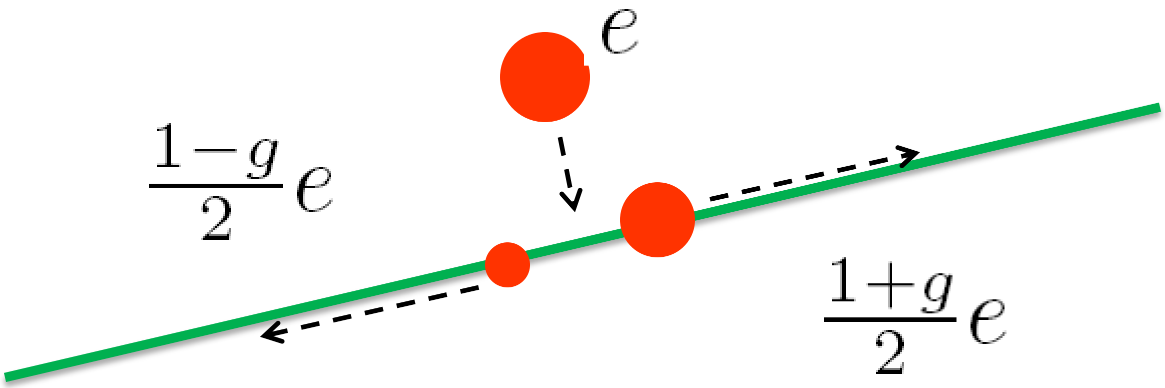
PSI 2014
"Quantum Theory of Many Particles"
(평창, 2014년 8월 28-29일)

Quantum Theory of Low Dimensional Systems: Bosonization

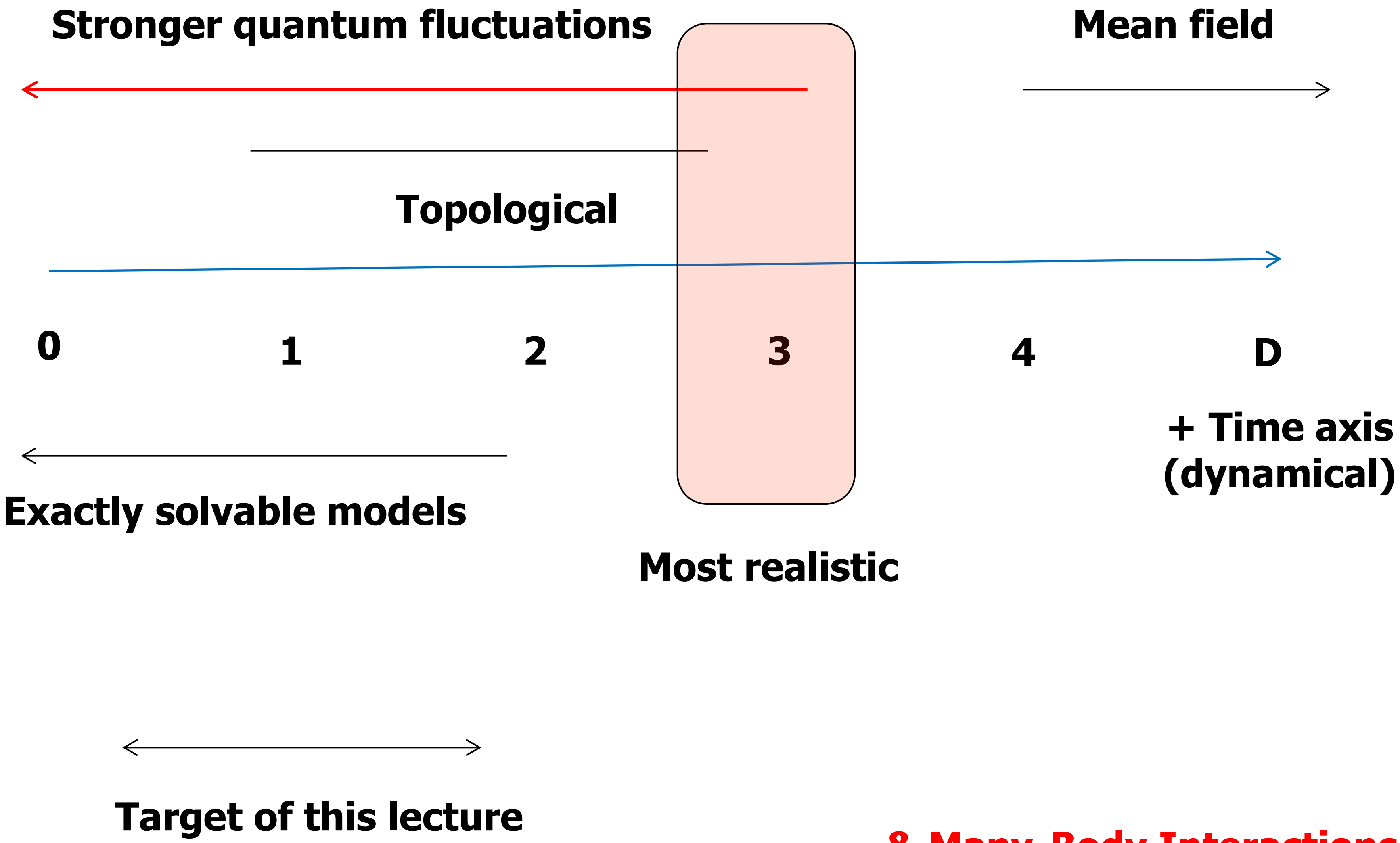


Heung-Sun Sim

Physics, KAIST

Overview

Target of this lecture: low dimension



Stronger quantum fluctuations

Mean field

Topological

0

1

2

3

4

D

+ Time axis
(dynamical)

Exactly solvable models

Most realistic

Target of this lecture

& Many-Body Interactions

Condensed matter

Many-body interactions generate difficulty in most cases…

A= single-ptl kinetic + exchange statistics → Solvable. Berry phase
A + disorders → Solvable. Anderson localization. Diffusive…
A + interactions → Solvable only in certain cases, e.g., in low dim.
A + interactions + disorders → !!!!

BUT, give rise to surprising effects.

Why we study low-dimensional many-body systems:

- **Exactly solvable models**
  Luttinger liquids / Kondo physics / spin chains /
  Kitaev models…

- **Experimentally feasible**
  Quantum Hall systems / quantum wires /
  quantum dots / optical lattices…

- **Beyond Landau framework** (Fermi liquid theory/Phase transition)

  Non-Fermi liquids

  Topological order
  – no local (single-ptl) order parameter

  Quantum criticality without symmetry breaking
  – quantum impurities

  Exotic quasiparticles
  – anyons, fractionalization

Bosonization (+ refermionization): Usefulness

Bosonization offers:

- Exact solutions for 1D interacting electrons (under certain conditions)
    - Luttinger liquids

- Exact solutions for quantum impurity problems:
    - (non-Fermi, quantum critical, topological effects)
    - Impurity scattering + electron interaction,
    - multi-channel Kondo,
    - Y-junctions, etc

- Tool for describing chiral edge channel along quantum Hall edges:
    - Electron interferometry
    - Anyons
    - Topological quantum computation

Bosonization is compatible with

Numerical tools (NRG, DMRG, ED,…)

Contents

I. Backgrounds:

Second quantization
Fermi liquid
1D electrons

II. Bosonization

Construction (Rigorous approach by von Delft and Schoeller)

III. Quantum impurities (+ refermionization for senior phD students)

Impurity in Luttinger liquids
Kondo impurity
Tunneling between the fractional quantum Hall edges

IV. Fractionalization

Spin-charge separation
Charge fractionalization
Anyons

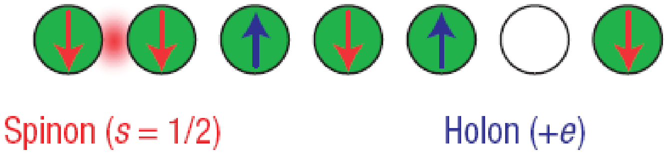
References:

von Delft and Schoeller, "Bosonization for Beginners --- Refermionization for Experts" (1998).
Schonhammer, "Interacting fermions in one dimension: the Tomonaga-Luttinger model" (1997).
Zarand and von Delft, "Simple Bosonization Solution of the 2-channel Kondo model" (1998).

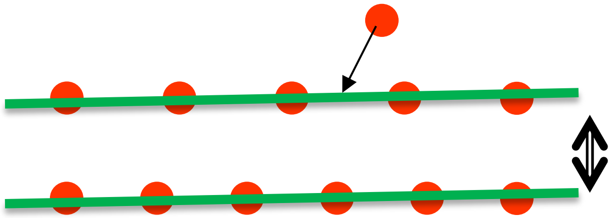
Luttinger liquids :
Low-energy properties of 1D interacting electrons

Bosonization (plasmons)

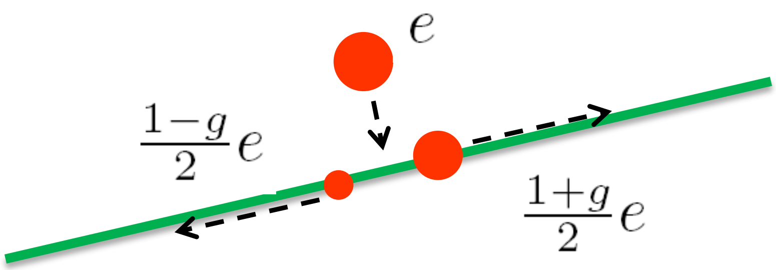
Spin-charge separation



Orthogonality catastrophe (tunneling exponent)



Charge fractionalization



I. Backgrounds

Second quantization

Fermi liquid

1D electrons

Second quantization - 1

Identical particles --- how to construct their wave functions?

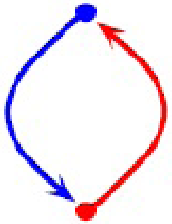
$$\psi(r_1, \cdots, r_i, \cdots, r_j, \cdots, r_N)$$

(1) Indistinguishability: N-particle probability

$$|\psi(r_1, \cdots, r_i, \cdots, r_j, \cdots, r_N)|^2 = |\psi(r_1, \cdots, r_j, \cdots, r_i, \cdots, r_N)|^2$$

(2) Particle exchange statistics

$$\psi(r_1, \cdots, r_i, \cdots, r_j, \cdots, r_N) = \lambda\psi(r_1, \cdots, r_j, \cdots, r_i, \cdots, r_N)$$



$\lambda = 1,$ bosons

$\lambda = -1,$ fermions

(anti)symmetrization

It is inconvenient to handle Schrodinger equations for 10^{23} identical particles (with correct exchange statistics) in the first quantization formulation.

Second quantization - 2

Steps for constructing N-particle states: Occupation number representation

1. Choose a complete orthonormal basis set $\{\nu\}$ of single-particle states

2. Consider the N-particle basis set of $\boxed{\{|n_1, n_2, \cdots, n_\nu, \cdots\rangle\}}$

   Each basis state satisfies the permutation symmetry

$$\sum n_i = N$$

$$\langle n_1', n_2', \cdots | n_1, n_2, \cdots\rangle$$
$$= \delta_{n_1', n_1}\delta_{n_2', n_2}\cdots$$

3. Construct any arbitrary N-particle wave function from the superposition of the N-particle basis states.

Examples: two noninteracting identical particles in a box

$$\psi_F(r_1, r_2) = \frac{1}{\sqrt{2}}(\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1))$$

$$\begin{aligned}
\psi_{B1}(r_1, r_2) &= \psi_a(r_1)\psi_a(r_2), \\
\psi_{B2}(r_1, r_2) &= \psi_b(r_1)\psi_b(r_2), \\
\psi_{B3}(r_1, r_2) &= \frac{1}{\sqrt{2}}(\psi_a(r_1)\psi_b(r_2) + \psi_a(r_2)\psi_b(r_1))
\end{aligned}$$

Second quantization - 3

Number operators

$$\hat{n}_\nu |n_1, n_2, \cdots, n_\nu, \cdots\rangle = n_\nu |n_1, n_2, \cdots, n_\nu, \cdots\rangle$$

$n_\nu \geq 0 \longrightarrow$ it is natural to write $n_\nu = d_\nu^\dagger d_\nu$

$\hat{n}_\nu = b_\nu^\dagger b_\nu$ for bosons, and $\hat{n}_\nu = c_\nu^\dagger c_\nu$ for fermions

Creation / Annihilation operators

Bosons $[b_i, b_j^\dagger]_+ \equiv b_i b_j^\dagger - b_j^\dagger b_i = \delta_{ij},$

Fermions $[c_i, c_j^\dagger]_- \equiv \{c_i, c_j^\dagger\} = c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$

both $[d_i, d_j]_\pm = 0,$

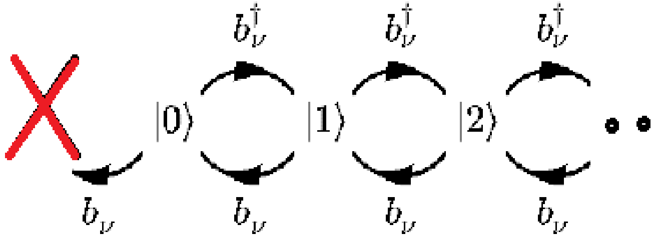
$$\begin{aligned} b_\nu |n_\nu\rangle &= \sqrt{n_\nu} |n_\nu - 1\rangle, \\ b_\nu^\dagger |n_\nu\rangle &= \sqrt{n_\nu + 1} |n_\nu + 1\rangle. \end{aligned}$$

