dt}(y|x) = P_t(y|x)(1-dt) + P_t(y+1|x)dt/2 + P_t(y-1|x)dt/2 \, . \tag{2.2}$$

Hence we obtain the differential equation

$$\frac{d}{dt} P_t(y|x) = \frac{1}{2} P_t(y+1|x) + \frac{1}{2} P_t(y-1|x) - P_t(y|x) \tag{2.3}$$

with the initial condition

$$P_0(y|x) = \delta_{y,x} \, . \tag{2.4}$$

The differential equation can be solved in terms of the Fourier series

$$P_t(y|x) = \int_{-\pi}^{\pi} d\theta A(\theta) \cos[(y-x)\theta] e^{-\kappa(\theta)t} \, , \tag{2.5}$$

where the differential equation and the initial condition yield that

$$A(\theta) = \frac{1}{2\pi} \tag{2.6}$$

and

$$\kappa(\theta) = (1 - \cos\theta)\ . \tag{2.7}$$

The integral in (2.5) cannot be evaluated in terms of elementary functions. Note that the modified Bessel functions $\{I_n(z)\}$ have the generating function

$$e^{z\cos(\theta)} = I_0(z) + 2\sum_{n=1}^{\infty} I_n(z)\cos(n\theta)\ . \tag{2.8}$$

Comparing (2.5) and (2.8), we find that

$$P_t(y|x) = e^{-t} I_{y-x}(t)\ . \tag{2.9}$$

It is instructive to derive its asymptotic form in the $t \to \infty$ limit. Due to the exponential factor, the integral is dominated by the contribution near $\theta = 0$, which allows one to use $\kappa(\theta) \simeq \theta^2/2$ and extend the integration interval to $(-\infty, \infty)$. Therefore, we obtain the Gaussian distribution

$$P_t(y|x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}\ . \tag{2.10}$$

This result is easily understood from the central limit theorem. Since the jumping rate is μ, the RWer makes $\sim \mu t$ jumps on average. Each jump is independent of each other and the variance is given by $\sigma_d^2 = 1$. Hence, the overall displacement $y - x$ should converge to the Gaussian distribution with the variance $\sigma_d^2 \mu t = t$ with $\mu = 1$.

The RWer is spread over the length scale

$$\xi(t) \sim t^{1/2}\ . \tag{2.11}$$

Usually, the power-law scaling relation between time and length is characterized by the dynamic exponent z as $\xi \sim t^{1/z}$. The dynamic exponent for the random walks is given by $z = 2$ [diffusion].

2.3 First passage probabilities

How long does it take for the RWer to find a target site? The first passage probability is defined as

$$F_t(y|x)dt = \text{ Prob [RWer starting at } x \text{ reaches } y \text{ for the first time]} \tag{2.12}$$

during the time interval $t \sim t + dt$. For $y \neq x$, it satisfies

$$P_t(y|x) = \int_0^t dt' F_{t'}(y|x) P_{t-t'}(y|y) \; . \tag{2.13}$$

In order to find the solution for F_t, one needs to introduce the Laplace transformations

$$P_s(y|x) \equiv \int_0^\infty dt e^{-st} P_t(y|x) \; , \tag{2.14}$$

$$F_s(y|x) \equiv \int_0^\infty dt e^{-st} F_t(y|x) \; . \tag{2.15}$$

Multiplying both sides of (2.13) of e^{-st} and integrate over t, and using the convolution theorem, we obtain that

$$F_s(y|x) = \frac{P_s(y|x)}{P_s(y|y)} \; . \tag{2.16}$$

Note that the Laplace transformation of the modified Bessel function is given by

$$\int_0^\infty e^{-st} I_n(t) dt = \frac{1}{\sqrt{s^2-1}\left(s+\sqrt{s^2-1}\right)^{|n|}} \; . \tag{2.17}$$

The Laplace transformation for $P_t(y|x)$ is obtained by replacing s with $s+1$:

$$P_s(y|x) = \frac{1}{\sqrt{s^2+2s}\left(s+1+\sqrt{s^2+2s}\right)^{|y-x|}} \,. \tag{2.18}$$

Consequently, we obtain

$$F_s(y|x) = \frac{1}{\left(s+1+\sqrt{s^2+2s}\right)^{|y-x|}} \,. \tag{2.19}$$

for $y \neq x$.

It is subtle to define the first passage probability when $y = x$. In this case, we adopt the convention that the first visit to the starting site x should mean the first return to x after the RWer leaves x. Then, the first passage probability is given by

$$F_t(x|x) = \int_0^t dt' \mu e^{-\mu t'} \left(F_{t-t'}(x|x+1) + F_{t-t'}(x|x-1)\right)/2 \,. \tag{2.20}$$

The convolution theorem yields that ($\mu = 1$)

$$F_s(x|x) = \frac{1}{(s+1)(1+s+\sqrt{s^2+2s})} \,. \tag{2.21}$$

Can the RWer find the target eventually? The probability of finding is given by

$$P_{finding}(y|x) = \int_0^\infty dt F_t(y|x) = F_{s=0}(y|x) = 1 \,. \tag{2.22}$$

The RWer can find the target for sure. It is said that the random walks in 1D is *recurrent* instead of being *transient*.

The mean first passage time (MFPT) is given by

$$T(y|x) = \int_0^\infty dt\; tF_t(y|x) = -\left.\frac{\partial F_s(y|x)}{\partial s}\right|_{s=0} . \tag{2.23}$$

Near $s = 0$, F_s behaves as $F_s(y|x) \simeq 1 - a\sqrt{s}$ with a positive constant a depending on $|y-x|$. Therefore, the MFPT is infinity!

The RWer can find a target for sure, but it takes infinite time on average. The divergence of the MFPT is caused by the worst case in which the RWer is wandering in the wrong place. Is it possible to find a clever RW strategy to achieve a finite MFPT? Recently, Evans and Majumdar suggested that the RW search can by optimized by introducing a stochastic resetting [EM11].

In order to obtain $F_t(y|x)$, we have to perform the inverse Laplace transform of $F_s(y|x)$ in principle. However, one can find F_t easily by noting that $\frac{\partial}{\partial s}F_s(y|x) = -|y-x|P_s(y|x)$ for $y \neq x$. It yields that

$$F_t(y|x) = \frac{|y-x|}{t}P_t(y|x) = \frac{|y-x|}{t}e^{-t}I_{y-x}(t) \simeq \frac{|y-x|}{\sqrt{2\pi t^3}}e^{-(y-x)^2/2t} . \tag{2.24}$$

Even without the exact solution, one can find the $t^{-3/2}$ scaling law from the $\sqrt{s}$ dependence of F_s.

2.4 Record statistics

As an application of the 1D RWs, I would like to mention the record statistics shortly [And54, MZ08, WMS12]. In this section, we consider the RWs in discrete time:

$$x_i = x_{i-1} + d_i,\ i = 1, 2, \cdots \tag{2.25}$$

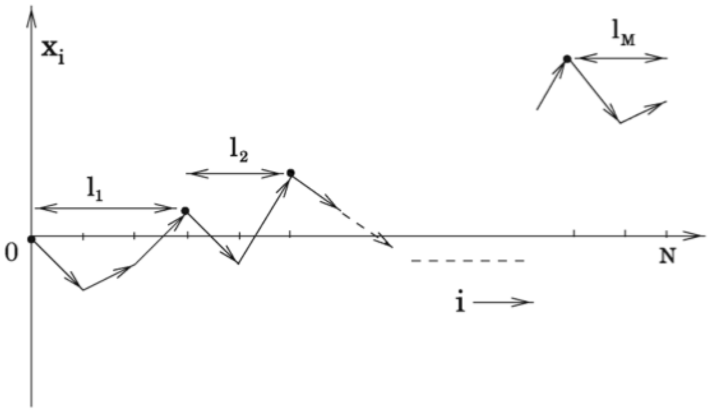

Figure 2.1: Records and the ages. This figure is taken from [MZ08].

where $x_0 = 0$ and the displacements d_i's are independent and identically distributed (i.i.d.) drawn from a distribution $w(d)$.

Consider a time trajectory $\{x_i\}$ at $0 \leq i \leq N$. RWer. If x_k is *strictly* greater than all the previous values, i.e., $x_k > x_i$ for all $i < k$, then x_k is called the *record*. x_0 will be regarded as the first record. The age of the n-th record, l_n, is defined as the time interval between the n-th and $(n+1)$-th records. See Fig. 2.1.

We are interested in the total number of records up to time N which will be denoted as R_N. In order to obtain the distribution function for R_N, consider the joint pdf $P(\{l_1, l_2, \cdots, l_R\}, R|N)$ of the ages $\{l_i\}$ and the number of records $R_N = R$ for the time interval N. A given record x_i is broken when the position x_j overtakes the previous record for the first time. Therefore, the pdf for the age of a record is given by the *first passage time* distribution F_l of the random walks. It is defined as

$$F_l = \text{Prob}[x_1 \leq x_0, x_2 \leq x_0, \cdots, x_{l-1} \leq x_0, x_l > x_0] \ . \qquad (2.26)$$

Notice the difference from the definition of the first passage event in the previous section. The last record is not broken up to l time steps. Hence the pdf for its age is given by the *persistence probability* q_l defined as

$$q_l = \text{Prob}[x_1 \leq x_0, x_2 \leq x_0, \cdots, x_l \leq x_0] \ . \qquad (2.27)$$

The persistence probability and the first passage probability is related through

$$F_l = q_{l-1} - q_l \ . \qquad (2.28)$$

The record breaking events are independent of each other, which yields

$$P(\{l_i\}, R|N) = F_{l_1} F_{l_2} \cdots F_{l_{R-1}} q_{l_R} \delta_{l_1+\cdots+l_R=N} \ . \qquad (2.29)$$

Therefore, the whole record statistics is determined by the persistence probability q_l. To a given jump distribution $w(d)$, one can calculate the persistence probability explicitly. Instead, we will use the theorem of Sparre Anderson for the generating function of q_l [And54]:

$$\tilde{q}(z) = \sum_{l=0}^{\infty} q_l z^l = \exp\left[\sum_{k=1}^{\infty} \frac{z^k}{k} \text{Prob}[x_k \leq 0]\right] \ , \qquad (2.30)$$

where $\text{Prob}[x_k \leq 0]$ denotes the probability that the position of the RWer after k steps is equal to or less than the initial position. The generating function for the first passage probability is given by

$$\tilde{F}(z) = \sum_{l=0}^{\infty} F_l z^l = 1 - (1-z)\tilde{q}(z) \ . \qquad (2.31)$$

The probability distribution of the number of records up to N time steps is given by

$$P(R|N) = \sum_{l_1+l_2+\cdots+l_R=N} P(\{l_i\}, R|N) \ . \tag{2.32}$$

Using the expression in (2.29), we finally obtain the generating function as

$$\tilde{P}(R,z) = \sum_{N=0}^{\infty} P(R|N) z^N = [\tilde{F}(z)]^{R-1} \tilde{q}(z) \ . \tag{2.33}$$

We assume that the jump distribution $w(d)$ is continuous and symmetric ($w(d) = w(-d)$). Then, $\text{Prob}[x_k \leq 0] = 1/2$ at all values of $k > 0$ irrespective of the shape of $w(d)$ (this is not the case when d is discrete). In this case, we can find easily that

$$\tilde{q}(z) = (1-z)^{-1/2} \tag{2.34}$$

$$\tilde{F}(z) = 1 - \sqrt{1-z} \tag{2.35}$$

$$\tilde{P}(R,z) = (1-z)^{-1/2}[1-\sqrt{1-z}]^{R-1} \tag{2.36}$$

The Taylor expansion yields that

$$P(R|N) = {}_{2N-R+1}C_N 2^{-2N+R-1} \simeq \frac{1}{\sqrt{\pi N}} e^{-(R-1)^2/(4N)} \tag{2.37}$$

Using the generating function, one can can calculate the average number of records. It is given by

$$\langle R \rangle_N = (2N+1)_{2N}C_N 2^{-2N} \to \frac{2}{\pi}\sqrt{N} \tag{2.38}$$

The $\sqrt{N}$ scaling implies that the record breaking events become rare and rare.