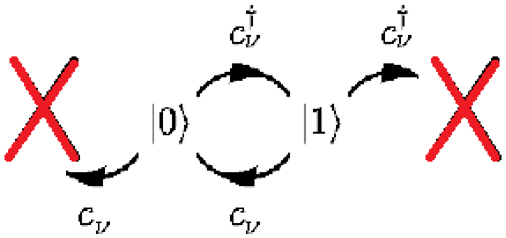
$$(c_\nu^\dagger)^2 |0\rangle = 0$$

$$\begin{aligned} c_\nu^\dagger |n_\nu = 0\rangle &= |n_\nu = 1\rangle \\ c_\nu^\dagger |n_\nu = 1\rangle &= 0. \end{aligned}$$

$$\begin{aligned} c_\nu |n_\nu = 1\rangle &= |n_\nu = 0\rangle \\ c_\nu |n_\nu = 0\rangle &= 0. \end{aligned}$$

$|0\rangle$ is the vacuum state





Second quantization - 4

Bosons : n = 0,1,2,3,...

basis $|n_{\nu_1}, n_{\nu_2}, n_{\nu_3}, \ldots\rangle$

$$|n_\nu\rangle = \frac{1}{\sqrt{n_\nu!}}(b_\nu^\dagger)^{n_\nu}|0\rangle,$$

$$|n_1, n_2, \ldots\rangle = \prod_i \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}}|0, 0, \cdots\rangle$$

$$\begin{array}{c|l} 0 & |0,0,0,0,..\rangle \\ 1 & |1,0,0,0,..\rangle,\ |0,1,0,0,..\rangle,\ |0,0,1,0,..\rangle,\ .. \\ 2 & |2,0,0,0,..\rangle,\ |0,2,0,0,..\rangle,\ |1,1,0,0,..\rangle, |0,0,2,0,..\rangle, \end{array}$$

Fermions : n = 0, 1

$$|n_1, n_2, \ldots\rangle = (c_1^\dagger)^{n_1}(c_2^\dagger)^{n_2}(c_3^\dagger)^{n_3}\cdots|0, 0, 0, \cdots\rangle$$

$$\begin{array}{c|l} 0 & |0,0,0,0,..\rangle \\ 1 & |1,0,0,0,..\rangle,\ |0,1,0,0,..\rangle,\ |0,0,1,0,..\rangle,\ .. \\ 2 & |1,1,0,0,..\rangle,\ |0,1,1,0,..\rangle,\ |1,0,1,0,..\rangle, |0,0,1,1,..\rangle, \end{array}$$

Now the exchange statistics symmetry is satisfied in $|n_1, n_2, \ldots\rangle$ **.**

$$\Phi_{n_1,n_2,\cdots}(x_1, x_2, \cdots, x_N) = \langle x_1, x_2, \cdots, x_N|n_1, n_2, \cdots\rangle$$

$$[d_i, d_j]_\pm = 0$$

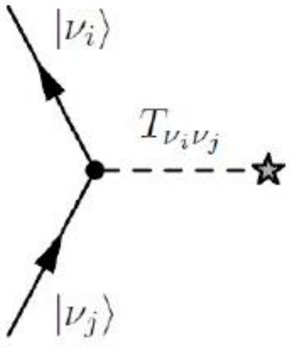
Second quantization - 5

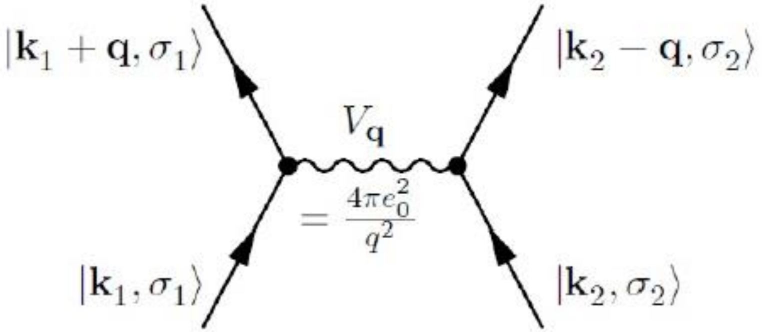
$$\hat{H} = \hat{T} + \hat{V} = \sum_{ij} \langle i|T|j\rangle d_i^\dagger d_j + \frac{1}{2} \sum_{ijkl} \langle ij|V|lk\rangle d_i^\dagger d_j^\dagger d_k d_l$$

$$\hat{T} = \sum_k T_k d_k^\dagger d_k = \sum_k T_k \hat{n}_k, \qquad T_{k,k'} = \langle k|T|k\rangle \delta_{kk'}$$

the sum of (kinetic energy T_k of state k) × (occupation number n_k)

diagonalizes $\langle \{n_\nu\}|\hat{H}|\{n_{\nu'}\}\rangle$





Second quantization - 6

Homework 1. $d_1^\dagger d_2 d_1 d_2^\dagger |n_1, n_2\rangle = c|n_1, n_2\rangle$. What is the c-number c?

Homework 2.

Consider noninteracting electrons in a periodic 1D lattice (the lattice spacing is a) with

$$H = \epsilon \sum_i c_i^\dagger c_i - t \sum_i (c_{i+1}^\dagger c_i + h.c.)$$

Here, i is the index of lattice sites, ϵ is the onsite energy (real), and t is the hopping energy (real). Diagonalize it and find the energy dispersion relation.

Fermi liquid theory - 1

Electrons are interacting with each other. Their interaction energy is comparable with the kinetic energy in usual metals.

Why does Drude conductivity (which is derived, based on noninteracting electrons) work well in usual metals?

Fermi liquid theory – 2: Quasiparticles

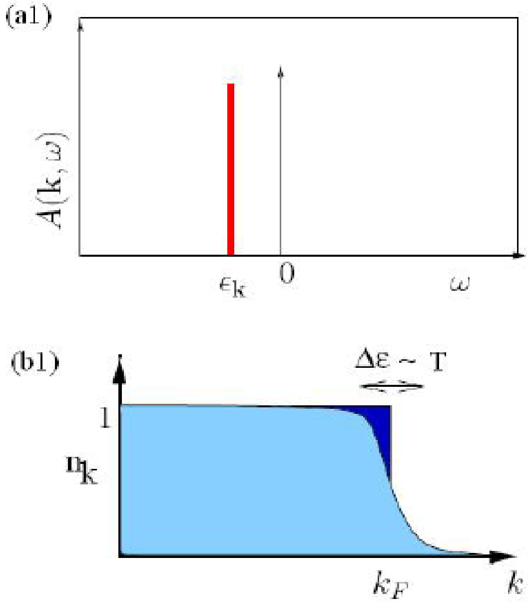
Low-lying excitations of long-lived quasiparticles

- * Adiabatic connection (via one-to-one correspondence) between the quasiparticles and bare non-interacting electrons

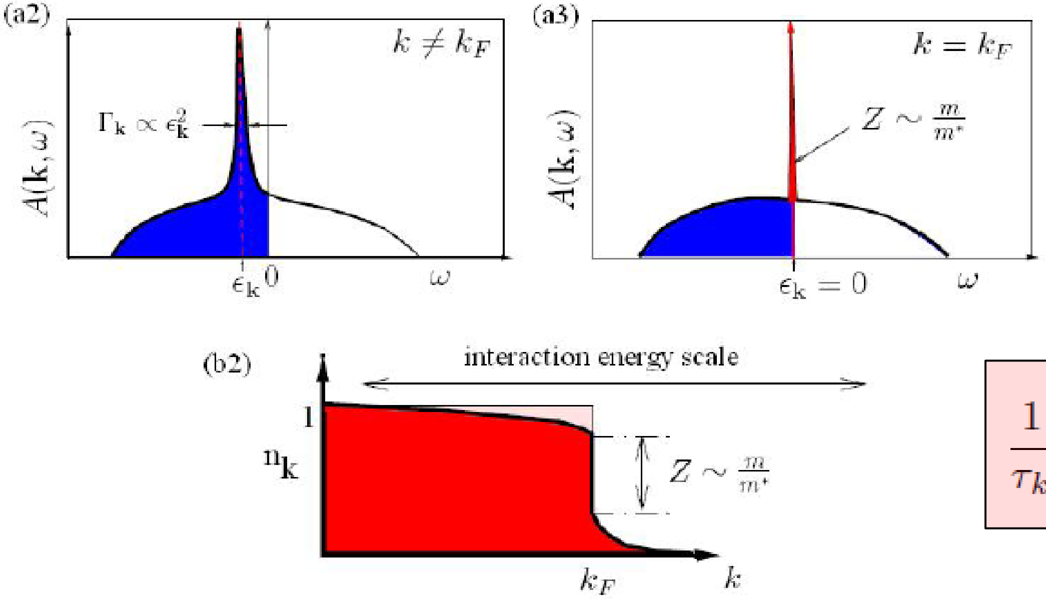
- * Pauli exclusion plays an important role in their stability.

- * They are robust against perturbation.

Electron gas



Fermi liquid



$$\boxed{\frac{1}{\tau_k} \propto T^2}$$

Fermi liquid theory – 3: quasiparticle weight Z

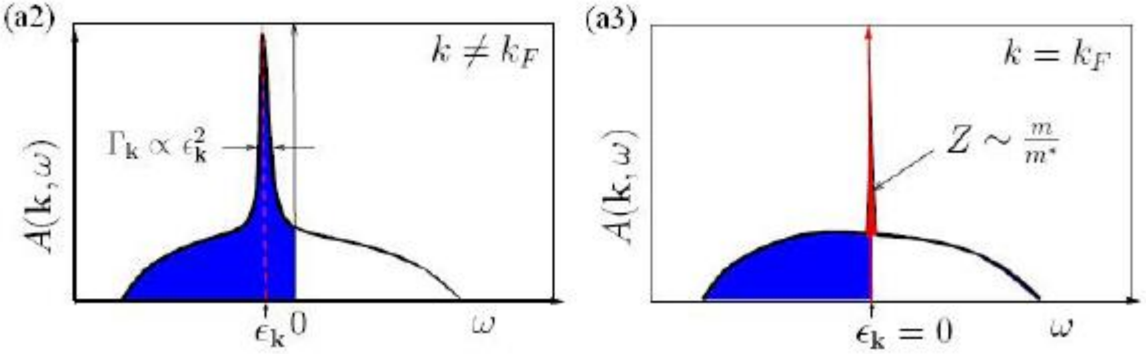
$$G^R(k\sigma,\omega) \;=\; \frac{1}{\omega-\xi_k-\Sigma^R(k,\omega)} = \frac{1}{\omega-[\xi_k+\Re\Sigma^R(k,\omega)]-i\Im\Sigma^R(k,\omega)}$$

$$\equiv \; \frac{Z}{\omega-\tilde{\xi}_k+i/(2\tilde{\tau}(k,\omega))}$$

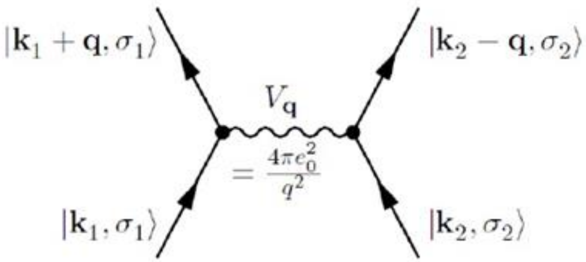
$$\tilde{\xi}_k = \hbar v_f(k-\tilde{k}_f) = \frac{\hbar^2}{m^*}(k-\tilde{k}_f)\tilde{k}_f$$

effective mass

$$A(k,\omega) \;=\; 2\pi Z\delta(\omega-\tilde{\xi}_k)+\underline{A'(k,\omega)} \quad \longleftarrow \quad \textbf{incoherent background}$$



Fermi liquid theory – 4: Quasiparticle life time



Fermi golden rule (2nd order perturbation theory)

$$\frac{1}{\tau_k} = \frac{2}{V^2}\sum_{k'q}\Gamma_{k+q\sigma,k-q\sigma';k'\sigma',k\sigma}[n_k n_{k'}(1-n_{k+q})(1-n_{k'-q})-(1-n_k)(1-n_{k'})n_{k+q}n_{k'-q}]$$

$$\Gamma_{k+q\sigma,k-q\sigma';k'\sigma',k\sigma} = \frac{2\pi}{\hbar}|\langle k+q\sigma,k'-q\sigma'|W^{RPA}(q)|k'\sigma',k\sigma\rangle|^2\delta(\xi_k+\xi_{k'}-\xi_{k+q}-\xi_{k'-q})$$

scattering rate from $|k\sigma,k'\sigma'\rangle$ to $|k+q\sigma,k'-q\sigma'\rangle$