These results are universal in the sense that they are independent of the shape of $w(d)$ if it is symmetric and d is a continuous random variable. Record statistics for more general cases is studied in [MZ08, WMS12].

[Homework 1] Perform the Monte Carlo simulations to measure the persistence probability q_l of random walks using the jump distributions (i) uniform distribution $w(d) = 1/2$ for $-1 \leq d \leq 1$ and 0 otherwise, (ii) Lorenzian distribution $w(d) = 1/(d^2+1)/\pi$, (iii) $w(d) = \frac{1}{2}(\delta(d+1) + \delta(d-1))$, and (iv) $w(d) = \frac{1}{3}(\delta(d+1) + \delta(d) + \delta(d-1))$.

[Homework 2] The record statistics may be used for a test of randomness of a given time series x_t. Take an empirical time series, such as the daily price of a stock, and investigate its record statistics and discuss its randomness. The reference [WMS12] may be useful.

3

1D Ising model with $T = 0$ Glauber dynamics

If one cools down a thermal system from a high-temperature disordered phase to a low-temperature ordered phase, the system evolves into the ordered state displaying universal dynamics scaling. Figure 3.1 illustrates the ordering dynamics in 2D Ising model. In this chapter, we will study the the 1D Ising model whose ordering dynamics is exactly solvable.

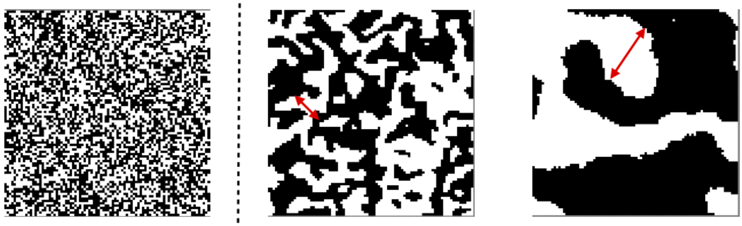

Figure 3.1: Coarsening dynamics of the 2D Ising model.

3.1 Model

Consider the Ising spins $\{s_i = \pm 1\}$ in one dimensional chain $i = 1, \cdots, L$. The energy is given by

$$E = -J \sum_i s_i s_{i+1} \tag{3.1}$$

with a positive coupling constant J. The periodic boundary condition $s_{L+i} = s_i$ is assumed. The thermal equilibrium properties of the model are simulated by the Monte Carlo dynamics. Specifically, one considers the Glauber dynamics [Gla63] in which each spin s_i flips to $s_i' = -s_i$ at the rate

$$w(s_i \to s_i' | s_{i-1}, s_{i+1}) = \mu \frac{e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}} \tag{3.2}$$

where $\beta = 1/(k_B T)$ and ΔE is the energy change upon flip. The one-dimensional Ising spins order only at zero temperature. We focus on the zero-temperature dynamics. At zero temperature, the transition rate becomes simple

$$w = \begin{cases} \mu & , \text{ if } \Delta E < 0 \quad (e.g., + - + \to + + +) \\ \frac{1}{2}\mu & , \text{ if } \Delta E = 0 \quad (e.g., + - - \to + + -) \\ 0 & , \text{ if } \Delta E > 0 \quad (e.g., + + + \to + - +) \,. \end{cases} \tag{3.3}$$

The same transition rule is achieved by requiring that each spin $s_i(t)$ copies the state of one of its neighbors at rate μ.

$$s_i(t+dt) = \begin{cases} s_i(t) & \text{with prob. } 1-\mu dt \\ s_{i-1}(t) & \text{with prob. } \frac{1}{2}\mu dt \\ s_{i+1}(t) & \text{with prob. } \frac{1}{2}\mu dt \end{cases} \tag{3.4}$$

Hence, the 1D Ising model with the zero-temperature Glauber dynamics is equivalent to the *voter model*.

The 1D Ising model can be also mapped to a reaction diffusion model. Introduce a random variable

$$n_i = (1 - s_i s_{i+1})/2 \,, \tag{3.5}$$

which takes the value $+1$ (being occupied by a particle X) when there is a domain wall between sites i and $i+1$ and 0 (being empty O) otherwise. The particles hop at a rate μ ($OX \overset{\mu/2}{\longleftrightarrow} XO$) and annihilate in pairs upon collision ($X + X \to O$). That is to say, the 1D Ising with the zero-temperature Glauber dynamics is equivalent to the *annihilating random walks*.

In the mean field theory, the mean density of domain walls $\rho = \langle n_i \rangle$ evolves in time following the rate equation

$$\frac{d\rho}{dt} = -2\mu\rho^2 \,. \tag{3.6}$$

Hence, the mean field theory predicts that

$$\rho = \frac{\rho_0}{1 + 2\mu\rho_0 t} \simeq \frac{1}{2\mu t} \,. \tag{3.7}$$

The mean field results will be compared with the exact solution.

3.2 Master equation approach

Let $P(\boldsymbol{s}, t)$ be the probability that the spins are in state $\boldsymbol{s} = (s_1, \cdots, s_L)$ at time t. It obeys the master equation

$$\dot{P}(\boldsymbol{s}, t) = \frac{d}{dt} P(\boldsymbol{s}, t) = -\sum_{i=1}^{L} w_i(\boldsymbol{s}) P(\boldsymbol{s}, t) + \sum_i w_i(\boldsymbol{s}'_i) P(\boldsymbol{s}'_i, t) \tag{3.8}$$

where $\boldsymbol{s}'_i \equiv (s_1, \cdots, s_{i-1}, -s_i, s_{i+1}, \cdots, s_L)$ and $w_i(\boldsymbol{s})$ denotes the transition rate from $\boldsymbol{s}$ to $\boldsymbol{s}'_i$. It is given by

$$w_i(\boldsymbol{s}) = \frac{\mu}{2} \left(1 - \frac{1}{2} s_i (s_{i-1} + s_{i+1}) \right) . \tag{3.9}$$

The mean value of s_k at time t is defined as $m_k(t) = \langle s_i \rangle_t = \sum_{\boldsymbol{s}} s_k P(\boldsymbol{s}, t)$. Its time evolution is governed by

$$\dot{m}_k = \sum_{\boldsymbol{s}} s_k \dot{P}(\boldsymbol{s}, t) = -2 \sum_{\boldsymbol{s}} s_k w_k(\boldsymbol{s}) P(\boldsymbol{s}, t) . \tag{3.10}$$

The last equation is obtained by inserting the master equation directly, or from a good intuition. The rate in (3.9) yields that

$$\dot{m}_k = -\mu \sum_{\boldsymbol{s}} \left(s_k - \frac{1}{2} s_k^2 (s_{k-1} + s_{k+1}) \right) P(\boldsymbol{s}, t) . \tag{3.11}$$

Note that $s_k^2 = 1$. Therefore, we obtain

$$\dot{m}_k = \frac{\mu}{2} \left(m_{k-1} - 2m_k + m_{k+1} \right) . \tag{3.12}$$

This is the diffusion equation which is exactly solvable.

In general, the equations of motion for $\langle s_k \rangle$ involve higher order correlation functions $\langle s_k s_{k'} \cdots \rangle$ (hierarchy). In this model, due to the specific property of the hopping rate, the higher order term $\langle s_k s_k s_{k'} \rangle$ reduces to $\langle s_{k'} \rangle$ and the equations of motion are *closed* and *solvable*.

What about the two point correlation functions $G_{i,j}(t) \equiv \langle s_i s_j \rangle_t = \sum_{\boldsymbol{s}} s_i s_j P(\boldsymbol{s}, t)$? The equations of motion are

$$\dot{G}_{i,j}(t) = -2\mu \langle s_i s_j (w_i(\boldsymbol{s}) + w_j(\boldsymbol{s})) \rangle \ . \tag{3.13}$$

Using the transition rates, we obtain

$$\dot{G}_{i,j} = \frac{\mu}{2} \left(G_{i-1,j} + G_{i+1,j} + G_{i,j-1} + G_{i,j+1} - 4G_{i,j} \right) \tag{3.14}$$

with the boundary condition $G_{i,i} = 1$. The equations of motion are also closed and solvable.

3.3 Duality

Let $\{s_i(0)\}$ be the initial spin configuration at time $t = 0$. The time evolution is specified by a so-called copy map. If s_i copies s_j at time t, we write

$$s_i(t) \to s_j(t) \ . \tag{3.15}$$

Figure 3.2 shows a realization of the copy map. The spin state $s_i(t)$ at site i at time t is determined by tracing back the copy map in the *time-reversed direction*. We denote the destination by i_t. Then, one finds that

$$s_i(t) = s_{i_t}(0) \ . \tag{3.16}$$

In Fig. 3.2, $s_1(t) = s_2(t) = s_3(t) = s_2(0)$ and $s_4(t) = s_5(t) = s_4(0)$.

Note that the trajectory connecting (i, t) to $(i_t, 0)$ is equivalent to a trajectory of the RWer with jumping rate μ. Initially at time t, there are N RWers. When two RWers collide, they merge into a single RWer. Therefore, we conclude that the 1D Ising model

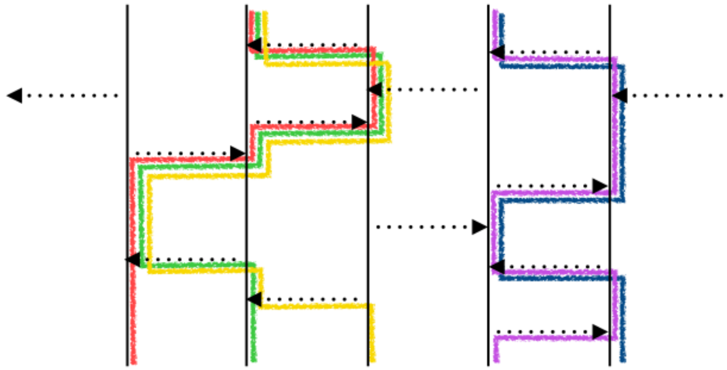

Figure 3.2: Copy map

with $T = 0$ Glauber dynamics is dual to the coalescing RWs in the opposite time direction [DZ96]. Statistical properties of a single RWer were studied in the previous chapter. They are useful for understanding the coalescing RWers problem.

3.4 Spin-spin correlation function

The ordering dynamics is characterized with the equal-time correlation function

$$\Phi_t(i, j) \equiv \text{ Prob } [s_i(t) = s_j(t)] \, . \tag{3.17}$$

For convenience, we assume that the spins are distributed independently and randomly with equal probability initially at $t = 0$.

In order to find $s_i(t)$ and $s_j(t)$, we consider two RWers who perform the coalescing RWs starting from i and j, respectively, for a time interval t. If they meet before t ($i_t = j_t$), then $s_i(t) = s_j(t)$. If they do not ($i_t \neq j_t$), then the two spins may be equal to each other depending on the initial spin states $s_{i_t}(0)$ and $s_{j_t}(0)$. Since we assume the random initial condition, such a probability is given by 1/2. Thus, the duality yields that

$$\Phi_t(i,j) = \frac{1}{2}C_t(i,j) + (1 - C_t(i,j)) = \frac{1}{2} + \frac{1}{2}(1 - C_t(i,j)) \quad (3.18)$$

where $C_t(i,j)$ is the *surviving probability* that two RWers starting at i and j never meet up to t.

Since each RWer hops at rate μ, the relative position performs the RWs with the hopping rate 2μ. Hence the surviving probability is written in terms of the first passage probability defined in (2.12) as

$$C_t(i,j) = \int_t^\infty dt' F_{t'}(i|j) \; . \quad (3.19)$$

In (2.24), we solved the first passage probability for the RWer with the hopping rate 1. Using the solution and rescaling the time unit properly, we obtain that

$$C_t(i,j) = \int_t^\infty dt' \frac{l}{t'} e^{-2\mu t'} I_l(2\mu t') \quad (3.20)$$

with $l = |j - i|$. The modified Bessel function is approximated as

$$I_l(x) \simeq \frac{e^t}{\sqrt{2\pi x}} e^{-l^2/2x} \quad (3.21)$$

for large $x \gg l^2$. So, in the large t limit, the surviving probability becomes

$$\begin{aligned} C_t(i,j) &\simeq \int_t^\infty dt' \frac{l}{\sqrt{4\pi\mu t'^3}} e^{-l^2/4\mu t'} \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{l}{\sqrt{4\mu t}}} dx e^{-x^2} \\ &= \operatorname{erf}\left(\frac{l}{\sqrt{4\mu t}}\right) \simeq \begin{cases} 0 & \text{if } l \ll \sqrt{4\mu t} \\ 1 & \text{if } l \gg \sqrt{4\mu t} \end{cases} . \end{aligned} \tag{3.22}$$

We finally obtain that

$$\begin{aligned} \Phi_t(i,j) &\simeq \frac{1}{2} + \frac{1}{2}\operatorname{erfc}\left(\frac{l}{\sqrt{4\mu t}}\right) \\ &\simeq \begin{cases} 1 \text{ (ordering)} & \text{if } l \ll \sqrt{4\mu t} \\ \frac{1}{2} \text{ (random)} & \text{if } l \gg \sqrt{4\mu t} \end{cases} \end{aligned} \tag{3.23}$$

Spins within the distance $|i-j| \ll t^{1/2}$ are perfectly correlated while those separated by the distance $|i-j| \gg t^{1/2}$ remain uncorrelated. Therefore the size of the ordered domain grows algebraically as

$$\xi \sim t^{1/2} . \tag{3.24}$$

3.5 Domain wall density

A domain wall is present between sites i and $i+1$ if $s_i(t) \neq s_{i+1}(t)$. Thus, the mean density of domain wall is given by

$$\langle n_i \rangle(t) = (1 - \Phi_t(i,i+1)) = \frac{1}{2} C_t(i,i+1) = \rho(t) , \tag{3.25}$$

where we used $\Phi_t(i,i+1) = \frac{1}{2} + \frac{1}{2}(1 - C_t(i,i+1))$ from (3.18). Especially when $l = (i+1) - i = 1$, $C_t(i,i+1)$ has the closed-form expression

$$C_t(i+1,i) = \int_t^\infty dt' \frac{1}{t'} e^{-2\mu t'} I_1(2\mu t') = e^{-2\mu t}(I_0(2\mu t) + I_1(2\mu t)) \ . \tag{3.26}$$

Therefore the domain wall density is given by

$$\rho(t) = \frac{e^{-2\mu t}}{2} \left(I_0(2\mu t) + I_0(2\mu t) \right) \ . \tag{3.27}$$

For large t, the modified Bessel function is approximated as in (3.21). Therefore we obtain that

$$\rho(t) \simeq \frac{1}{\sqrt{4\pi\mu t}} (1 + \mathcal{O}(t^{-1})) \ . \tag{3.28}$$

The exact solution should be compared with the mean field result in (3.7). What makes the mean field theory wrong?

[Homework 3] The analytic results in the chapter was obtained in the infinite lattice with $L = \infty$. Perform the numerical Monte Carlo simulations to check if the result in (3.27) is still valid in finite lattices. We will need a finite size scaling analysis.

[Homework 4] Due to the translational invariance, the spin-spin correlation function is the function of the relative distance: $G_{i,j}(t) = g(l = i - j, t)$. The equations of motion in (3.14) becomes $\dot{g}(l) = \mu(g(l-1) - 2g(l) + g(l+1))$ with the boundary condition $g(l = 0, t) = 1$. When the spins are distributed randomly initially, $g(l, 0) = 1 - \delta_{l,0}$. Solve this equation to obtain the result in (3.27) [Gla63].

[Homework 5] Investigate the coarsening dynamics for the q-state voter model [DZ96].

4

Zero range process

Condensation occurs in various equilibrium and nonequilibrium systems. The Bose Einstein condensation is an example in equilibrium statistical physics, and the condensation in vibrating granular systems is another one in nonequilibrium statistical physics. In this chapter, we introduce an exactly solvable model for the condensation (see the movie).

4.1 Model

Consider a lattice of L sites. There are N particles on the lattice with the occupation number $n_i \in \{0, 1, \cdots\}$ satisfying

$$N = \sum_{i=1}^{L} n_i \ . \tag{4.1}$$

Particles jump from site to site. In general, jumping rate may depend on the occupation number at the departure site and the target site [SPI70]. The zero range process (ZRP) is defined as the stochastic process where the hopping rate depends only on the occupation number at the departure site. That is to say, a particle jumps from site i to j at the rate

$$\text{Rate}[\text{one particle jumps from } i \text{ to } j] = u(n_i) T_{ji} \ , \tag{4.2}$$

where $u(n)$ is the jumping rate function and the hopping matrix T_{ji} is the probability to select the target site j satisfying

$$\sum_j T_{ji} = 1. \tag{4.3}$$

It is crucial that the jumping rate is independent of the occupation number at the target site.

The rate function $u(n)$ is determined by the on-site (zero-ranged) interaction among particles. Noninteracting particles jumps independently, which yields that $u(n) \propto n$. Jumping rates increasing faster (slower) than the linear function correspond to repulsive (attractive) interactions among particles.

The hopping matrix T_{ji} defines a random walk problem. If one takes $T_{ji} = 1/L$, than a particle can jump to any sites with the equal probability (mean field). If one is interested in the

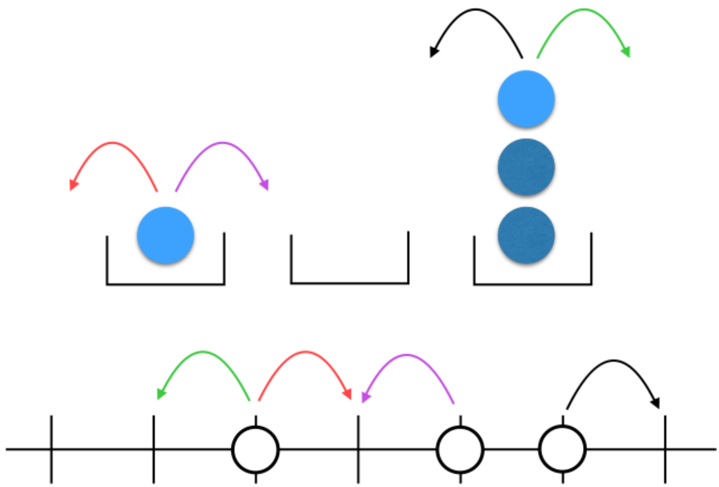

Figure 4.1: Mapping between the ZRP and asymmetric simple exclusion process

local hopping in a one-dimensional lattice under the periodic boundary condition, one can choose

$$T_{ji} = p\delta_{j,i+1} + q\delta_{j,i-1} \tag{4.4}$$

with $q = 1 - p$. We will see that the stationary state of the ZRP is solvable for any choice of T and $u(n)$.

4.2 ZRP and ASEP

Especially, in the one-dimensional lattice, the ZRP is mapped to another important stochastic process, the asymmetric simple exclusion process (ASEP). See Fig. 4.1 The connection between them is listed below.

ZRP	ASEP
L sites	L particles
occupation number	# of vacancies in the right-hand side.
N particles	N vacant sites

The hopping rate for the ASEP particles depends on the number of vacancies in the direction of the movement. This connection may be useful in modelling a road traffic. The particle density in the two models are related through

$$\rho_{ASEP} = \frac{L}{L+N} = \frac{1}{1+\rho_{ZRP}} . \tag{4.5}$$

4.3 Factorized stationary state

Let $P(\boldsymbol{n}, t)$ be the probability that the system is in a configuration $\boldsymbol{n} = (n_1, \cdots, n_L)$ at time t. The master equation is given by

$$\frac{d}{dt}P(\boldsymbol{n}) = \sum_{i,j} \left[-T_{ji}u(n_i)P(\boldsymbol{n}) + T_{ji}u(n_i+1)P(\boldsymbol{n}'_{ji}) \right] , \tag{4.6}$$

where we used the shorthand notation

$$\boldsymbol{n}'_{ji} = (n_1, \cdots, n_j - 1, \cdots, n_i + 1, \cdots, n_L) . \tag{4.7}$$

A transition from $\boldsymbol{n}'_{ji}$ to $\boldsymbol{n}$ means a single particle hopping from i to j.

We seek for the stationary state solution $P_{st}(\boldsymbol{n})$ that satisfies

$$\sum_{ij} T_{ij}u(n_j)P_{st}(\boldsymbol{n}) = \sum_{ij} T_{ji}u(n_i+1)P_{st}(\boldsymbol{n}'_{ji}) \tag{4.8}$$

It looks horrible to find the solution. But, the fact that the jumping rate depends only on the occupation number at the departure site may allow for the product solution [Eva00]

$$P_{st}(\boldsymbol{n}) = f_1(n_1)f_2(n_2)\cdots f_L(n_L) = \prod_{l=1}^{L} f_l(n_l) \tag{4.9}$$

with the functions $f_j(n)$ undetermined yet. Inserting the factorized state into the stationarity condition in (6.7), we obtain

$$\begin{aligned}\sum_{ij} T_{ij}u(n_j)\left[\prod_{l=1}^{L} f_l(n_l)\right] &= \sum_{ji} u(n_i+1)\left[\prod_{l=1}^{L} f_l(n_l)\right] \\ &\quad \times \frac{f_j(n_j-1)f_i(n_i+1)}{f_j(n_j)f_i(n_i)} \ .\end{aligned} \tag{4.10}$$

Canceling the common factor in $[\cdot]$ and using $\sum_i T_{ij} = 1$, we get

$$\sum_j u(n_j) = \sum_{ji} T_{ji}\left\{u(n_i+1)\frac{f_i(n_i+1)}{f_i(n_i)}\right\}\left(\frac{f_j(n_j-1)}{f_j(n_j)}\right) \ . \tag{4.11}$$

The equation should be valid for any occupation number distribution. This is possible when

$$\frac{f_i(n+1)}{f_i(n)} = \frac{\omega_i}{u(n+1)} \tag{4.12}$$

with the unknown constants $\{\omega_i\}$. It transforms (4.11) to

$$\sum_j u(n_j) = \sum_{ij} T_{ji}\omega_i \frac{u(n_j)}{\omega_j} \ . \tag{4.13}$$

It holds for any occupation numbers if ω_i's satisfy

$$\omega_j = \sum_i T_{ji}\omega_i \ . \tag{4.14}$$

This condition means that $\{\omega_i\}$ is given by the stationary state distribution of the corresponding random walk problem with the hopping matrix T_{ji}. In the mean field graph, or in the 1D lattice with/without bias, the solution is simply given by ω_i = const.. In general graphs, $\{\omega_i\}$ may have complex structure [NSL05].