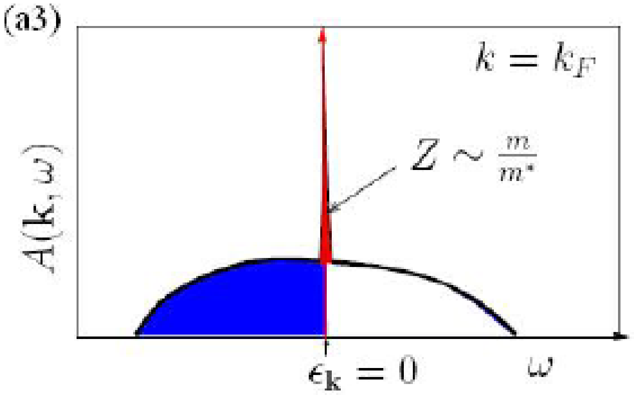
$$\frac{1}{\tau_k} \sim \int_{-\infty}^{0} d\xi_{k'}\int_{0}^{\infty} d\xi_{k'-q}\Theta(\xi_k+\xi_{k'}-\xi_{k'-q}) = \int_{-\infty}^{0} d\xi_{k'}(\xi_k+\xi_{k'})\Theta(\xi_k+\xi_{k'}) \sim \xi_k^2 \boxed{\propto T^2}$$

$\xi_k > 0 \qquad \xi_{k'} < 0$

$\xi_{k+q},\ \xi_{k'-q} > 0$

$W^{RPA}(q) \sim 1/(q^2+k_s^2) \sim k_s^{-2}$

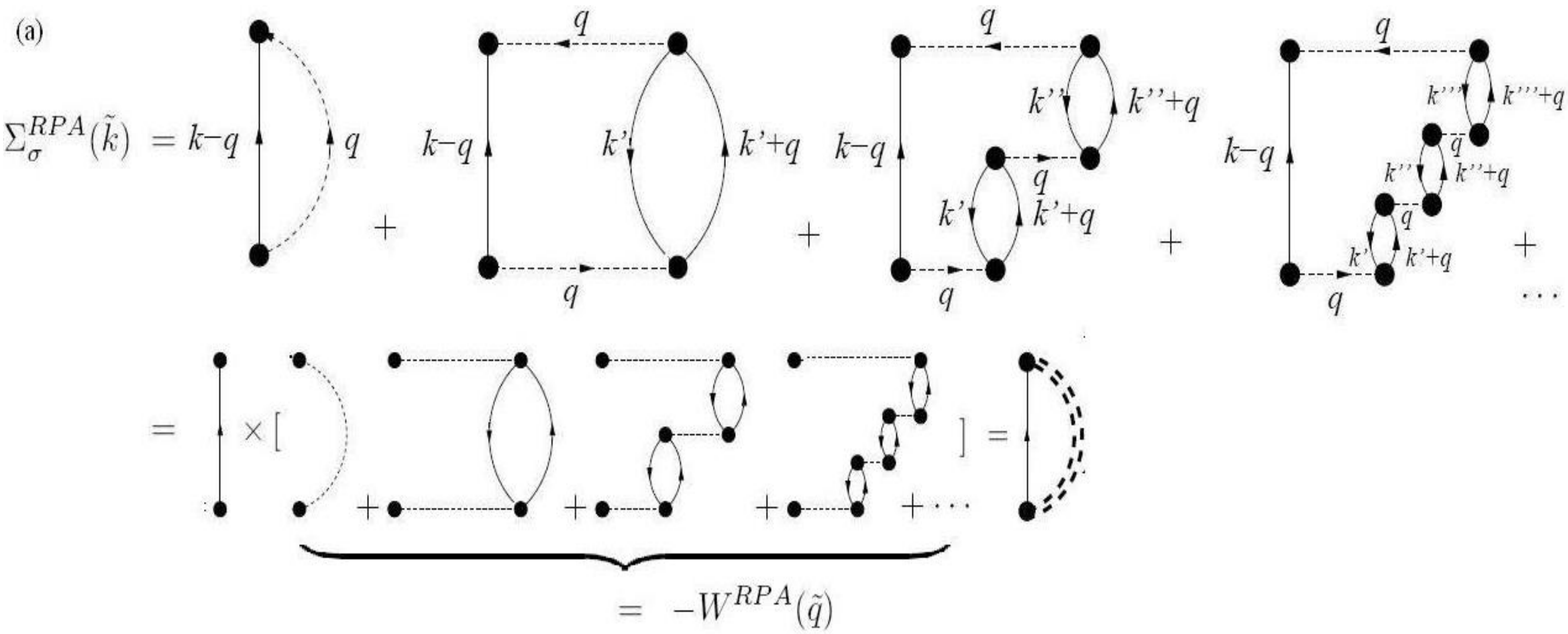
for small q



Supplement: RPA

Random phase approximation (RPA):

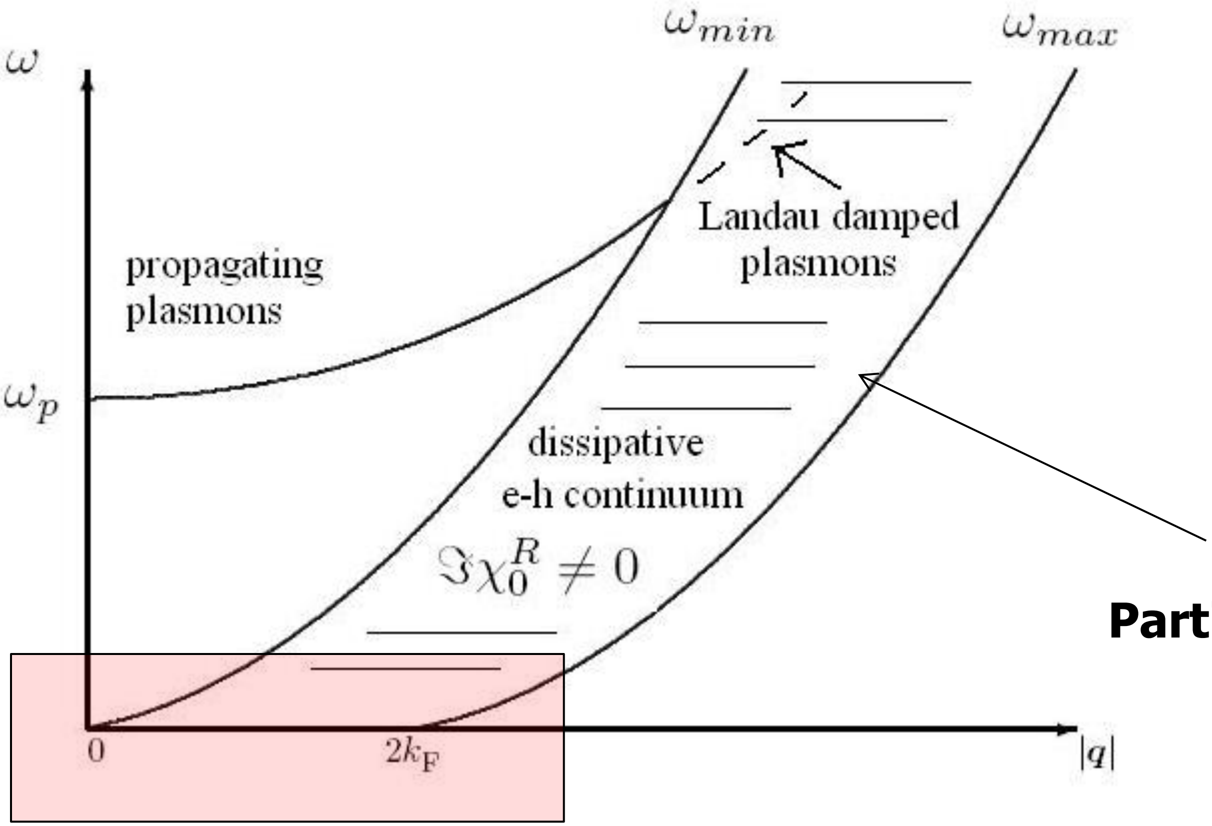
- **good in the high density limit.**
- **particle-hole pair bubbles.**
- **screened Coulomb interaction, plasmon (charge density) excitations**



$$W^{RPA}(q,0) \;\rightarrow_{q\to 0}\; \frac{e^2}{\epsilon_0(q^2+k_s^2)}$$

k_s is the Thomas-Fermi screening wave number

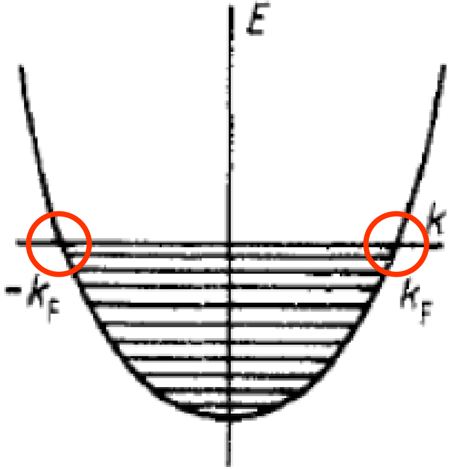
Fermi liquid theory – 5: Excitations in 3D

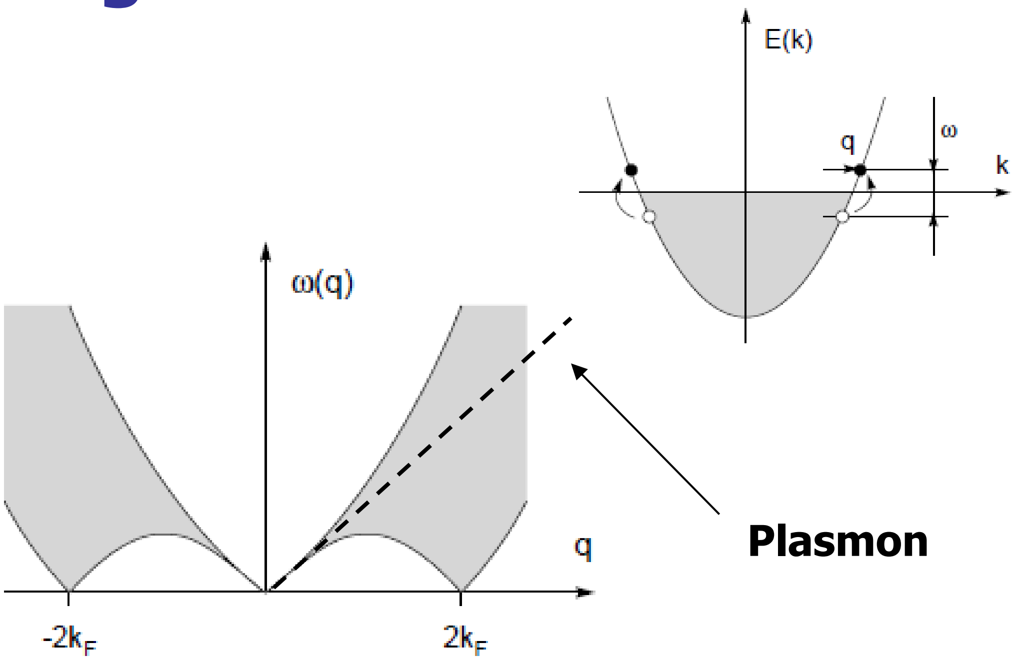


Particle-hole pair excitations

low-energy excitations: particle-hole pairs

1D electrons – 1:
1D differs from higher dimensions





Plasmon

Single-particle energy band.

Particle-hole excitation spectrum

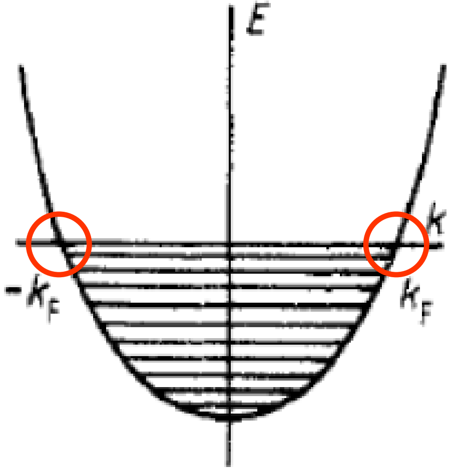
$$c^{\dagger}_{k+q} c_k |0\rangle$$

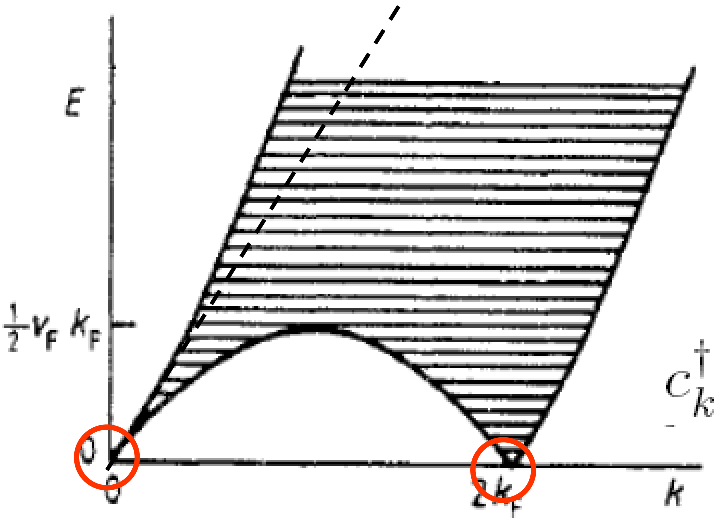
Fermi "surface" consists of two isolated points.

Less phase space to two-particle interaction processes

Particle-like excitations are not well separated from RPA plasmons.

1D electrons – 2: Breakdown of Fermi liquid theory





$c^\dagger_{k+q} c_k |0\rangle$

Quasiparticle life time: $\tau_k^{-1} \propto \xi_k \propto T$

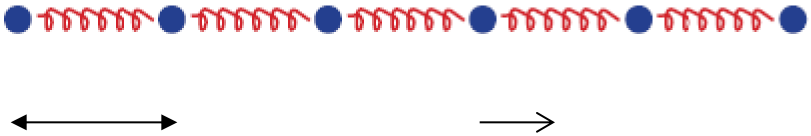
Cf. 3D: $\tau \propto T^{-2}$

→ **Break down of the "noninteracting" quasiparticle picture**

Homework: Prove $\tau_k^{-1} \propto \xi_k \propto T$ **in 1D, using Fermi golden rule + 2nd order perturbation.**

1D electrons – 3:
Strong interaction limit: Wigner solids

Let's consider the opposite limit of strong e-e interactions: 1D Wigner solids



$a = 1/n_0$

lattice spacing

$a\theta_j/\pi$

$$r_j = r_j^0 + a\theta_j/\pi$$

n_0 **: average electron density**

$$k_F = \pi n_0$$

$$ja \to x \qquad \theta_j \to \theta(x)$$

$$\delta n(x) = -\frac{1}{\pi}\frac{\partial\theta}{\partial x}$$

Lagrangian for phonons in 1D Wigner solid

$$\mathcal{L} = \underbrace{\int dx \frac{ma}{2\pi^2}\dot{\theta}(x)^2}_{\textbf{kinetic}} - \underbrace{\int dx \frac{V_0}{2}\delta n(x)^2}_{\textbf{interaction (short range)}}$$

$$= \boxed{\frac{\hbar}{2\pi g}\int dx \left[\frac{1}{v_\rho}(\partial_t\theta)^2 - v_\rho(\partial_x\theta)^2\right]}$$

$$g \equiv \sqrt{\frac{\pi\hbar v_F}{V_0}}$$

$$v_\rho \equiv \sqrt{\frac{V_0}{ma}} = \frac{v_F}{g}$$

$$v_F = \hbar k_F/m = \pi\hbar n_0/m$$

1D electrons – 4: Stability of Wigner solids

$$\mathcal{L} = \frac{\hbar}{2\pi g}\int dx \left[\frac{1}{v_\rho}(\partial_t\theta)^2 - v_\rho(\partial_x\theta)^2\right]$$

- **How quantum/thermal fluctuations destroy the broken symmetry phase?**
  - **Mermin Wagner: no true long-range order below lower critical dimension**
  - **Quasi-long range solid order** $\langle n_{q=2\pi/a}\rangle \sim (a/L)^g \to 0 \text{ as } L \to \infty$
  - **"Wigner solid" only at strong interaction (very small g).**

- **Particle exchange statistics**
  - **In the solid limit, electrons never exchange. The statistics is irrelevant.**
  - **Low-energy excitations are phonons (bosons), although the system is composed of electrons.**

Summary I

- Fermi liquid theory breaks down for 1D electrons.

- Low-energy excitations of 1D electrons are described by bosons.

Phonons of 1D Wigner crystal ↔ **Low-energy excitations of 1D noninteracting electrons**

Plasmons
(bosonic excitation)

Bosonization

$$\psi_\eta(x) = F_\eta\, a^{-1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\phi_\eta(x)}$$

***Rigorous construction of bosonization:**

Ref.: von Delft and Schoeller, condmat/9805275

Bosonization -1: Prerequisite

Fermion fields (M species; spin up,down/left-moving, right)

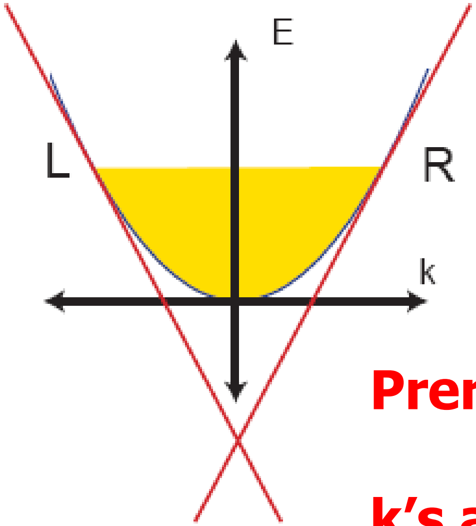
$$\{c_{k\eta}, c^\dagger_{k'\eta'}\} = \delta_{\eta\eta'}\delta_{kk'} \ , \qquad k \in [-\infty, \infty], \qquad \eta = 1, \ldots, M$$

$$k = \frac{2\pi}{L}(n_k - \tfrac{1}{2}\delta_b) \ , \qquad \text{with} \quad n_k \in \mathbb{Z} \quad \text{and} \quad \delta_b \in [0, 2)$$

parameter for BC

$$\psi_\eta(x + L/2) = e^{i\pi\delta_b}\,\psi_\eta(x - L/2)$$

$$\psi_\eta(x) \quad \equiv \quad \left(\tfrac{2\pi}{L}\right)^{1/2} \sum_{k=-\infty}^{\infty} e^{-ikx} c_{k\eta}$$



Prerequisite for bosonization :

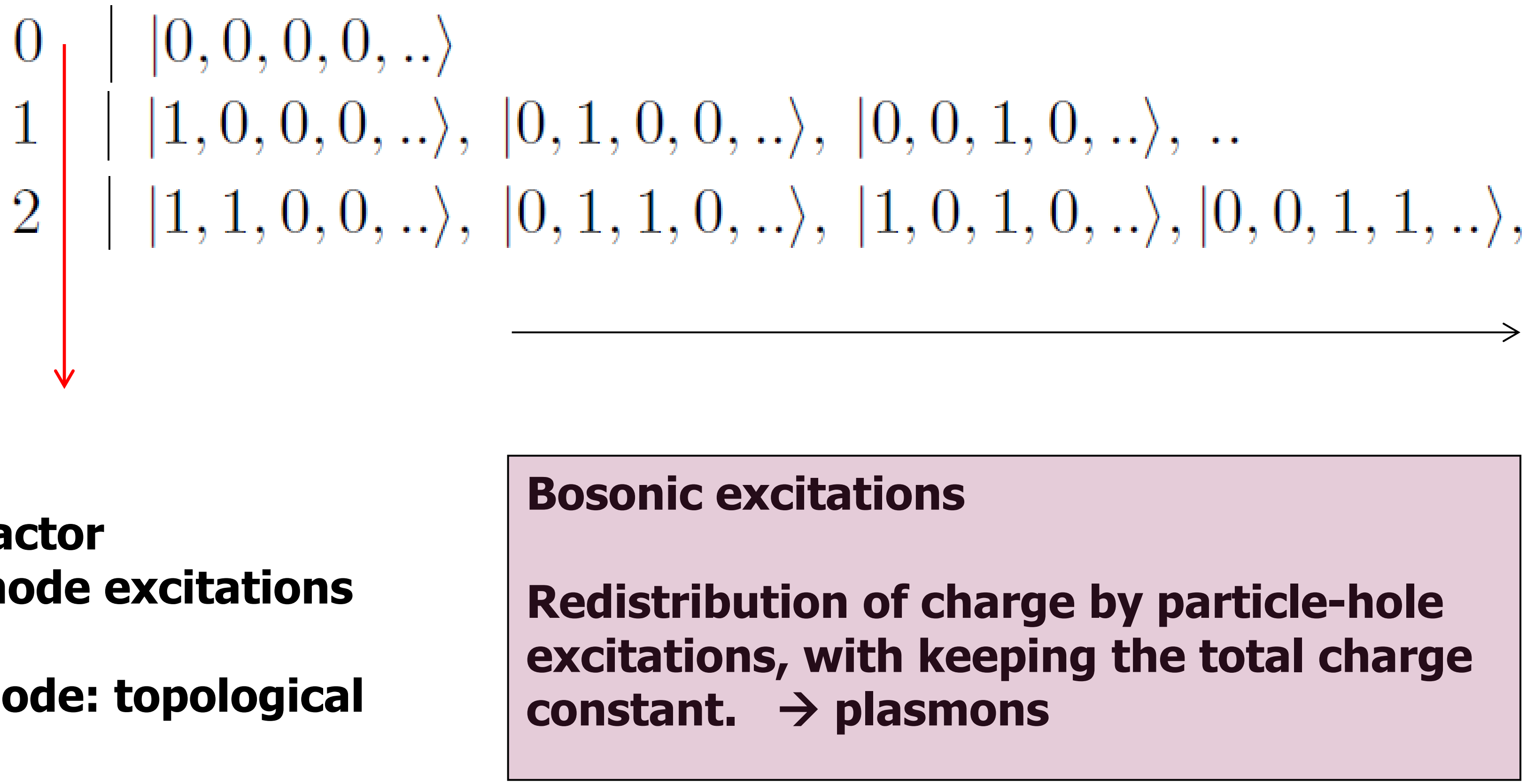
k's are discrete and unbounded.

Number operator (normal ordering: relative to vacuum – important for unbounded k's)

$$\widehat{N}_\eta \equiv \sum_{k=-\infty}^{\infty} {}^*_*c^\dagger_{k\eta} c_{k\eta}{}^*_* = \sum_{k=-\infty}^{\infty} \left[c^\dagger_{k\eta} c_{k\eta} - {}_0\langle\vec{0}|c^\dagger_{k\eta} c_{k\eta}|\vec{0}\rangle_0 \right]$$

Bosonization -2: Structure

Fermion Fock space



Total number

$$\begin{array}{c|l}
0 & |0,0,0,0,..\rangle \\
1 & |1,0,0,0,..\rangle,\ |0,1,0,0,..\rangle,\ |0,0,1,0,..\rangle,\ .. \\
2 & |1,1,0,0,..\rangle,\ |0,1,1,0,..\rangle,\ |1,0,1,0,..\rangle, |0,0,1,1,..\rangle,
\end{array}$$

Klein factor
Zero-mode excitations

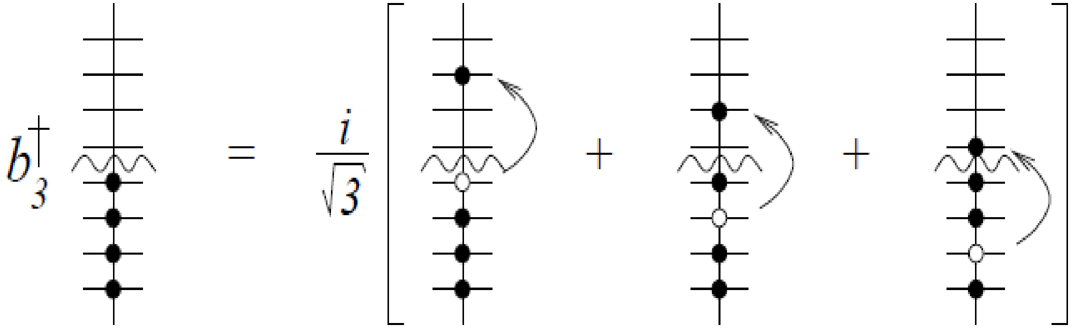
Zero mode: topological

Bosonic excitations

Redistribution of charge by particle-hole excitations, with keeping the total charge constant. → plasmons

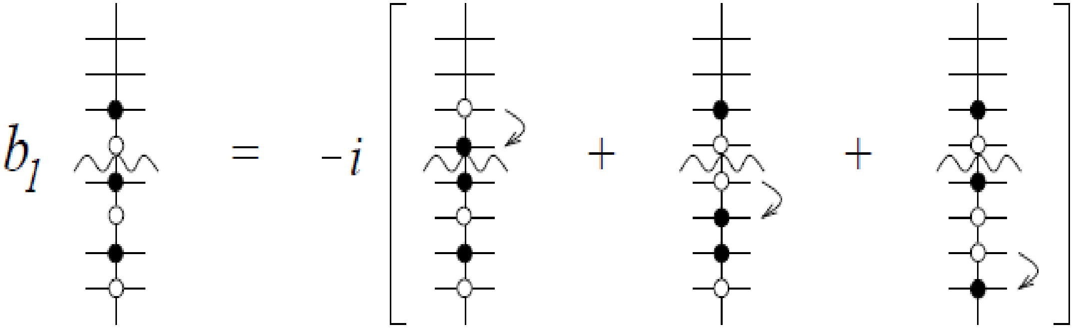
Bosonization -3: Bosonic excitations

Particle-hole excitations (bosonic)



$$b^\dagger_{q\eta} \equiv \frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k+q\;\eta} c_{k\eta}$$

$$q \equiv \frac{2\pi}{L} n_q > 0 \qquad n_q \in \mathbb{Z}^+$$



$$[b_{q\eta}, b_{q'\eta'}] = [b^\dagger_{q\eta}, b^\dagger_{q'\eta'}] = 0$$

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = \delta_{\eta\eta'}\delta_{qq'}$$

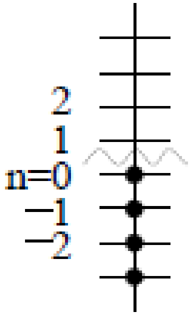
$$[N_{q\eta}, b_{q'\eta'}] = [N_{q\eta}, b^\dagger_{q'\eta'}] = 0$$

$$b_{q\eta}|\vec{N}\rangle_0 = 0$$

* $|\vec{N}\rangle = f(b^\dagger)|\vec{N}\rangle_0$ **span the complete N-particle Hilbert space**