From (4.12), we find that

$$f_i(n) = \begin{cases} 1 & \text{for } n = 0 \\ \frac{\omega_i}{u(0)} \cdots \frac{\omega_i}{u(n)} = \prod_{m=1}^{n} \left[\frac{\omega_i}{u(m)} \right] & \text{for } n \geq 1 \end{cases} \tag{4.15}$$

and

$$P_{st}(\boldsymbol{n}) = \frac{1}{Z(L,N)} \delta(N - \sum_j n_j) \prod_{i=1}^{L} f_i(n_i) \tag{4.16}$$

with the partition function

$$Z(L,N) = \sum_{\boldsymbol{n}} \delta(N - \sum_j n_j) \prod_{i=1}^{L} f_i(n_i) \; . \tag{4.17}$$

4.4 Grand canonical ensemble approach

From now on, we focus on the special but still general cases with $\omega_i = 1$. In these cases, $f_i(n) = f(n)$. Consider the occupation number distribution at a single site

$$p(n) \equiv \mathrm{Prob}[n_1 = n] \tag{4.18}$$

of the occupation number n_1 at site 1 in the stationary state. It is given by

$$\begin{aligned} p(n) &= \frac{1}{Z(L,N)} \sum_{n_2,\cdots,n_L} \delta(n + \sum_{j=2}^{L} n_j - N) f(n) \prod_{l=2}^{L} f(n_l) \\ &= f(n) \frac{Z(L-1, N-n)}{Z(L,N)} \end{aligned} \tag{4.19}$$

We will evaluate the partition function by using the formalism of equilibrium statistical mechanics. Introduce the effective free energy $F(L,N)$ defined by

$$Z(L,N) = e^{-F(L,N)} \ . \tag{4.20}$$

Then, we find

$$\begin{aligned} p(n) &= f(n) \exp[-F(L-1, N-n) + F(L,N)] \\ &\simeq f(n) \exp[p_0 - \mu n] \\ &= f(n) z^n / (\sum_{m=0}^{\infty} f(m) z^m) \end{aligned} \tag{4.21}$$

where $p_0 = -\partial F/\partial L$ is the pressure, $\mu = \partial F/\partial N$ is the chemical potential, and $z = e^{-\mu}$ is the fugacity. The pressure term just gives a normalization factor. The chemical potential μ or the fugacity z is fixed by the total particle density $\rho \equiv N/L$:

$$\rho = \sum_{n=0}^{L} n p(n) = z \frac{\mathcal{F}'(z)}{\mathcal{F}(z)} = z \frac{\partial}{\partial z} \ln \mathcal{F}(z) \tag{4.22}$$

with the generation function

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} f(n) z^n \ . \tag{4.23}$$

We have completed the grandcanonical formalism for the ZRP. The canonical formalism can be found in [EM08]. Here is the summary for the case with $\omega_i = 1$.

1. Given hopping matrix T_{ij}, find the stationary state solution ω_i for the corresponding random walk problem from (4.14).
2. Given hopping rate function $u(n)$, construct the function $f(n)$ defined in (4.15).
3. Find the generating function $\mathcal{F}(z)$ in (4.23).
4. Find the value of the fugacity z or the chemical potential μ using (4.22).
5. Finally one obtains the mass distribution function $p(n)$ from (4.21).

4.5 $u(n) = 1$ and $\omega_i = 1$ case

As I said before, $u(n) = n$ indicates that particles are noninteracting. The jumping rate $u(n) = 1$ means that there is an attraction among the particles at the same site. We will solve this problem by following the recipe step by step.

The function $f(n) = 1$, so the generating function is given by

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \, . \tag{4.24}$$

The density-fugacity relation is then given by

$$\rho = \frac{z}{1-z} \quad \longrightarrow \quad z = \frac{\rho}{1+\rho} \, . \tag{4.25}$$

For any value of ρ, we have the corresponding fugacity value z. Therefore, the particle number distribution is given by

$$p(n) = cf(n)z^n = c\left(\frac{\rho}{1+\rho}\right)^n = c\, e^{-n\ln\frac{1+\rho}{\rho}} \tag{4.26}$$

with the normalization constant $c = 1 - z = 1/(1+\rho)$.

4.6 Macroscopic condensation

Consider the special jumping rate function

$$u(n) = 1 + \frac{b}{n} = \frac{b+n}{n}\ . \tag{4.27}$$

We still assume that $\omega_i = 1$ (mean field, or 1D lattice with-/without bias). Then, we find that

$$f(n) = \frac{1}{b+1}\frac{2}{b+2}\cdots\frac{n}{b+n} = \frac{\Gamma(n+1)\Gamma(b+1)}{\Gamma(n+b+1)} \tag{4.28}$$

with the gamma function $\Gamma(x+1) = x!$. Its generating function is given by

$$\mathcal{F}(z) = \sum_{n=0}^{\infty}\frac{\Gamma(n+1)\Gamma(b+1)}{\Gamma(n+b+1)}z^n = {}_2F_1(1,1,b+1,z) \tag{4.29}$$

with the hypergeometric function ${}_2F_1$. This infinite series converges in the interval $|z| < 1$.

In order to obtain some intuition, we approximate $f(n)$ for large n.

$$f(n) = \prod_{m=1}^{n}\left(1+\frac{b}{m}\right) \sim n^{-b}\ . \tag{4.30}$$

Hence, the generating function is also approximated as

$$\mathcal{F}(z) = \sum_n n^{-b} z^n \ . \tag{4.31}$$

The approximate form provides all the relevant information. The generating function has the properties:

- $\mathcal{F}(z)$ is an increasing function of z in the interval $0 < z < 1$.
- $\lim_{z\to 1} \mathcal{F}(z) = \frac{b}{b-1}$ for $b > 1$ and ∞ for $b \le 1$.
- $\lim_{z\to 1} \mathcal{F}'(z)$ is finite for $b > 2$ and ∞ for $b \le 2$.
- $\lim_{z\to 1} \mathcal{F}'(z)/\mathcal{F}(z) = \frac{b}{(b-1)(b-2)}$ for $b > 2$ and ∞ for $b \le 2$.

Using these properties, we can draw the schematic plot of the function $z\mathcal{F}'(z)/\mathcal{F}(z)$ that appears in (4.22). See Fig. 4.2.

4.6.1 Weak attraction regime: $b \le 2$

In this regime, for any value of ρ, we can find the corresponding fugacity value $z = z(\rho) < 1$. Therefore, the occupation number distribution is given by the power-law truncated by the exponential decay:

$$p(n) \sim n^{-b} z(\rho)^n = n^{-b} e^{-n/(\ln(1/z(\rho)))^{-1}} \ . \tag{4.32}$$

4.6.2 Strong attraction regime: $b > 2$

We can find the fugacity value $z = z(\rho) < 1$ only when

$$\rho \le \rho_c = \frac{1}{b-2} \ . \tag{4.33}$$

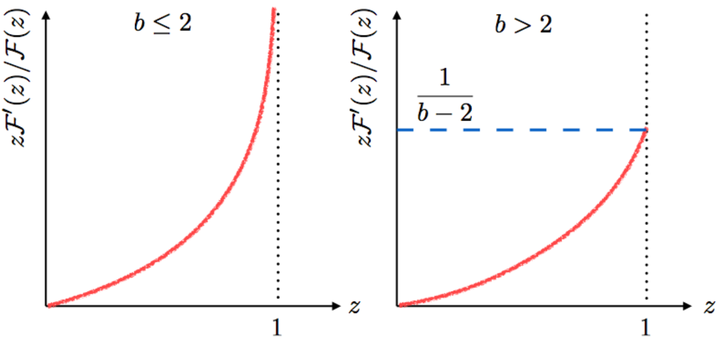

Figure 4.2: Schematic plots of the function $z\mathcal{F}'(z)/\mathcal{F}(z)$ when $b \le 2$ or $b > 2$. For $b > 2$, the function is bounded above by $1/b - 2$.

In this case, the occupation number distribution has the exponential tail.

When $\rho > \rho_c$, the fugacity value is given by $z = 1$. Therefore the occupation number distribution follows the power law

$$p(n) \sim n^{-b} . \tag{4.34}$$

The average occupation number that is accommodated by the power law is $\rho_c = 1/(b-2)$. Then, what happens for the excess particles of number $(\rho - \rho_c)L$? They are condensed into a single site to form a macroscopic condensate! Actually, the particle condensation in the ZRP has the similar mathematical structure as the Bose Einstein condensation. The phase diagram is given in Fig. 4.3.

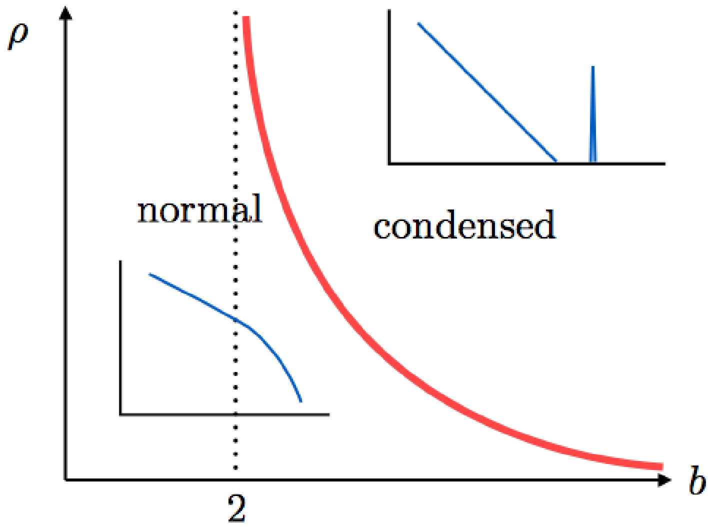

Figure 4.3: Phase diagram of the ZRP.

4.7 Concluding remarks

There are many generalized studies of the ZRP. One may consider the target process in which the jumping rate depends only on the occupation number at the target site [LG07]. The target process may be dual to the ZRP or not depending on the detailed balance. More generally, the jumping rate may depend on the occupation numbers at the departure and the target sites simultaneously [EH05]. Still there exist some solvable cases. The ZRP on complex networks display interesting condensation phenomena [NSL05]. Dynamical aspects of the condensation are also interesting.

[Homework 6] As a simple exercise, calculate the occupation number distribution $p(n)$ when the jumping rate is given by $u(n) = n$. The particles are on a 1D lattice with the hopping matrix $T_{ji} = (\delta(j, i+1) + \delta(j, i-1))/2$. Following the recipe, you will find the solution. Can you interpreted your calculation result in a simple way?