$|\vec{N}\rangle_0$

N-particle ground state



Bosonization -4: Anomalous commutator

Homework : Prove $[b_{q\eta}, b^{\dagger}_{q'\eta'}] = \delta_{\eta\eta'}\delta_{qq'}$ **Correct only for unbound k's in 1D**

$$b^{\dagger}_{q\eta} \equiv \frac{i}{\sqrt{n_q}}\sum_{k=-\infty}^{\infty} c^{\dagger}_{k+q\;\eta}c_{k\eta} \qquad b_{q\eta} \equiv \frac{-i}{\sqrt{n_q}}\sum_{k=-\infty}^{\infty} c^{\dagger}_{k-q\;\eta}c_{k\eta}$$

$$[AB,C] = ABC - CAB = ABC + ACB - ACB - CAB = A\{B,C\} - \{A,C\}B$$

$$[b_{q\eta}, b^{\dagger}_{q'\eta'}] = \delta_{\eta\eta'}\sum_{k=-\infty}^{\infty}\frac{1}{n_q}\left(c^{\dagger}_{k+q-q'\,\eta}c_{k\eta} - \underline{c^{\dagger}_{k+q\eta}c_{k+q'\eta}}\right)$$

= 0 ?? (wrong for $q = q'$**)**

← $k \to k - q'$

For $q \neq q'$ the two terms are both normal-ordered

$${}^{*}_{*}ABC\ldots{}^{*}_{*} = ABC\ldots - {}_0\langle\vec{0}|ABC\ldots|\vec{0}\rangle_0$$

$$= \delta_{\eta\eta'}\delta_{qq'}\sum_k \frac{1}{n_q}\left\{\left[{}^{*}_{*}c^{\dagger}_{k\eta}c_{k\eta}{}^{*}_{*} - {}^{*}_{*}c^{\dagger}_{k+q\eta}c_{k+q\eta}{}^{*}_{*}\right] + \left({}_0\langle\vec{0}|c^{\dagger}_{k\eta}c_{k\eta}|\vec{0}\rangle_0 - {}_0\langle\vec{0}|c^{\dagger}_{k+q\eta}c_{k+q\eta}|\vec{0}\rangle_0\right)\right\}$$

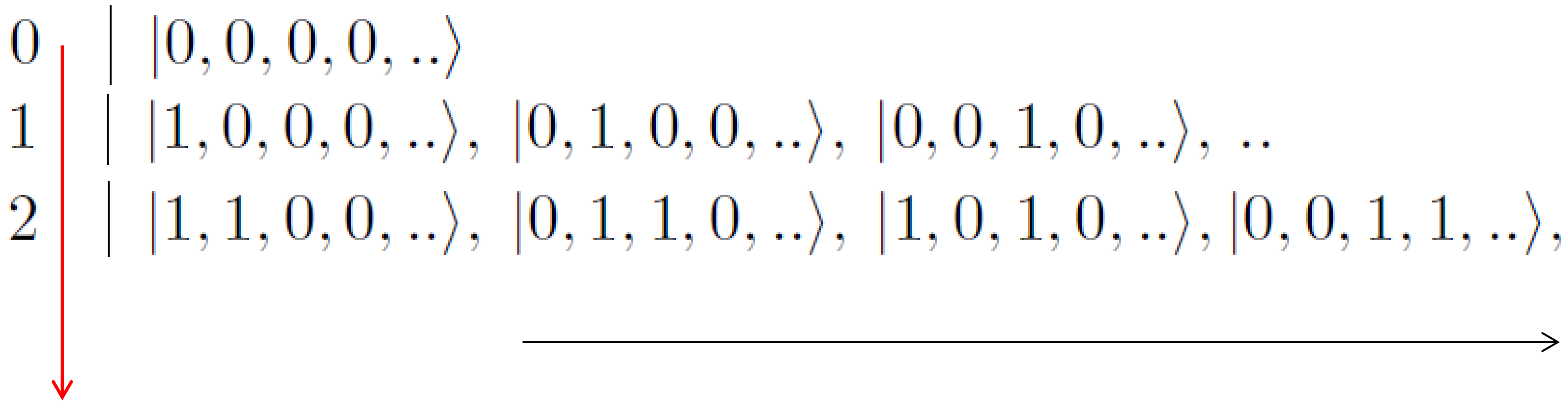
$$= \delta_{\eta\eta'}\delta_{qq'}$$

$$\frac{1}{n_q}\left(\sum_{n_k=-\infty}^{0} - \sum_{n_k=-\infty}^{-n_q}\right) = \frac{1}{n_q}n_q = 1$$

$$q \equiv \frac{2\pi}{L}n_q$$

Bosonization -2: Structure

Fermion Fock space



$$\begin{array}{c|l} 0 & |0,0,0,0,..\rangle \\ 1 & |1,0,0,0,..\rangle,\ |0,1,0,0,..\rangle,\ |0,0,1,0,..\rangle,\ .. \\ 2 & |1,1,0,0,..\rangle,\ |0,1,1,0,..\rangle,\ |1,0,1,0,..\rangle, |0,0,1,1,..\rangle, \end{array}$$

Klein factor
Zero-mode excitations

Zero mode: topological

Bosonic excitations

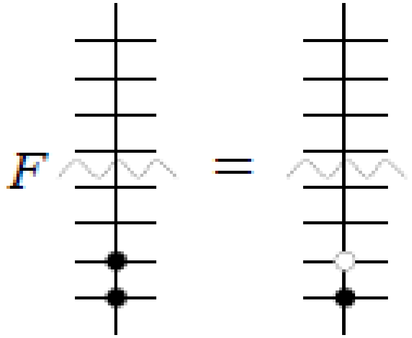
Redistribution of charge by particle-hole excitations, with keeping the total charge constant. → plasmons

Bosonization -5: Klein factor

Klein factors $F_\eta^\dagger$ **(zero-mode excitations; fermionic)**

- **Treat Klein factors carefully.**
- **Many papers, which ignored them, turned out to be wrong.**
- **Their commutators can capture topological effects.**

$$F\,|\text{-}2\rangle_0 = |\text{-}3\rangle_0$$



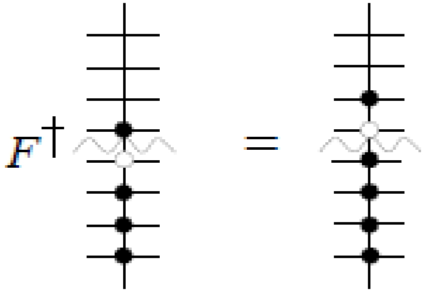
$$\boxed{F_\eta^{-1} = F_\eta^\dagger \equiv e^{i\theta_\eta}}$$

$$\{F_\eta^\dagger, F_{\eta'}\} = 2\delta_{\eta\eta'} \qquad F_\eta F_\eta^\dagger = F_\eta^\dagger F_\eta = 1$$

$$\{F_\eta^\dagger, F_{\eta'}^\dagger\} = \{F_\eta, F_{\eta'}\} = 0$$

$$[b_{q\eta}, F_{\eta'}^\dagger] = [b_{q\eta}^\dagger, F_{\eta'}^\dagger] = [b_{q\eta}, F_{\eta'}] = [b_{q\eta}^\dagger, F_{\eta'}] = 0$$

$$F^\dagger\, b_1^\dagger\, |0\rangle_0 = b_1^\dagger\, |1\rangle_0$$



$$\boxed{[\hat{N}_\eta, F_{\eta'}] = -\delta_{\eta\eta'} F_\eta \qquad [\hat{N}_\eta, F_{\eta'}^\dagger] = \delta_{\eta\eta'} F_\eta^\dagger}$$

$$|\vec{N}\rangle = f(b^\dagger)|\vec{N}\rangle_0 \qquad \hat{T}_\eta \equiv (-)^{\sum_{\eta=1}^{\eta-1} \hat{N}_\eta}$$

$$F_\eta^\dagger|\vec{N}\rangle \equiv f(b^\dagger)c_{N_\eta+1}^\dagger|N_1,\ldots,N_\eta,\ldots N_M\rangle_0 \equiv f(b^\dagger)\hat{T}_\eta|N_1,\ldots,N_\eta+1,\ldots N_M\rangle_0$$

$$F_\eta|\vec{N}\rangle \equiv f(b^\dagger)c_{N_\eta}|N_1,\ldots,N_\eta,\ldots N_M\rangle_0 \equiv f(b^\dagger)\hat{T}_\eta|N_1,\ldots,N_\eta-1,\ldots N_M\rangle_0$$

Derivation of the bosonization identity

$$\psi_\eta(x) = F_\eta\, a^{-1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\phi_\eta(x)}$$

Bosonization -6: Bosonization identity

Relation between the boson field and the electron field?

$$\begin{aligned}[b_{q\eta'}, \psi_\eta(x)] &= \delta_{\eta\eta'}\alpha_q(x)\psi_\eta(x) \\ [b^\dagger_{q\eta'}, \psi_\eta(x)] &= \delta_{\eta\eta'}\alpha^*_q(x)\psi_\eta(x)\end{aligned}$$

$$\alpha_q(x) = \tfrac{i}{\sqrt{n_q}}e^{iqx}$$

$\longrightarrow$ $\psi_\eta(x)|\vec{N}\rangle_0$ **is an eigenstate of** $b_{q\eta}$ $\quad b_{q\eta'}\psi_\eta(x)|\vec{N}\rangle_0 = \delta_{\eta\eta'}\alpha_q(x)\psi_\eta(x)|\vec{N}\rangle_0$

$\longrightarrow$ **Coherent-state representation**

$$\psi_\eta(x)|\vec{N}\rangle_0 = \exp\left[\sum_{q>0}\alpha_q(x)b^\dagger_{q\eta}\right]F_\eta\hat{\lambda}_\eta(x)|\vec{N}\rangle_0 = e^{-i\varphi^\dagger_\eta(x)}F_\eta\hat{\lambda}_\eta(x)|\vec{N}\rangle_0$$

Find $\hat{\lambda}_\eta(x)$ **:**

$${}_0\langle\vec{N}|F^\dagger_\eta\psi_\eta(x)|\vec{N}\rangle_0 = \lambda_\eta(x)$$

$$= \left(\tfrac{2\pi}{L}\right)^{1/2}e^{-i\frac{2\pi}{L}(N_\eta - \frac{1}{2}\delta_b)x}$$

since ${}_0\langle\vec{N}|e^{-i\varphi^\dagger_\eta(x)} = {}_0\langle\vec{N}|$

$$\psi_\eta(x) \equiv \left(\tfrac{2\pi}{L}\right)^{1/2}\sum_{k=-\infty}^{\infty}e^{-ikx}c_{k\eta}$$

$$n_k = N_\eta \text{ [i.e. } k = \tfrac{2\pi}{L}(N_\eta - \tfrac{1}{2}\delta_b)]$$

Dynamical phase (zero mode)

In general,

$$\boxed{\psi_\eta(x) = F_\eta\, a^{-1/2}e^{-i\frac{2\pi}{L}(\hat{N}_\eta - \frac{1}{2}\delta_b)x}\underline{e^{-i\phi_\eta(x)}}}$$

Boson representation of a localized hole

$a > 0$ **: Infinitesimal regularization parameter (~ lattice spacing)**

Bosonization -7: Bosonization identity (2)

In general,

$$\boxed{\psi_\eta(x) = F_\eta\, a^{-1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} \underline{e^{-i\phi_\eta(x)}}} \qquad (*)$$

$a > 0$: **Infinitesimal regularization parameter (~ lattice spacing)**

Boson representation of a localized hole

$$\phi_\eta(x) \equiv \varphi_\eta(x) + \varphi_\eta^\dagger(x) = -\sum_{q>0} \underbrace{\frac{1}{\sqrt{n_q}}\,(e^{-iqx} b_{q\eta}}_{\varphi_\eta(x)} + e^{iqx} b_{q\eta}^\dagger)\, e^{-aq/2}$$

Particle density

$$\rho_\eta(x) \equiv {}^{*}_{*}\psi_\eta^\dagger(x)\psi_\eta(x){}^{*}_{*} = \frac{2\pi}{L}\sum_q e^{-iqx} \sum_k {}^{*}_{*}c_{k-q,\eta}^\dagger c_{k\eta}{}^{*}_{*} = \boxed{\partial_x \phi_\eta(x) + \frac{2\pi}{L}\widehat{N}_\eta}$$

Commutator
(**)

$$[\phi_\eta(x), \partial_{x'}\phi_{\eta'}(x')] \xrightarrow{L\to\infty} \delta_{\eta\eta'} 2\pi i \left[\frac{a/\pi}{(x-x')^2 + a^2} - \frac{1}{L}\right] \xrightarrow{a\to 0} \boxed{2\pi i\,\left[\delta(x-x') - \frac{1}{L}\right]}$$

Homework: Derive (*), by proving and using the following identities.

$$|\vec{N}\rangle = f(\{b_{q\eta'}^\dagger\})|\vec{N}\rangle_0$$

$$\begin{aligned} \psi_\eta(x) f(\{b_{q\eta'}^\dagger\}) &= f(\{b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\})\,\psi_\eta(x) \\ f(\{b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\}) &= e^{-i\varphi_\eta(x)}\, f(\{b_{q\eta'}^\dagger\})\, e^{i\varphi_\eta(x)}. \end{aligned}$$

Homework: Derive (**), by proving and using $e^{i\varphi_\eta^\dagger(x)} e^{i\varphi_\eta(x)} = e^{i(\varphi_\eta^\dagger + \varphi_\eta)(x)} e^{[i\varphi_\eta^\dagger(x), i\varphi_\eta(x)]/2}$

Bosonization -8: Kinetic Hamiltonian

Linear dispersion $\varepsilon(k) = v_F \hbar k$

$$H_0 \equiv \sum_\eta H_{0\eta} \qquad H_{0\eta} \equiv \sum_{k=-\infty}^{\infty} k \, {}^*_* c^\dagger_{k\eta} c_{k\eta} {}^*_* = \int_{-L/2}^{L/2} \frac{dx}{2\pi} \, {}^*_* \psi^\dagger_\eta(x) i \partial_x \psi_\eta(x) {}^*_*$$

$\downarrow$ **Bosonization form**

$$[H_{0\eta}, b^\dagger_{q\eta'}] = q \, b^\dagger_{q\eta} \, \delta_{\eta\eta'} \longrightarrow H_0 b^\dagger_{q\eta} |E\rangle = (E+q) b^\dagger_{q\eta} |E\rangle$$

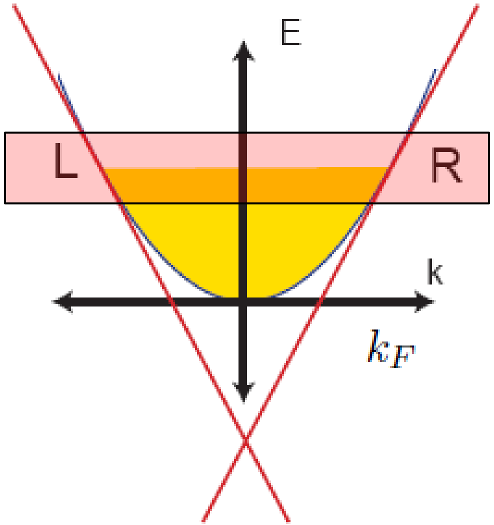
$|\vec{N}\rangle = f(b^\dagger)|\vec{N}\rangle_0$ **span the N-particle Hilbert space**

$$E^{\vec{N}}_{0\eta} = {}_0\langle \vec{N} | H_{0\eta} | \vec{N} \rangle_0 = \frac{2\pi}{L} \frac{1}{2} N_\eta (N_\eta + 1 - \delta_b)$$

$$\longrightarrow \quad \boxed{\begin{aligned} H_{0\eta} &= \sum_{q>0} q \, b^\dagger_{q\eta} b_{q\eta} + \frac{2\pi}{L} \frac{1}{2} \widehat{N}_\eta (\widehat{N}_\eta + 1 - \delta_b) \\ &= \int_{-L/2}^{L/2} \frac{dx}{2\pi} \frac{1}{2} \, {}^*_* (\partial_x \phi_\eta(x))^2 {}^*_* + \left(\frac{2\pi}{L}\right) \frac{1}{2} \widehat{N}_\eta (\widehat{N}_\eta + 1 - \delta_b). \end{aligned}}$$

Bosonization -9: Spinless 1D electrons

1D wire of one left-moving and another right-moving spinless channels



$$\Psi_{phys}(x) \quad \text{“=”} \quad e^{-ik_F x}\tilde{\psi}_L(x) + e^{+ik_F x}\tilde{\psi}_R(x)$$

$$\tilde{\psi}_\nu(x) \equiv \tilde{\psi}_{L/R}(x) \quad \equiv \quad \left(\tfrac{2\pi}{L}\right)^{1/2} \sum_{\boxed{k=-\infty}}^{\infty} e^{\mp ikx} c_{k,L/R} \, .$$

$$\begin{aligned} H_{kin} &= \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_* \left[\psi_L^\dagger(x) i\partial_x \psi_L(x) \; + \; \psi_R^\dagger(x)(-i\partial_x)\psi_R(x) \right] {}^*_* \\ &= \sum_{\nu=L,R} \left[\frac{2\pi}{L}\frac{1}{2}\widehat{N}_\nu^2 + \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_* \frac{1}{2}(\partial_x \phi_\nu(x))^2 {}^*_* \right] = \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_* \frac{1}{2}\left[\tilde{\rho}_L^2 + \tilde{\rho}_R^2\right](x) {}^*_* \end{aligned}$$

$$\tilde{\psi}_{L/R}(x) \quad = \quad a^{-1/2} F_{L/R}\, e^{\mp i\frac{2\pi}{L}(\widehat{N}_{L/R} - \frac{1}{2}\delta_b)x} e^{-i\tilde{\phi}_{L/R}(x)}$$

$$\tilde{\rho}_{L/R}(x) \quad \equiv \quad {}^*_* \tilde{\psi}^\dagger_{L/R}(x)\tilde{\psi}_{L/R}(x) {}^*_* = \pm\partial_x\tilde{\phi}_{L/R}(x) + \frac{2\pi}{L}\widehat{N}_{L/R}$$

Bosonization -10: Interaction Hamiltonian

Short-range inter- (g2) and intra-channel (g4) interactions

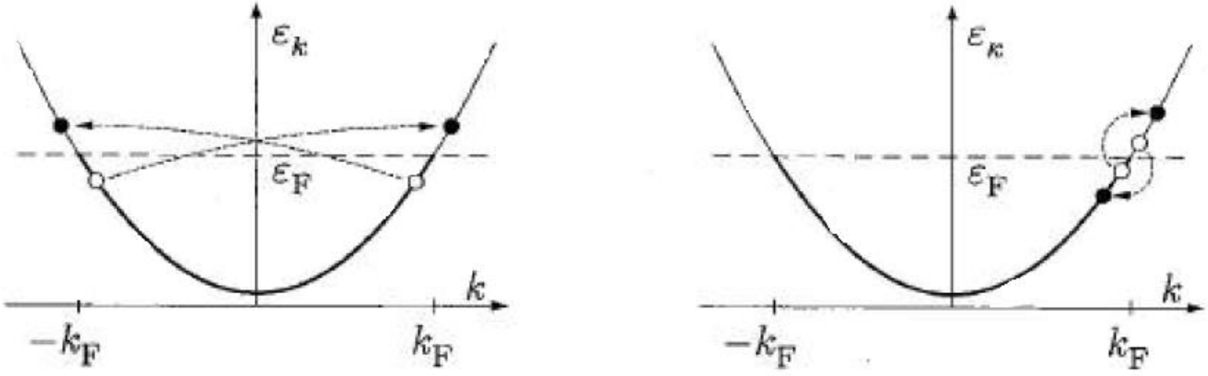
$$H_{kin} = \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^{*}_{*}\left[\psi_L^\dagger(x) i\partial_x \psi_L(x) \; + \; \psi_R^\dagger(x)(-i\partial_x)\psi_R(x)\right]{}^{*}_{*}$$

$$= \sum_{\nu=L,R}\left[\frac{2\pi}{L}\frac{1}{2}\widehat{N}_\nu^2 + \int_{-L/2}^{L/2}\frac{dx}{2\pi}{}^{*}_{*}\frac{1}{2}\left(\partial_x\phi_\nu(x)\right)^2{}^{*}_{*}\right] = \int_{-L/2}^{L/2}\frac{dx}{2\pi}{}^{*}_{*}\frac{1}{2}\left[\tilde\rho_L^2 + \tilde\rho_R^2\right](x){}^{*}_{*}$$

$$H_{int} = \int_{-L/2}^{L/2}\frac{dx}{2\pi}{}^{*}_{*}\left[g_2\,\tilde\rho_L(x)\tilde\rho_R(x) \; + \; \frac{1}{2}g_4\left(\tilde\rho_L^2(x) + \tilde\rho_R^2(x)\right)\right]{}^{*}_{*}$$

$$H_{int} = \frac{1}{L}\sum_{k,k',q,s,s'} V(q) c^\dagger_{k+q,s} c^\dagger_{k'-q,s'} c_{k',s'} c_{k,s}$$

$$g_2(q)\rho_\alpha(q)\rho_{-\alpha}(-q) + g_4(q)\rho_\alpha(q)\rho_\alpha(-q)$$



Other interaction g_1, g_3 terms (problematic for integrability) appear with spins and/or specific filling. For example,

$$e^{i2k_Fx}\tilde\psi_L^\dagger\tilde\psi_R\tilde\rho_\nu \qquad e^{i4k_Fx}\tilde\psi_L^\dagger\tilde\psi_L^\dagger\tilde\psi_R\tilde\psi_R$$

Bosonization -11: Diagonalization

Short-range inter- (g2) and intra-channel (g4) interactions

$$H_0 = \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_* \frac{1}{2}\left[\tilde{\rho}_L^2 + \tilde{\rho}_R^2\right](x) {}^*_* + \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_* \left[g_2\, \tilde{\rho}_L(x)\tilde{\rho}_R(x) + \frac{1}{2} g_4 \left(\tilde{\rho}_L^2(x) + \tilde{\rho}_R^2(x)\right)\right] {}^*_*$$

$$= \frac{v}{4}\int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_* \left[\frac{1}{g}\left(\tilde{\rho}_L + \tilde{\rho}_R\right)^2 + g\left(\tilde{\rho}_L - \tilde{\rho}_R\right)^2\right](x) {}^*_*$$

interaction parameter $g \equiv \left[\frac{1+g_4-g_2}{1+g_4+g_2}\right]^{1/2}$

$$v \equiv \left[(1+g_4)^2 - g_2^2\right]^{1/2}$$

$$= \frac{2\pi}{L}\frac{v}{2}\left\{\left(\frac{1}{g}+g\right)\sum_{\nu=L,R}\left[\frac{1}{2}\widehat{N}_\nu^2 + \sum_q n_q b^\dagger_{q\nu} b_{q\nu}\right] + \left(\frac{1}{g}-g\right)\left[\widehat{N}_L\widehat{N}_R - \sum_q n_q\left(b_{qR}b_{qL} + b^\dagger_{qR}b^\dagger_{qL}\right)\right]\right\}$$

$$= v\frac{2\pi}{L}\sum_{\nu=\pm}\left[g^\nu \widehat{N}_\nu^2 + \sum_q n_q B^\dagger_{q\nu}B_{q\nu}\right]$$

$$\widehat{N}_+ = \frac{1}{2}(\widehat{N}_L - \widehat{N}_R)\,, \quad \widehat{N}_- = \frac{1}{2}(\widehat{N}_L + \widehat{N}_R)$$

$$q = \frac{2\pi}{L}n_q$$

$$= v\sum_{\nu=\pm}\left[\frac{2\pi}{L}g^\nu \widehat{N}_\nu^2 + \int_{-L/2}^{L/2}\frac{dx}{2\pi} {}^*_* \frac{1}{2}\left(\partial_x\Phi_\nu(x)\right)^2 {}^*_*\right] \equiv H_{0+} + H_{0-}$$

$$\Phi_\pm(x) \equiv -\sum_{q>0}\frac{1}{\sqrt{n_q}}e^{-aq/2}\left[e^{-iqx}B_{q\pm} + e^{+iqx}B^\dagger_{q\pm}\right]$$

Bogoliubov transformation $B_{q\pm} = \frac{1}{\sqrt{8}}\left\{\left(\frac{1}{\sqrt{g}} + \sqrt{g}\right)\left(b_{qL} \mp b_{qR}\right) \pm \left(\frac{1}{\sqrt{g}} - \sqrt{g}\right)\left(b^\dagger_{qL} \mp b^\dagger_{qR}\right)\right\}$

$$[B_{q\nu}, B^\dagger_{q'\nu'}] = \delta_{\nu\nu'}\delta_{qq'}$$

Supplement: Another derivation (field theoretical/less rigorous) -1

1. Kac-Moody algebra (anomalous commutators)

$$\rho_R(q) = \sum_{k>0} c_k^\dagger c_{k+q} \qquad \rho_L(q) = \sum_{k<0} c_k^\dagger c_{k+q}$$

(*) $$\boxed{[\rho_R(q), \rho_R(-q')] = \delta_{qq'} \frac{qL}{2\pi} \longrightarrow [a_q, a_{q'}^\dagger] = \delta_{qq'}}$$

$$[\rho_L(q), \rho_L(-q')] = -\delta_{qq'} \frac{qL}{2\pi}$$

$$[\rho_R(q), \rho_L(-q')] = 0$$

$$a_q = \sqrt{\frac{2\pi}{qL}} \rho_R(q) \qquad (q > 0)$$

$$a_q^\dagger = \sqrt{\frac{2\pi}{qL}} \rho_R(-q)$$

$$b_q = \sqrt{\frac{2\pi}{qL}} \rho_L(-q)$$

$$b_q^\dagger = \sqrt{\frac{2\pi}{qL}} \rho_L(q)$$

Plasmon

2. Kinetic Hamiltonian (spinless electrons)

$$[H_0, \rho_R(q)] = -qv_F \rho_R(q)$$

$$\longrightarrow \quad H_0 = \frac{2\pi v_F}{L} \sum_{q>0} [\rho_R(-q)\rho_R(q) + \rho_L(q)\rho_L(-q)]$$ **+ zero-mode parts**

Homework: prove (*).