[Homework 7] Compare the two jumping rates $u(n) = (1 + 1/n^2)/2$ and $u(n) = (1 + 1/\sqrt{n})/2$ with $\omega_i = 1$. Which one displays the macroscopic condensation? You may perform Monte Carlo simulations for these examples.

5 ASEP under the periodic boundary condition

The asymmetric simple exclusion process (ASEP) is one of the most important model systmes in nonequilibrium statistical mechanics. The ASEP was first introduced as a model for protein synthesis by ribosomes sliding along an mRNA chain [MGP68]. Since then, the ASEP has been adopted for various noneqilibrum phenomena. Its role is analogous to the "Ising model" for equilibrium statistical physics. In this chapter, we study the ASEP in 1D lattice under the periodic condition (PBC). The key mathematical tool for this subject is the **Bethe ansatz**.

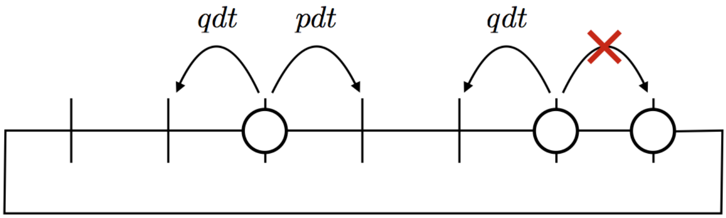

Figure 5.1: ASEP dynamics under the PBC.

5.1 Model

Each particle hops to the right with rate p and to the left with rate q. Choose the time unit in such a way that $p + q = 1$. Particles are subject to the exclusion interaction so that the hopping to an occupied site is forbidden.

The model with $p = q = 1/2$ is called the symmetric simple exclusion process (SSEP). The dynamics obeys the detailed balance, so the system reaches the thermal equilibrium. On the other hand, when $p \neq q$, the system reaches the nonequilibrium steady state. In particular, the model with $q = 0$ is called the totally asymmetric simple exclusion process (TASEP). Figure 5.1 explains the dynamic rule of the ASEP.

A configuration is specified by $\boldsymbol{n} = (n_1, \cdots, n_L)$ where $n_i = 1$ (0) if site i is occupied (empty). The PBC means that $n_{i+L} = n_i$, and the total number of particles

$$N = \sum_i n_i \tag{5.1}$$

is conserved.

The ASEP may be regarded as a model for transport. The particle current is given by

$$\begin{aligned} J &= \frac{1}{L}\sum_{i=1}^{L}\{p\langle n_i(1-n_{i+1})\rangle - q\langle(1-n_i)n_{i+1}\rangle\} \\ &= (p-q)\overline{\langle n_i(1-n_{i+1})\rangle}\ , \end{aligned} \tag{5.2}$$

where $\langle\cdot\rangle$ means the ensemble average and $\bar{\cdot}$ means the spatial average.

5.2 BCSOS model

The ASEP under the PBC can be mapped to the single-step model or the BCSOS model for fluctuating interfaces (see Fig. 5.2). We introduce a step variable $\tau_i \equiv 1-2n_i = \pm 1$ and a height variable $h_{i+1/2} - h_{i-1/2} = \tau_i$. The hopping of particles corresponds to the deposition (p) or the evaporation (q) of rhombic bricks.

The height variables are subject to the BCSOS constraint

$$|h_{i+1/2} - h_{i-1/2}| = 1 \tag{5.3}$$

and the boundary condition

$$h_{L+1/2} = h_{1/2} + \sum_{i=1}^{L}\tau_i = h_{1/2} + \sum_i (1-2n_i) = h_{1/2} + (L-2N) \tag{5.4}$$

The interface also satisfies the PBC at half filling ($N = L/2$). For other filling fractions ($N \neq L/2$), the interface is tilted on average and satisfies the helical boundary condition.

The average height is given by

$$\langle\bar{h}\rangle = \left\langle \frac{1}{L}\sum_{i=1}^{L}\left(h_{1/2} + \sum_{j=1}^{i}\tau_i\right)\right\rangle = \langle h_{1/2}\rangle + \frac{1}{L}\sum_{i=1}^{L} i\langle\tau_i\rangle\ . \tag{5.5}$$

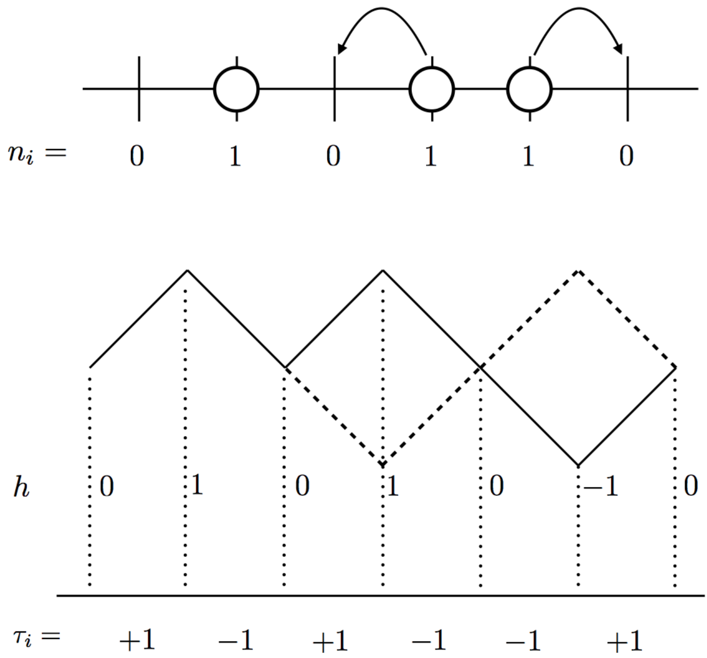

Figure 5.2: Mapping of the ASEP

The interface grows or evaporates with the average velocity

$$v = 2J = \frac{p-q}{2}\left(1 - \overline{\langle \tau_i \tau_{i+1} \rangle}\right) \tag{5.6}$$

The fluctuation of the interface is measured by the width defined as

$$W^2(L,t) \equiv \left\langle \overline{(h-\bar{h})^2} \right\rangle = \left\langle \overline{h^2} - \bar{h}^2 \right\rangle \; . \tag{5.7}$$

It can be written in terms of the step variable as

$$W^2 = \frac{L^2-1}{6L} + \frac{2}{L}\sum_{i>j}(L-i+1-ij)\langle \tau_i \tau_j \rangle \; . \tag{5.8}$$

Physical properties of the interface can be calculated from the correlation functions $\langle \tau_i \tau_j \rangle$.

The width of the fluctuating interface is known to follow the dynamic scaling law

$$W^2(L,t) = L^{2\alpha} F(t/L^z) = \begin{cases} t^\beta & \text{for } t \ll L^z \\ L^\alpha & \text{for } t \gg L^z \end{cases} \tag{5.9}$$

with the roughness exponent α, the growth exponent $\beta = \alpha/z$, and the dynamic exponent z. These exponents will be obtained from the exact solution of the ASEP.

5.3 Quantum spin chain

We want to write down the master equation for the ASEP with the step variable $\boldsymbol{\tau}(\tau_1, \cdots, \tau_N)$. Introduce a notation

$$\boldsymbol{\tau}'_{i,i+1} = (\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \tau_i, \tau_{i+2}, \cdots, \tau_L) \tag{5.10}$$

for a configuration with ith and $i+1$th step variables being exchanged. Then, the master equation for the probability distribution $P(\boldsymbol{\tau},t)$ is given by

$$\begin{aligned}\dot{P}(\boldsymbol{\tau},t) &= p\sum_i\left\{-\frac{(1-\tau_i)(1+\tau_{i+1})}{4}P(\boldsymbol{\tau})+\frac{(1-\tau_{i+1})(1+\tau_i)}{4}P(\boldsymbol{\tau}'_{i,i+1})\right\}\\ &+ q\sum_i\left\{-\frac{(1+\tau_i)(1-\tau_{i+1})}{4}P(\boldsymbol{\tau})+\frac{(1+\tau_{i+1})(1-\tau_i)}{4}P(\boldsymbol{\tau}'_{i,i+1})\right\}.\end{aligned}$$

The master equation is written in a compact form by introducing the quantum mechanics notation. Let $\sigma_i^{x,y,z}$ be the Pauli matrices for the *lattice fermion* at site i acting on the 2^L-dimensional Hilbert space spanned by 2^N states

$$\{|\pm,\cdots,\pm\rangle\}=\{|\pm\rangle\otimes\cdots\otimes|\pm\rangle\}\,. \tag{5.11}$$

Regard the step variable τ_i as the z component of the spin at site i. We introduce a state vector

$$|\Psi(t)\rangle=\sum_{\boldsymbol{\tau}}P(\boldsymbol{\tau},t)|\boldsymbol{\tau}\rangle \tag{5.12}$$

Then, the master equation for $P(\boldsymbol{\tau},t)$ becomes the imaginary-time Schrödinger equation

$$\frac{d}{dt}|\Psi\rangle=-H|\Psi\rangle \tag{5.13}$$

with the Hamiltonian operator

$$H=\sum_i\left\{-p\sigma_i^+\sigma_{i+1}^- - q\sigma_i^-\sigma_{i+1}^+ + \frac{(p+q)}{4}(1-\sigma_i^z\sigma_{i+1}^z)\right\} \tag{5.14}$$

with the raising/lowering operator $\sigma^\pm$ (see 8.2).

The Hamiltonian for the SSEP ($p = q = 1/2$) is given by

$$H_{SSEP} = -\frac{1}{4}\sum_i (2\sigma_i^+\sigma_{i-1}^- + 2\sigma_i^-\sigma_{i+1}^+ + \sigma_i^z\sigma_{i+1}^z - 1) \tag{5.15}$$

$$= -\frac{1}{4}\sum_i (\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z - 1) \tag{5.16}$$

It is also called the XXX spin chain Hamiltonian, which is Hermitian.

For the TASEP ($p = 1$ and $q = 0$), the Hamiltonian becomes

$$H_{TASEP} = -\frac{1}{4}\sum_i (4\sigma_i^+\sigma_{i+1}^- + \sigma_i^z\sigma_{i+1}^z - 1)\ . \tag{5.17}$$

It is non-Hermitian.