Supplement: Another derivation (field theoretical/less rigorous) -2

C. L. Kane, M. P. A. Fisher, Phys. Rev. Lett. 68 (1992) 1220

Kane-Fisher notation

$$\theta_{kf}(x) \;:=\; \frac{1}{2\sqrt{\pi}}\left[\tilde{\phi}_L(x) - \tilde{\phi}_R(x)\right]$$

$$\phi_{kf}(x) \;:=\; \frac{1}{2\sqrt{\pi}}\left[\tilde{\phi}_L(x) + \tilde{\phi}_R(x)\right]$$

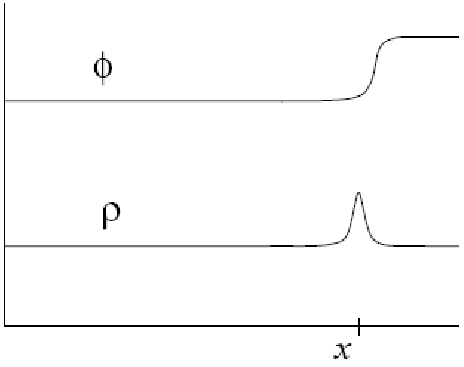
$$[\phi_{kf}(x), \theta_{kf}(x')] = -\tfrac{1}{2} i\epsilon(x - x')$$

$$\partial_x \phi_{kf} \;=\; \partial_t \theta_{kf}$$

$$\epsilon(x) \equiv \begin{cases} \pm 1 & \text{for} \quad x \gtrless 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

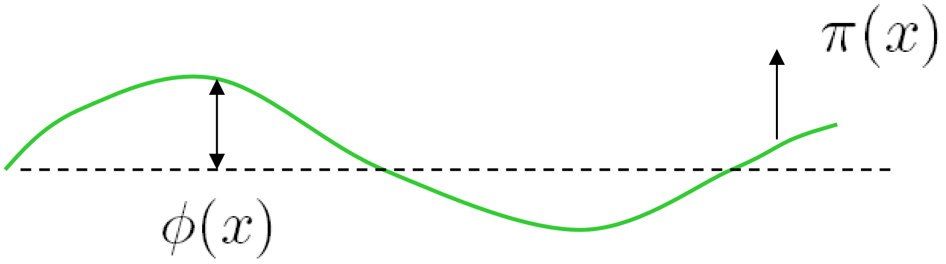
$$\partial_x \theta_{kf}(x) := \frac{1}{2\sqrt{\pi}}[\tilde{\rho}_L(x) + \tilde{\rho}_R(x)]$$ **Total charge density**

$$\partial_x \phi_{kf}(x) := \frac{1}{2\sqrt{\pi}}[\tilde{\rho}_L(x) - \tilde{\rho}_R(x)]$$ **Current density**



$$H_0 = \frac{v}{2}\int_{-L/2}^{L/2} dx \, {}^{*}_{*}\left[\frac{1}{g}(\partial_x \theta_{kf}(x))^2 \;+\; g(\partial_x \phi_{kf}(x))^2\right]{}^{*}_{*}$$

Quantum string



$$H = \int dx \frac{T}{2}(\nabla_x \phi(x))^2 + \frac{1}{2\rho}\pi(x)^2$$

$$[\phi(x), \pi(y)] = i\hbar\delta(x - y)$$

Summary II

Bosonization identity

$$\psi_\eta(x) \;=\; F_\eta\, a^{-1/2} e^{-i\frac{2\pi}{L}(\hat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\phi_\eta(x)}$$

Hamiltonian (spinless electrons)

$$H_0 = \int_{-L/2}^{L/2} \tfrac{dx}{2\pi} \,{}^*_*\, \tfrac{1}{2}\left[\tilde{\rho}_L^2 + \tilde{\rho}_R^2\right](x) \,{}^*_*\; + \int_{-L/2}^{L/2} \tfrac{dx}{2\pi} \,{}^*_*\left[g_2\, \tilde{\rho}_L(x)\tilde{\rho}_R(x) \;+\; \tfrac{1}{2} g_4 \left(\tilde{\rho}_L^2(x) + \tilde{\rho}_R^2(x)\right)\right]{}^*_*$$

$$= \;v \sum_{\nu=\pm}\left[\tfrac{2\pi}{L} g^\nu \widehat{N}_\nu^2 + \int_{-L/2}^{L/2} \tfrac{dx}{2\pi} \,{}^*_*\, \tfrac{1}{2}\left(\partial_x \Phi_\nu(x)\right)^2 \,{}^*_*\right] \;\equiv\; H_{0+} + H_{0-}$$

$$g \equiv \left[\tfrac{1+g_4-g_2}{1+g_4+g_2}\right]^{1/2}$$

$$v \equiv \left[(1+g_4)^2 - g_2^2\right]^{1/2}$$

$$\Phi_\pm(x) \;\equiv\; -\sum_{q>0} \tfrac{1}{\sqrt{n_q}} e^{-aq/2}\left[e^{-iqx} B_{q\pm} + e^{+iqx} B_{q\pm}^\dagger\right]$$

$$B_{q\pm} \;=\; \tfrac{1}{\sqrt{8}}\left\{\left(\tfrac{1}{\sqrt{g}} + \sqrt{g}\right)\left(b_{qL} \mp b_{qR}\right) \pm \left(\tfrac{1}{\sqrt{g}} - \sqrt{g}\right)\left(b_{qL}^\dagger \mp b_{qR}^\dagger\right)\right\}$$

III. Quantum Impurities

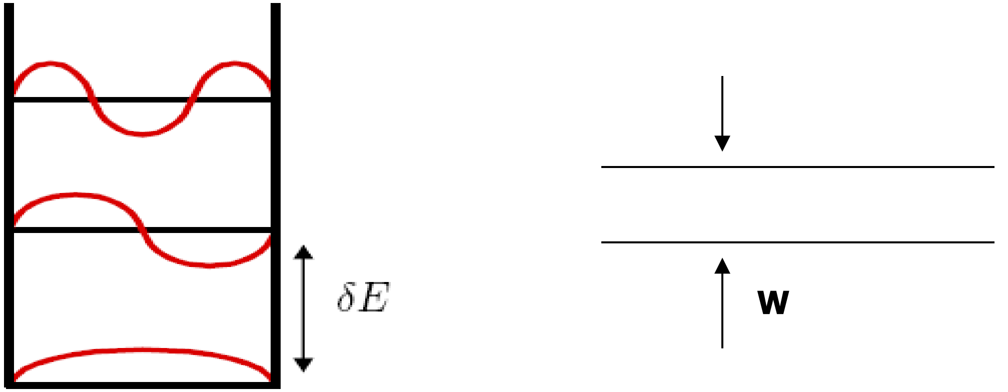
Impurity scattering in Luttinger liquids

Kondo impurity

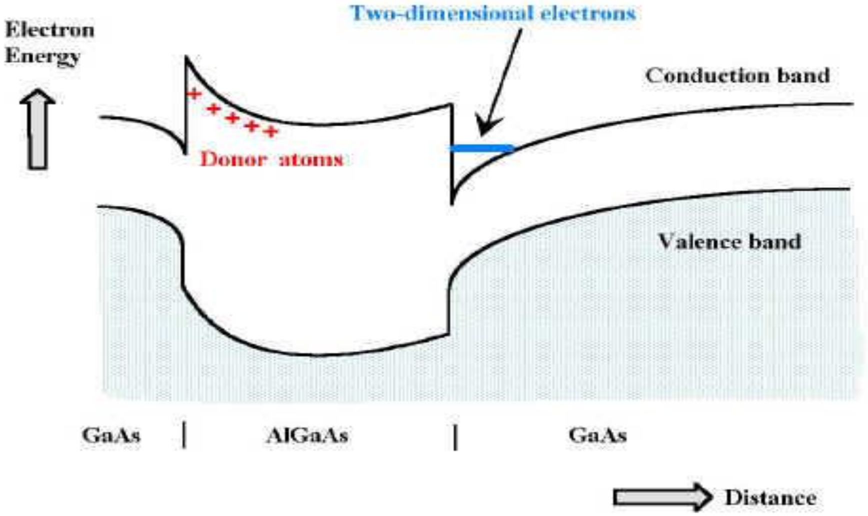
Anyon tunneling

Semiconductor quantum wires

To be 1D: $E < \delta E$

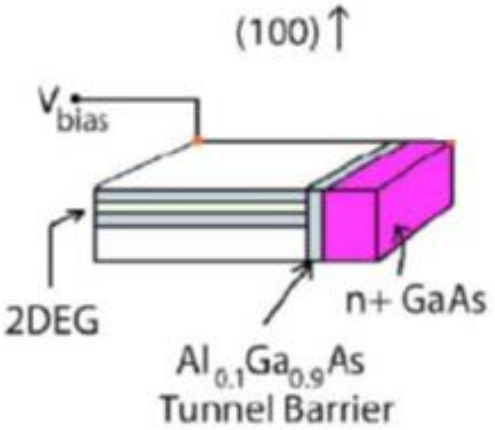


2D electrons



- **Width of triangular well (~ 10nm) < Fermi wave length**
- **High mobility**

Cleaved edge overgrowth technique: 1D electrons



δE **~ 20 meV**

Ref.: Yacoby et al., 1996

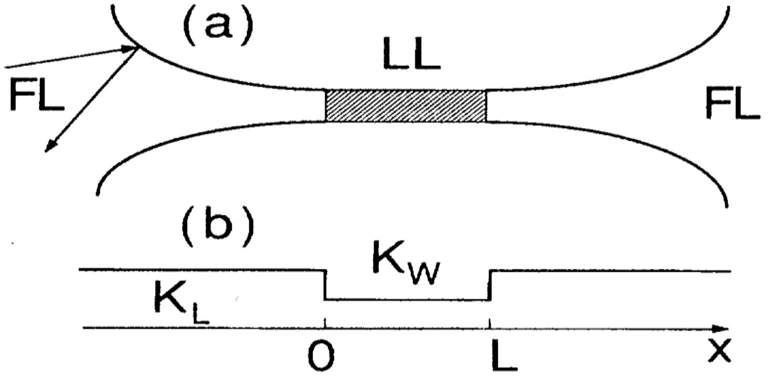
Ballistic conductance

Noninteracting case: Landauer formalism $G = \frac{e^2}{h}$

$$I = \int_0^{eV} dE \underbrace{\frac{dk}{2\pi dE}}_{\text{DOS}} e \underbrace{\frac{dE}{\hbar dk}}_{\text{velocity}} = \frac{e^2}{h} V$$

LL between FL reservoirs : DC conductance is not renormalized by interactions!

Ref.: D. Maslov and AD Stone, 1994



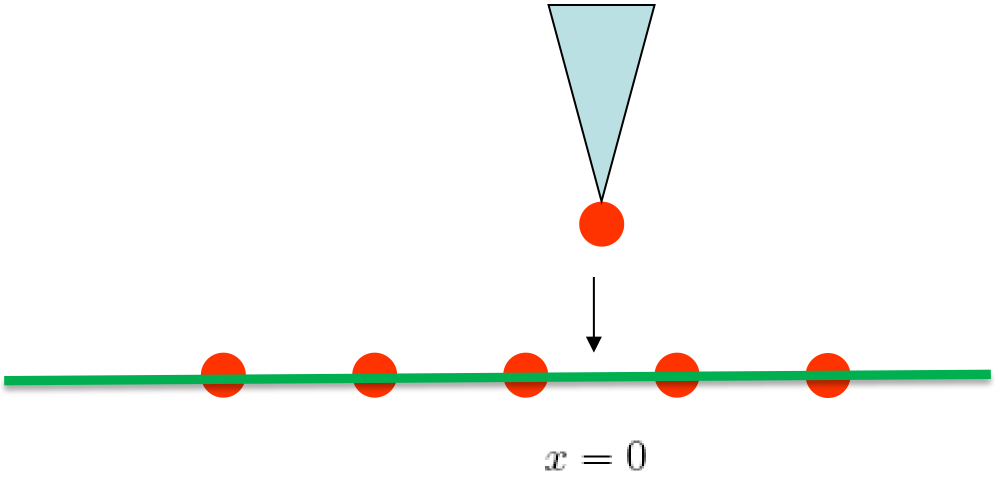
$$G = \frac{e^2}{h}$$

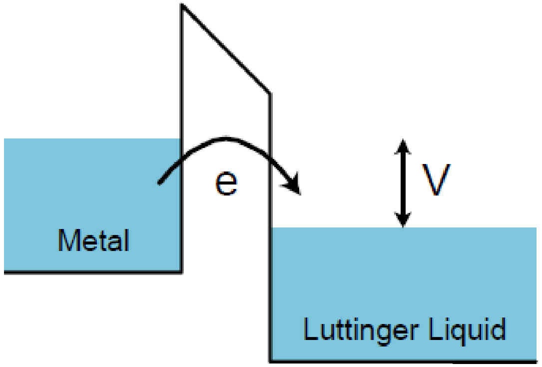
Electron transfer from LL to FL (instead of quasiparticle)