The time-independent stationary state satisfies

$$H|\Psi_{st}\rangle = 0\ . \tag{5.18}$$

That is to say, the stationary state is the right eigenstate of the Hamiltonian with eigenvalue 0. If the eigenstate $|\Psi_{st}\rangle$ is found, then the steady state probability distribution is given by

$$P_{st}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}|\Psi_{st}\rangle. \tag{5.19}$$

The time-dependent solution is given by

$$|\Psi(t)\rangle = e^{-Ht}|\Psi(0)\rangle\ . \tag{5.20}$$

Let $|\Psi_l\rangle$ and E_l be the right eigenstate of H and the corresponding eigenvalue, respectively. Since the Hamiltonian is non-Hermitian, the eigenvalues may be complex in general. We label the eigenstates in such a way that eigenstates of H with the eigenvalues

$$E_1 = 0 < \Re[E_2] \leq \Re[E_3] \leq \cdots \leq \Re[E_{2^L}]\ . \tag{5.21}$$

Among them, the first eigenstate corresponds to the stationary state

$$|\Psi_{st}\rangle = |\Psi_1\rangle \text{ and } E_1 = 0. \tag{5.22}$$

One can decompose the initial state in terms of the eigenstates as

$$|\Psi(0)\rangle = \sum_{l=1}^{2^L} a_l |\Psi_l\rangle \ . \tag{5.23}$$

Then, in the long time limit, the state vector is approximated as

$$|\Psi(t)\rangle \simeq |\Psi_{st}\rangle + a_2 e^{-E_2 t}|\Psi_2\rangle + \cdots \ . \tag{5.24}$$

This spectral decomposition tells us that the relaxation time to reach the stationary state is determined by the energy gap $\Delta E = E_2 - E_1$ of the lowest excited state. The relation time is given by

$$t_{relax} = 1/\Re[\Delta E] \ . \tag{5.25}$$

5.4 Stationary state of the ASEP

For the ASEP, $|\Psi_{st}\rangle$, the right eigenstate of H with eigenvalue 0, is found from the symmetry property of the Hamiltonian. First of all, we note that the corresponding left eigenstate is given by the so-called disorder state

$$\langle D| \equiv \sum_{\boldsymbol{\tau}} \langle \boldsymbol{\tau}|. \tag{5.26}$$

Take the inner product of the disorder state with the Schrödinger equation (5.13). It yields that

$$\frac{d}{dt}\langle D|\Psi(t)\rangle = -\langle D|H|\Psi(t)\rangle \ . \tag{5.27}$$

Note that $\langle D|\Psi(t)\rangle = \sum_{\boldsymbol{\tau}} P(\boldsymbol{\tau}, t) = 1$ is independent of t. Hence, we conclude that $\langle D|$ is the left eigenstate of H with eigenvalue 0. Actually, $\langle D \ H = 0$ is the requirement for the Hamiltonian for any stochastic system **[stochasticity condition]**.

Our task is to find the *right* eigenstate of $H(p,q)$. From (5.14), $H(p,q) = H(q,p)^\dagger$. This property yields that

$$H(p,q)|D\rangle = (\langle D|H(p,q)^\dagger)^\dagger = (\langle D|H(q,p))^\dagger = 0 \qquad (5.28)$$

Therefore, we conclude that

$$|\Psi_{st}\rangle \propto |D\rangle = \sum_{\boldsymbol{\tau}} |\boldsymbol{\tau}\rangle \text{ and } P_{st}(\boldsymbol{\tau}) = \text{ const. .} \qquad (5.29)$$

Every microstate is equally-likely. Therefore the stationary state is fully random and uncorrelated.

Since any spatial correlations are absent in the stationary state, we have

$$\begin{aligned} \langle n_i n_j \rangle_{st} &= \langle n_i \rangle_{st} \langle n_j \rangle_{st} = \rho^2 \\ \langle \tau_i \tau_j \rangle_{st} &= \langle \tau_i \rangle_{st} \langle \tau_j \rangle_{st} = (1-2\rho)^2 \end{aligned}$$

for $i \neq j$.

The particle current in (5.2) is given by

$$J_{st} = (p-q)\rho(1-\rho) \ . \qquad (5.30)$$

The interface velocity in (5.6) is given by $v_{st} = 2J_{st}$.

Recall that the interface width is written in terms of the correlation functions in (5.8). For the untilted case ($\langle \tau_i \rangle = 1 - 2\langle n_i \rangle = 0$), the stationary state width is given by

$$W^2(L, t \to \infty) = \frac{L}{6} - \frac{1}{6L} \sim L^1 \ . \qquad (5.31)$$

Therefore the roughness exponent is given by

$$\alpha = 1/2 \qquad (5.32)$$

for all values of p and q.

5.5 Bethe ansatz solution for the TASEP

In order to study the long-time dynamic behaviors of the ASEP, we need to find the lowest excited states of the Hamiltonian. The Hamiltonian of the ASEP is exactly diagonalized via the Bethe ansatz [GS92b, Der98]. I would like to introduce the basic idea very briefly focusing on the TASEP ($p = 1$ and $q = 0$) with Hamiltonian

$$H_{TASEP} = \sum_i \left(-\sigma_i^+ \sigma_{i+1}^- + \frac{1 - \sigma_i^z \sigma_{i+1}^z}{4} \right) . \qquad (5.33)$$

The general ASEP case is mentioned later.

It is convenienet to change the basis from $\{|\boldsymbol{\tau}\rangle\}$ to $\{|x_1 x_2 \cdots x_N\rangle\}$ where $1 \leq x_1 < x_2 \cdots < x_N \leq L$ denotes the position of ith spins in the state $\tau = -1$ or equivalently the position of the ith particle. The number of particles N is conserved under the PBC. So the Hamiltonian is diagonalized in each sector with fixed $N = 1, 2, \cdots$ separately.

5.5.1 $N = 1$ sector

The eigenstate is written as $|\Psi\rangle = \sum_{x=1}^{L} \psi(x)|x\rangle$. When the Hamiltonian acts on $|x\rangle$, it yields

$$H|x\rangle = -|x + 1\rangle + |x\rangle . \qquad (5.34)$$

So, the eigenvalue equation $H|\Psi\rangle = E|\Psi\rangle$ becomes

$$E\psi(x) = \{-\psi(x - 1) + \psi(x)\} \qquad (5.35)$$

Since the system has the translational invariance, we try a plane wave solution

$$\psi(x) = z^x = e^{ikx} \qquad (5.36)$$

It solves the eigenvalue equation yielding the eigenvalue

$$E = 1 - 1/z \ . \tag{5.37}$$

The PBC $\psi(x+L) = \psi(x)$ imposes a constraint on z:

$$z^L = 1 \ . \tag{5.38}$$

There are L possible values of $z_l = e^{ik_l}$ with

$$k_l = \frac{2\pi l}{L} \text{ with } l = -L/2, \cdots, L/2 - 1 \tag{5.39}$$

Corresponding eigenvalues are

$$E_l = 1 - e^{-ik_l} \text{ with } \Re[E_l] = 1 - \cos k_l. \tag{5.40}$$

5.5.2 $N = 2$ sector

Write the eigenstate as

$$|\Psi\rangle = \sum_{1 \le x_1 < x_2 \le L} \psi(x_1, x_2) |x_1 x_2\rangle \ . \tag{5.41}$$

When the Hamiltonian acts on $|x_1 x_2\rangle$, one obtains that

$$H|x_1 x_2\rangle = -|(x_1+1)x_2\rangle - |x_1(x_2+1)\rangle + 2|x_1 x_2\rangle \tag{5.42}$$

for $x_2 \neq x_1 + 1$ and

$$H|x_1 x_2\rangle = -|x_1(x_2+1)\rangle + 1|x_1 x_2\rangle \tag{5.43}$$

for $x_2 = x_1 + 1$. Note that state vectors transform in different ways depending on whether $x_2 = x_1 + 1$ or not because of the exclusion interaction.

The eigenvalue equation become

$$\begin{aligned} E\psi(x_1, x_2) &= \{\psi(x_1, x_2) - \psi(x_1 - 1, x_2)\} \\ &+ \{\psi(x_1, x_2) - \psi(x_1, x_2 - 1)\} \end{aligned} \quad (5.44)$$

for $x_2 \neq x_1 + 1$ and

$$E\psi(x_1, x_2) = \psi(x_1, x_2) - \psi(x_1 - 1, x_2) \quad (5.45)$$

for $x_2 = x_1 + 1$.

We try again with the trial wave function

$$\psi(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2} \quad (5.46)$$

consisting of two plane waves. It solves (5.44) with the eigenvalue

$$E = \sum_{i=1}^{2} (1 - 1/z_i) \quad (5.47)$$

for arbitrary values of z's and A's. Do (5.46) and (5.47) also solve (5.45)? The answer is YES if (5.46) satisfies $\psi(x_1, x_1 + 1) = \psi(x_1, x_1)$, which yields the condition for A's

$$\frac{A_{21}}{A_{12}} = -\frac{z_2 - 1}{z_1 - 1} \equiv B(z_2, z_1) \ . \quad (5.48)$$

The PBC $\psi(x_1, x_2) = \psi(x_2, x_1 + L)$ imposes the additional constraint on z_1 and z_2:

$$A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_2^{x_2} = A_{12} z_1^{x_2} z_2^{x_1 + L} + A_{21} z_2^{x_2} z_2^{x_1 + L} \ . \quad (5.49)$$

Because this equation should hold for any x's, we obtain the so-called Bethe ansatz equations (BAE)

$$z_1^L = \frac{A_{12}}{A_{21}} = B(z_1, z_2), \quad z_2^L = \frac{A_{21}}{A_{12}} = B(z_2, z_1) \ . \quad (5.50)$$

The BAE is supposed to have ${}_LC_2 = L(L-1)/2$ independent solutions for $\{z_1, z_2\}$. One obvious solution is $(z_1 \to 1, z_2 \to 1)$ which corresponds to the stationary state solution $\psi(x_1, x_2) =$ const. with $E = 0$. For all the other solutions, $z_1 \neq z_2$.

5.5.3 $N = 3$ sector

Write the eigenstate as

$$|\Psi\rangle = \sum_{x_1<x_2<x_3} \psi(x_1, x_2, x_3)|x_1x_2x_3\rangle \tag{5.51}$$

The eigenvalue equation becomes

$$\begin{aligned}
E\psi(x_1, x_2, x_3) &= -\psi(x_1-1, x_2, x_3) - \psi(x_1, x_2-1, x_3) - \psi(x_1, x_2, x_3-1) \\
&\quad +3\psi(x_1, x_2, x_3) \qquad \text{if } x_3 > x_2+1, x_2 > x_1+1 && (5.52) \\
&= -\psi(x_1-1, x_2, x_3) - \psi(x_1, x_2, x_3-1) + 2\psi(x_1, x_2, x_3) \\
&\qquad \text{if } x_3 > x_2+1, x_2 = x_1+1 && (5.53) \\
&= -\psi(x_1-1, x_2, x_3) - \psi(x_1, x_2-1, x_3) + 2\psi(x_1, x_2, x_3) \\
&\qquad \text{if } x_3 = x_2+1, x_2 > x_1+1 && (5.54) \\
&= -\psi(x_1-1, x_2, x_3) + \psi(x_1, x_2, x_3) \\
&\qquad \text{if } x_3 = x_2+1, x_2 = x_1+1. && (5.55)
\end{aligned}$$

The Bethe ansatz claims that the wave function is given by

$$\psi(x_1, x_2, x_3) = A_{123}z_1^{x_1}z_2^{x_2}z_3^{x_3} + A_{213}z_2^{x_1}z_1^{x_2}z_3^{x_3} + \cdots A_{321}z_3^{x_1}z_2^{x_2}z_1^{x_3},$$

where the sum is over all six permutation of z_1, z_2, and z_3. It indeed satisfies (5.52) leading the to eigenvalue

$$E = \sum_{i=1}^{3}(1 - 1/z_k) \tag{5.56}$$

The same wavefunction also satisfies (5.53) and (5.54) if we require that $\psi(x_1, x_1, x_3) = \psi(x_1, x_1+1, x_3)$ and $\psi(x_1, x_2, x_2) = \psi(x_1, x_2, x_2+1)$. They yield the constraint for the amplitudes:

$$A_{213} = B(z_2, z_1) A_{123} \quad , \quad A_{132} = B(z_3, z_2) A_{123} \tag{5.57}$$

$$A_{231} = B(z_3, z_1) A_{213} \quad , \quad A_{312} = B(z_3, z_1) A_{132} \tag{5.58}$$

$$A_{321} = B(z_3, z_2) A_{231} \tag{5.59}$$

Finally, the PBC $\psi(x_1, x_2, x_3 + L) = \psi(x_3, x_1, x_2)$ yields the BAE

$$z_1^N = B(z_1, z_2) B(z_1, z_3) \tag{5.60}$$

$$z_2^N = B(z_2, z_1) B(z_2, z_2) \tag{5.61}$$

$$z_3^N = B(z_3, z_1) B(z_3, z_2) \; . \tag{5.62}$$

We expect that there are ${}_L C_3$ independent solutions.

5.6 Bethe ansatz for the general ASEP

We are ready to generalize the Bethe ansatz to the general ASEP in an arbitrary N sector. For convenience, we assume that $p + q = 1$.

The Bethe ansatz states that the ASEP Hamiltonian in (5.14) in the N particle sector has the eigenvalue

$$E = \sum_{j=1}^{N} \left(1 - p/z_j - q z_j\right) \tag{5.63}$$

where $\{z_j\}$ satisfies the BAE

$$z_j^L = \prod_{l \neq j} \left[- \frac{z_j - q z_j z_l - p}{z_l - q z_j z_l - p} \right] \; . \tag{5.64}$$

The stationary state solution is given by $z_1 = z_2 \cdots = z_N = 1$ with $E = 0$.

It seems to be terribly hard to solve the nonlinear coupled BAE for excited states. However, some useful techniques have been developed to find the low-lying excited states governing the slowest relaxation processes. We mention the results of those studies without derivations.

For the TASEP model at half-filling [GS92b, GS92a], showed that

$$\Re[\Delta E] \sim \frac{1}{L^{3/2}} . \tag{5.65}$$

This means that the relaxation time for the growing interfaces diverges with the system size as (see (5.25))

$$t_{relax} \sim L^{3/2} . \tag{5.66}$$

This result explicitly shows that the growing interfaces belongs to the KPZ universality class with $z = 3/2$.

The energy gap for the general ASEP at any filling fraction was calculated by Prof. Doochul Kim [Kim95]. The KPZ scaling holds for any finite filling fraction and for any nonzero growing velocity ($p \neq q$). For the SSEP, the energy gap scales as

$$\Re[\Delta E] \sim \frac{1}{L^{2}} . \tag{5.67}$$

This means that the relaxation time for equilibrium interfaces scales as

$$t_{relax} \sim L^{2} . \tag{5.68}$$

That is to say, the equilibrium interfaces belong to the EW universality class with $z = 2$. The crossover scaling from the EW to the KPZ classes is also studied.

The Bethe ansatz method is also applied to find the fluctuations in the particle current for the TASEP [Der98] and the ASEP [LK99].

[Homework 8] Find the eigenvalue spectrum of the SSEP ($p = q = 1/2$) and the TASEP ($p = 0$ and $q = 1$) by solving the BAE in (5.64) for systems with $L = 4$. You may use a numerical tool such as a Mathematica or write your own program to find the solution.

[Homework 9] The XXZ model with the Hamiltonian

$$\begin{aligned} H_{XXZ} &= -\sum_{i=1}^{L}[2\sigma_i^+\sigma_{i+1}^- + 2\sigma_i^-\sigma_{i+1}^+ + \Delta\sigma_i^z\sigma_{i+1}^z] \\ &= -\sum_{i}[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z] \end{aligned}$$

is solvable exactly via the Bethe ansatz. Especially, $\Delta = 0$ case is called the *free fermion model*. Try to drive the BAE for the free fermion model following the similar procedure.

6
TASEP in the open boundary condition

In this chapter, we will learn the matrix product ansatz that can solve the stationary state of stochastic systems. Especially, we focus on the TASEP under the open boundary condition [DEHP93].

6.1 Model

In the open boundary condition (OBC), particles enter the lattice at site $i = 1$ (entrance) at rate α, and leave the lattice at site $i = L$ (exit) at rate β. Figure 6.1 explains the dynamic rule of the ASEP under the OBC. We focus on the TASEP limit with $p = 1$ and $q = 0$.

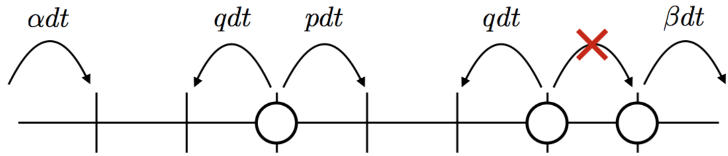

Figure 6.1: ASEP dynamics under the periodic boundary condition.

Consider how the occupation number $\{n_i\}$ changes in time. Inside the bulk ($1 < i < L$), the occupation number n_i increases when a particle jumps in from site $(i-1)$ and decreases when a particle jumps out to site $(i+1)$. So the mean value changes as

$$\frac{d}{dt}\langle n_i \rangle = \langle n_{i-1}(1-n_i)\rangle - \langle n_i(1-n_{i+1})\rangle \ . \tag{6.1}$$

At the entrance and the exit, we have the similar equations

$$\frac{d}{dt}\langle n_1 \rangle = \alpha\langle(1-n_1) - \langle n_1(1-n_2)\rangle \tag{6.2}$$

$$\frac{d}{dt}\langle n_L \rangle = -\beta\langle n_L\rangle + \langle n_{L-1}(1-n_L)\rangle \ . \tag{6.3}$$

Each term in the equations of motion corresponds to the particle current J:

$$J_{in} = \alpha\langle(1-n_1)\rangle \tag{6.4}$$

$$J_{i,i+1} = \langle n_i(1-n_{i+1})\rangle \quad \text{for } 1 \le i < L \tag{6.5}$$

$$J_{out} = \beta\langle n_L\rangle \tag{6.6}$$

In the stationary state, all the local current should be the same

$$J = J_{in} = J_{i,i+1} = J_{out}. \tag{6.7}$$

The main issue is to find the mean particle density distribution $\rho_i = \langle n_i \rangle$ and the current J in the stationary state in the parameter space (α, β).

6.2 Mean field theory

In the previous chapter, we found that the ASEP in the PBC has no spatial correlation in the stationary state. Under the OBC, the system is perturbed only at the boundary, so the mean field theory may be useful [DDM92].

Ignoring the spatial correlations, we approximate the current as

$$J_{i,i+1} = \langle n_i(1 - n_{i+1}) \rangle \simeq \langle n_i \rangle \times \langle (1 - n_{i+1}) \rangle = \rho_i(1 - \rho_{i+1}) \,. \tag{6.8}$$

In the stationary state, $J_{i,i+1}$ is equal to a constant J, which yields the recursion relation or the map

$$\rho_{i+1} = 1 - \frac{J}{\rho_i} \equiv h_J(\rho_i) \,. \tag{6.9}$$

The map $h_J(x)$ is depicted in Fig. 6.2.

When $J < 1/4$, there are two fixed points at

$$\rho_\pm = \frac{1}{2}(1 \pm \sqrt{1 - 4J}) \,, \tag{6.10}$$

which are the roots of

$$\rho = h_J(\rho) \;\text{ or }\; \rho^2 - \rho + J = 0 \,. \tag{6.11}$$

The fixed point ρ_- is attractive and ρ_+ is repulsive in the forward direction. The map does not have any fixed point when $J > 1/4$. Hence such a case is not realized.

The mean field solution is obtained in the following way:

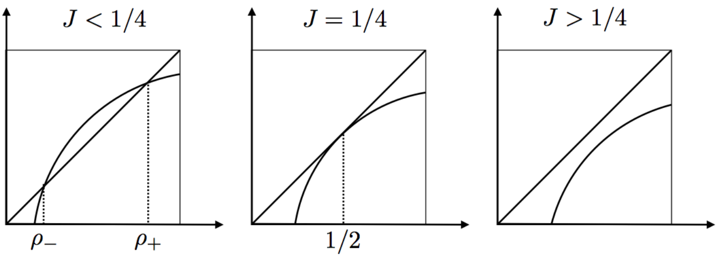

Figure 6.2: Plot of the map function $h_J(x) = 1 - J/x$

1. Assume a current J.
2. Find the starting value for the map: $\rho_1 = 1 - J/\alpha$ or $\rho_L = J/\beta$.
3. Iterate the map: $\rho_1 \to \cdots \to \rho_L(J)$ or $\rho_L \to \rho_1(J, \rho_L)$.
4. Determine J self-consistently using that $J = \beta\rho_L(J)$ or $J = \alpha(1 - \rho_1(J))$

6.2.1 Low density (LD) phase

Start the iteration from $\rho_L = J/\beta < \rho_+$ in the reverse direction. The iteration converges exponentially fast to $\rho_1 = \rho_- < 1/2$. The self-consistent equation for J becomes

$$J = \alpha(1 - \rho_-) = \frac{\alpha}{2}(1 + \sqrt{1 - 4J}) \tag{6.12}$$

The solution is given by

$$J = \alpha(1 - \alpha) \tag{6.13}$$

$$\rho_1 = \alpha \tag{6.14}$$

$$\rho_L = \alpha(1 - \alpha)/\beta \tag{6.15}$$

This solution is realized in the region

$$\alpha < 1/2 \text{ and } \beta > \alpha \,, \tag{6.16}$$

which comes from the conditions $\rho_L < \rho_+$ and $\rho_1 < 1/2$.

In this phase, the particle density remains constant to the low value $\rho_i \simeq \alpha$ except for the narrow region near the exit. Because the input flux is low enough and the output rate is larger than the input rate, the particle density in the bulk is governed by the input rate. This phase is called the LD phase.

6.2.2 High density (HD) phase

Start the iteration from $\rho_1 = 1 - J/\alpha > \rho_-$ in the forward direction. The iteration converges exponentially fast to $\rho_L = \rho_+ > 1/2$, and the self-consistent equation for J becomes

$$J = \beta\rho_+ = \frac{\beta}{2}(1 + \sqrt{1 - 4J}) \,. \tag{6.17}$$

It yields the solution

$$\begin{aligned} J &= \beta(1-\beta) && (6.18)\\ \rho_1 &= 1 - \beta(1-\beta)/\alpha && (6.19)\\ \rho_L &= 1 - \beta \,. && (6.20) \end{aligned}$$

This solution is realized in the region

$$\beta < 1/2 \text{ and } \alpha > \beta \,. \tag{6.21}$$

In this phase, the particle density remains constant to the higher value $\rho_i \simeq 1 - \beta$ except for the narrow region near the entrance. Because the output rate is small, the particles density in the bulk is governed by the output rate. This phase is called the HD phase.

6.2.3 Maximal current (MC) phase

Start the iteration from $\rho_1 = 1 - J/\alpha > \rho_-$ in the forward direction and $\rho_L = J/\beta < \rho_+$ in the reverse direction. The two solutions can match in the bulk only when there is a single fixed point at $\rho = 1/2$. Therefore, we find that

$$J = 1/4 \tag{6.22}$$

$$\rho_1 = 1 - 1/4\alpha \tag{6.23}$$

$$\rho_L = 1/4\beta \,. \tag{6.24}$$

This solution is realized in the region

$$\alpha \geq 1/2 \text{ and } \beta \geq 1/2 \tag{6.25}$$

In this phase, the bulk particle density $\rho_i = 1/2$ is independent of the input and output rates. The current is maximum among all phases.

6.2.4 Coexistence line

As an another possibility, one start the iteration from a site $1 \ll i \ll L$ in the bulk with $\rho_- < \rho_i < \rho+$. Then, the forward iteration converges to $\rho_L = \rho_+$ and the reverse iteration converges to $\rho_1 = \rho_-$. Then, the self-consistent equation becomes

$$J = \alpha(1 - \rho_-) = \beta\rho_+ \tag{6.26}$$

The solution is given by

$$J = \alpha(1 - \alpha) = \beta(1 - \beta) \tag{6.27}$$

$$\rho_1 = \alpha = \beta = \rho_L \,. \tag{6.28}$$

This phase is realized when

$$\alpha = \beta < 1/2 \,. \tag{6.29}$$

Along the coexistence line, the particle density is $\rho_i \simeq \alpha$ in the entrance side and $\rho_i \simeq \beta$ in the exit side. These two regions are separated by a shock.

The mean field result is summarized by the phase diagram shown in Fig. 6.3.

6.3 Matrix product ansatz

In this section, we will derive the exact stationary state for the TASEP using the so-called matrix product ansatz (MPA) [DEHP93]. I will explain the idea of the MPA using the TASEP with $L = 2$. Generalizations to the ASEP and arbitrary L is straightforward.

For a special case (as in the ZRP), the stationary state is given by the factorized product state $P(\boldsymbol{n}) = \delta(\sum_i n_i = N) \prod_{i=1}^{L} f_i(n_i)$. The product state is working for the ASEP under the PBC. However, such a product state does not satisfy the master equation under the OBC in general for any choices of the *functions* f_i. What happens if one use some matrices instead of functions in place of f_i?

For $L = 2$, there are 4 possible states. The MPA assumes that the stationary state distribution is given by

$$P_{st}(n_1, n_2) = \frac{1}{Z} \left\langle W| \prod_{i=1}^{2} \left[E\delta_{n_i,0} + D\delta_{n_i,1} \right] |V \right\rangle \tag{6.30}$$

where Z is the normalizing partition function, D and E are some matrices corresponding to occupied and vacant sites, respectively, and $\langle W|$ and $|V\rangle$ are some vectors. These elements are undetermined yet. For examples, $P_{st}(10) \propto \langle W|DE|V\rangle$, $P_{st}(01) \propto \langle W|ED|V\rangle$, and so on. The probability distribution

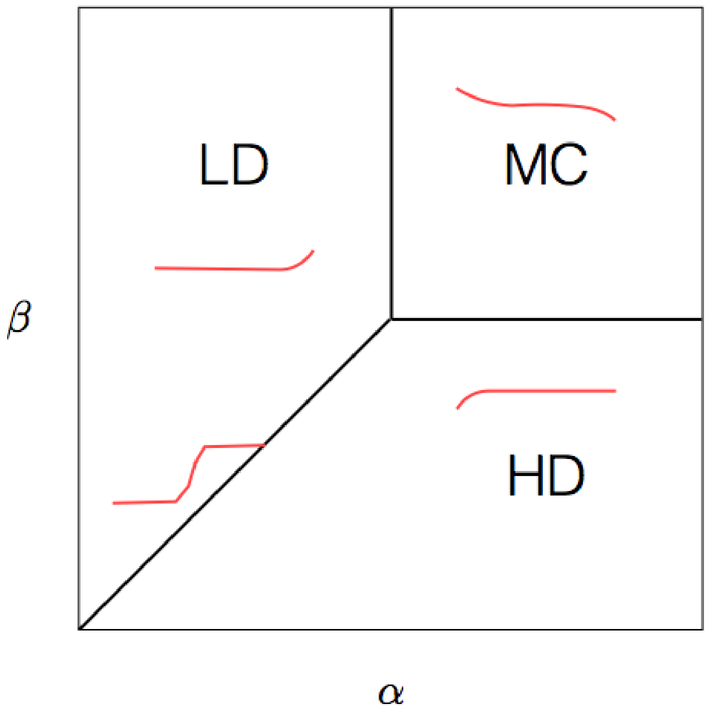

Figure 6.3: Phase diagram of the TASEP obtained from the mean field theory. Also shown are the particle density distribution.

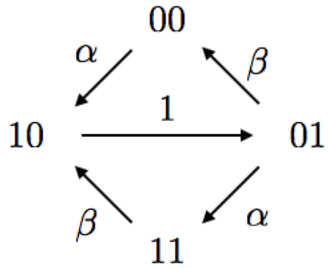

Figure 6.4: All possible transitions.

may be written as

$$|P_{st}\rangle = \frac{1}{Z}\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle \ . \tag{6.31}$$

We now investigate whether the matrix product state satisfies the master equation. Instead of writing down the master equation, we work with the transition diagrams in in Fig. 6.4. The stationary state condition yields the following relations:

$$\alpha\langle W|EE|V\rangle = \beta\langle W|ED|V\rangle \tag{6.32}$$

$$\langle W|DE|V\rangle = \alpha\langle W|EE|V\rangle + \beta\langle W|DD|V\rangle \tag{6.33}$$

$$(\alpha+\beta)\langle W|ED|V\rangle = \langle W|DE|V\rangle \tag{6.34}$$

$$\beta\langle W|DD|V\rangle = \alpha\langle W|ED|V\rangle \tag{6.35}$$

The first equation is satisfied if we require that

$$\langle W|E = \frac{1}{\alpha}\langle E| \ \ , \ \ D|V\rangle = \frac{1}{\beta}|V\rangle \ . \tag{6.36}$$

Applying the eigenvector condition to (6.33), one finds that

$$DE = D + E \ . \tag{6.37}$$

Remaining conditions in (6.34) and (6.35) are satisfied automatically from (6.36) and (6.37).

Once you find $\{E, D, \langle W|, |V\rangle\}$ satisfying the algebra, one can calculate all physical quantities. For example, the partition function is given by

$$Z(L, \alpha, \beta) = \langle W|(D+E)^L|V\rangle \ . \tag{6.38}$$

The average particle density at site i is given by

$$\rho_i = \frac{1}{Z}\langle W|(D+E)^{i-1}D(D+E)^{L-i}|V\rangle \ . \tag{6.39}$$

The two-point correlation function is given by

$$C_{i,j} = \frac{1}{Z}\langle W|(D+E)^{i-1}D(D+E)^{j-i-1}D(D+E)^{L-j}|V\rangle. \tag{6.40}$$

The particle current is given by

$$\begin{aligned} J_{i,i+i} &= \frac{1}{Z(L)}\langle W|(D+E)^{i-1}DE(D+E)^{L-i-1}|V\rangle \\ &= \frac{Z(L-1)}{Z(L)} \end{aligned} \tag{6.41}$$

6.4 Matrix representation

One way to solve the problem is to find the explicit form of the matrices and vectors. As a first trial, let's try to find a solution with commuting D and E. When they commute, we find that

$$\langle W|ED|V\rangle = \langle W|DE|V\rangle \tag{6.42}$$

This is possible only when

$$\frac{1}{\alpha\beta} = \frac{1}{\alpha} + \frac{1}{\beta} \Rightarrow \alpha + \beta = 1. \tag{6.43}$$

Then, we can choose $E = 1/\alpha$ and $D = 1/\beta$. This yields that

$$\rho_i = \alpha = 1 - \beta \text{ and } J = \alpha(1-\alpha) = \beta(1-\beta) \ . \tag{6.44}$$

It would be nice if one finds a finite dimensional representation for E and D. Unfortunately, the matrices should be $\infty \times \infty$ unless $\alpha + \beta = 1$. There are several possible choices for the matrices D and E. One example is given below:

$$D = \begin{pmatrix} 1/\beta & 1/\beta & 1/\beta & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, E = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{6.45}$$

and

$$\langle W| = (1, 1/\alpha, 1/\alpha^2, \cdots), |V\rangle = (1, 0, 0, \cdots)^\dagger \ . \tag{6.46}$$

6.5 Algebra method

The physical quantities can be calculated only from the algebra and the eigenvalue equations with knowing the explicit representation. In general, one wants to evaluate a following expression

$$\langle W|E^a D^b E^c \cdots |V\rangle \ . \tag{6.47}$$

If we can shift all E's to the $\langle W|$ side and D's to the $|V\rangle$ side, then the calculation becomes easy.

Suppose that you want to evaluate the partition function at

$L = 2$. It can be evaluated following the procedure

$$\begin{aligned} Z(2) &= \langle W|(D+E)^2|V\rangle \\ &= \langle W|(D^2 + ED + DE + E^2)|V\rangle \\ &= \langle W|(D^2 + ED + D + E + E^2)|V\rangle \\ &= \frac{1}{\beta^2} + \frac{1}{\alpha^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha} + \frac{1}{\beta} \,. \end{aligned}$$

This is summarized as below:

1. Eliminate all D's in front of E's using $DE = D + E$.
2. Replace E with $1/\alpha$ by acting it to $\langle W|$.
3. Replace D with $1/\beta$ by acting it to $|V\rangle$.

6.6 Phase diagram

The exact phase diagram obtained from the MPA is the same as that from the mean field analysis. The MPA analysis confirms that the mean field theory is successful in producing the overall phase diagram. But, the exact solution reveals that the mean field theory does not explain the particle distribution near the entrance and exit especially in the maximal current phase.

6.7 Concluding remarks

The MPA is quite successful in studying the stationary state of the stochastic particle dynamics in 1D. It works for the ASEP as well as the TASEP. It works when there are reactions among particles. It works when there are many different particle species.

7
Conclusion

8
Appendix

8.1 Modified Bessel functions

The modified Bessel function is given by

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) \tag{8.1}$$

where $J_\alpha(x)$ is the Bessel function.

Integral representation of the modified Bessel function of integer index n:

$$I_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos n\theta \; e^{x \cos\theta} \, d\theta \tag{8.2}$$

Asymptotic form for large x:

$$I_n(x) \simeq \frac{e^x}{\sqrt{2\pi x}} e^{-(4n^2-1)/8x} \, . \tag{8.3}$$

8.2 Pauli matrices

The Pauli matrices are given by

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.4}$$

Some of their properties are listed below:

- $(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = I$
- $[\sigma^\alpha, \sigma^\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma^\gamma$ with the Levi-Civita symbol $\varepsilon_{\alpha\beta\gamma}$.
- $\{\sigma^\alpha, \sigma^\beta\} = 2\delta_{\alpha,\beta} I$
- $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$: raising/lowering operator operators which act as $\sigma^+|-\rangle = |+\rangle$ and $\sigma^-|+\rangle = |-\rangle$.

The Dirac's spin exchange operator is defined as

$$\Pi_{ij} = \frac{\vec{\sigma}_i \cdot \vec{\sigma}_j + 1}{2} = \frac{\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z + 1}{2} \, . \tag{8.5}$$

Using the raising and lowering operator, it can be also written as

$$\Pi_{ij} = \sigma_i^+ \sigma_i^- + \sigma_i^- \sigma_j^+ + (1 + \sigma_i^z \sigma_j^z)/2 \tag{8.6}$$

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