(cf.) Chiral LL along FQHE edges

Tunneling exponent

Interaction parameter g is obtained from the bias dependence of tunneling current.
→ Simplest way of identifying Luttinger liquids in experiments





$$\frac{dI}{dV} \propto A_{LL}(V)A_{FL}(V)$$

$$A(x=0,\omega) \propto |\omega|^{\frac{1}{2}(g+\frac{1}{g}-2)}$$

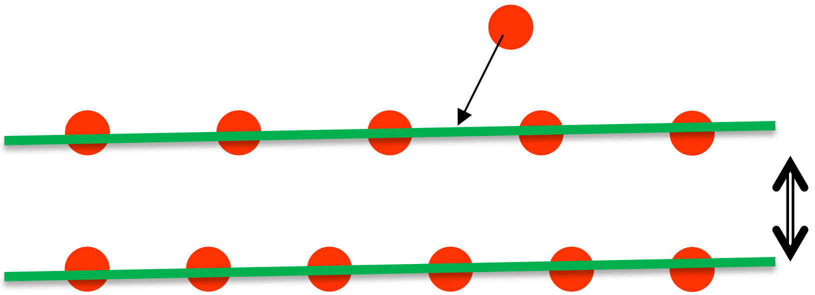
$$\frac{dI}{dV}|_{\mathrm{FL}\to\mathrm{LL}} \propto \boxed{V^{\frac{1}{2}(g+\frac{1}{g}-2)}} \qquad (T \ll V)$$

$g = 1 \longrightarrow$ **non-interacting limit. *dI/dV* is finite at *V = 0***

$g \neq 1 \longrightarrow$ ***dI/dV* vanishes at *V = 0***

Orthogonality catastrophe

$$\langle N+1|\psi^\dagger(x)|N\rangle \to 0 \qquad N \to \infty$$



Rearrangement of all electrons

Tunneling exponent - derivation

Homework: Tunneling density of states at x=0

$$\rho_{dos}(\omega) \quad \equiv \quad \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{i\omega t} \langle G|\Psi_{phys}(t)\Psi^{\dagger}_{phys}(0) + \Psi^{\dagger}_{phys}(0)\Psi_{phys}(t)|G\rangle$$

$$\Psi_{phys}(t) \equiv e^{iHt}\Psi_{phys}(x=0)e^{-iHt}$$

$$D_{phys}(t) \quad \equiv \quad \langle G|\Psi_{phys}(t)\Psi^{\dagger}_{phys}(0)|G\rangle \;=\; D_F(t)D_B(t) \;\sim\; (it)^{-\nu} \quad \text{for } t \to \infty$$

$$\rho_{dos}(\omega) \sim \omega^{\nu-1} \text{ for } \omega \to 0^{+} \qquad\qquad \nu = \tfrac{1}{2}(g + \tfrac{1}{g})$$

$$\Psi_{phys}(x=0) \quad = \quad a^{-1}e^{-\frac{i}{\sqrt{2g}}\Phi_-}\left(F_L e^{-i\sqrt{\frac{g}{2}}\Phi_+} + F_R e^{i\sqrt{\frac{g}{2}}\Phi_+}\right)$$

Ground state of H_{0-}

$$D_F(t) \quad = \quad \langle G_F|e^{iH_-t}\left(e^{-\frac{i}{\sqrt{2g}}\Phi_-(0)}\right)e^{-iH_-t}\left(e^{\frac{i}{\sqrt{2g}}\Phi_-(0)}\right)|G_F\rangle$$

$$\simeq \quad e^{\frac{1}{2g}\langle G_F|\Phi_-(t)\Phi_-(0)-\Phi_-(0)\Phi_-(0)|G_F\rangle} \;=\; (1+ivt/a)^{-\frac{1}{2g}}$$

$$D_B(t) \quad \simeq \quad a^{-1}(1+ivt/a)^{-\frac{g}{2}} + \text{c.c.}$$

1 $\quad L \to \infty$

$$F_\eta(\tau) \equiv e^{H_{0\eta}\tau}F_\eta e^{-H_{0\eta}\tau} = e^{-\frac{2\pi}{L}(\hat{N}_\eta - \delta_b/2)\tau}F_\eta$$

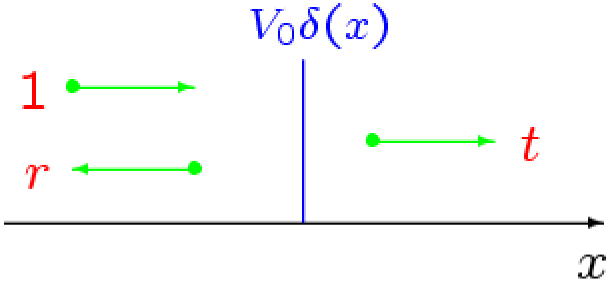
Quantum impurity problem:
Impurity scattering in Luttinger liquids

- Kinetic (ballistic) + Interaction → exactly solvable
- Kinetic (ballistic) + Impurity → exactly solvable

- Kinetic + Interaction + Impurity →
  not analytically solvable in general

Impurity scattering in Luttinger liquids - 1

Current through an impurity potential in a LL:



Noninteracting case:
(Landauer formalism)

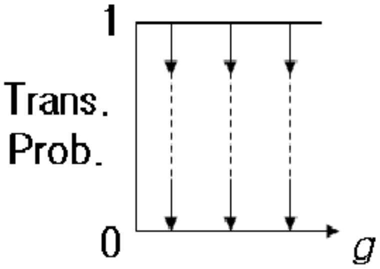
$$G = \frac{I}{V} = \frac{e^2}{h}\frac{1}{1+\left(\frac{V_0}{\hbar v_F}\right)^2}$$

$$0 \leq G \leq \frac{e^2}{h}$$

Interacting case:
(not exactly solvable
at finite temperature)

$$G \propto T^{\frac{2}{g}-2} \ (T \to 0)$$

g<1 (repulsive) : perfect reflection at T=0, LL-end to LL-end tunneling at finite T



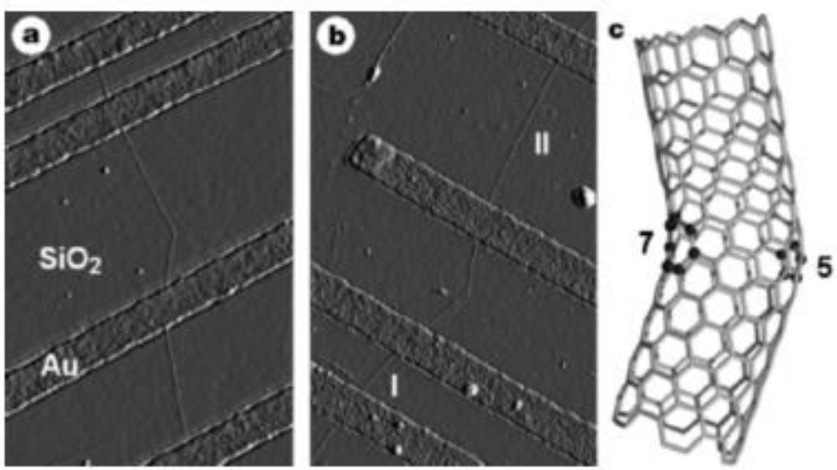
RG flow

Ref.: CL Kane and MPA Fisher, 1992

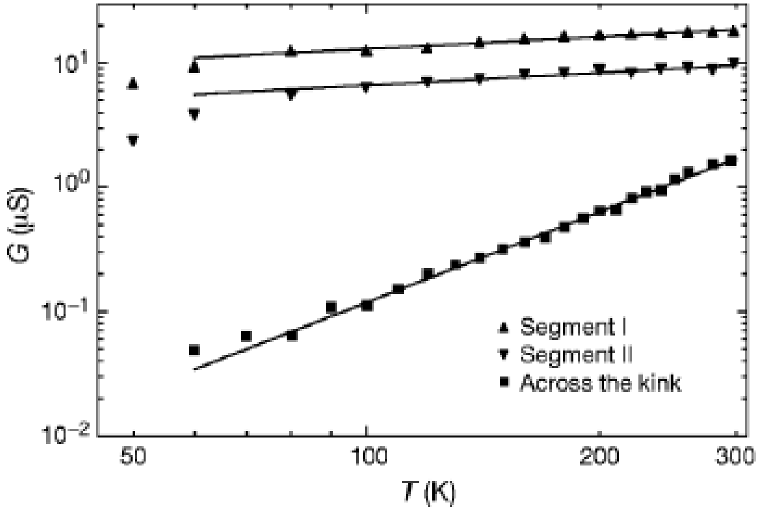
Impurity scattering in Luttinger liquids - 2

Tunneling through a barrier in a single carbon nanotube

Ref.: Yao et al. Nature 1999



$$G(T) \propto T^{\alpha}$$



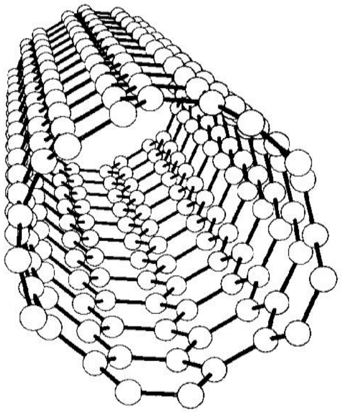
$$\alpha_{\text{bulk}} = (g^{-1} + g - 2)/8$$

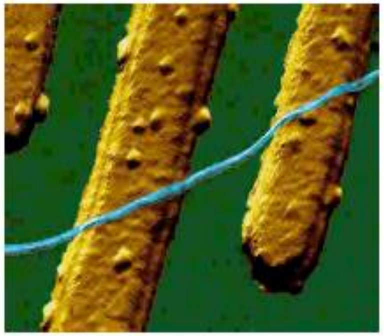
$$\alpha_{\text{end-end}} = (g^{-1} - 1)/2$$

$$g \approx 0.22$$

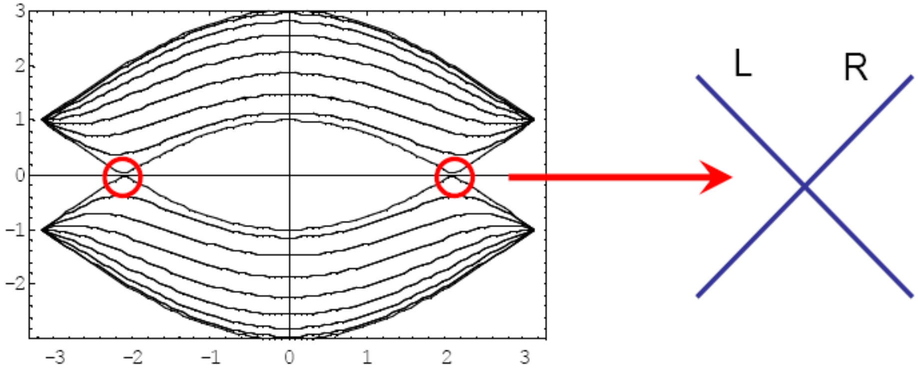
Carbon nanotube

Ref.: Egger and Gogolin PRL 1997





Nanotubes - e.g. Dekker (Delft)



Armchair CNT (metallic)

δE **~ 1 eV**

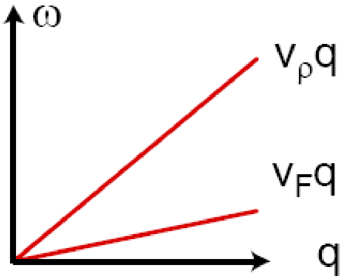
4 channels at Fermi energy: (2 Valleys) * (Spin up, down)

→ 1 charge mode (c), 3 neutral modes (n)

$$H_c = \frac{\hbar v_c}{2} \int dx [g_c P_c^2(x) + \frac{1}{g_c} (\partial_x \phi_c(x))^2]$$

$$H_n = \frac{\hbar v_n}{2} \int dx [g_n P_n^2(x) + \frac{1}{g_n} (\partial_x \phi_n(x))^2]$$

$$g_n \simeq 1 \qquad g_c \simeq 0.2 \qquad v_n = v_F \qquad v_c = \frac{v_F}{g}$$



Spin-charge separation

Impurity scattering in Luttinger liquids – 3 : Refermionization

Spinless Luttinger liquids + Impurity scattering at x=0

1. Forward scattering

$$H_F \quad = \quad \sum_{\nu=L,R} \tfrac{v\lambda_F}{2\pi} {}^*_*\tilde{\psi}^\dagger_\nu(0)\tilde{\psi}_\nu(0){}^*_* = \tfrac{v\lambda_F}{2\pi}\left(\tilde{\rho}_L(0)+\tilde{\rho}_R(0)\right) = \tfrac{v\lambda_F}{2\pi}\sqrt{2g}\,\rho_-$$

$$\rho_\pm \equiv \rho_\pm(0) = \tfrac{1}{\sqrt{2}} g^{\pm\frac{1}{2}}\left(\tilde{\rho}_L(0)\mp\tilde{\rho}_R(0)\right)$$

$$\longrightarrow \quad H_- = H_{0-} + H_F = v\left[\int_{-L/2}^{L/2}\tfrac{dx}{2\pi}\tfrac{1}{2}\,{}^*_*(\partial_x\Phi_-(x))^2{}^*_* \; + \; (\tfrac{2\pi}{L})\tfrac{1}{g}\widehat{N}_-^2 + c_-\left(\partial_x\Phi_- + \tfrac{2\pi}{L}\sqrt{\tfrac{2}{g}}\widehat{N}_-\right)\right]$$

$$c_- \; = \; \tfrac{\lambda_F}{2\pi}(2g)^{\frac{1}{2}}$$

$$H'_- \equiv U_- H_- U_-^{-1} = v\left[\int_{-L/2}^{L/2}\tfrac{dx}{2\pi}\tfrac{1}{2}\,{}^*_*(\partial_x\Phi_-(x))^2{}^*_* \; + \; \tfrac{2\pi}{L}\tfrac{1}{g}\widehat{N}_-^2 + c_-\tfrac{2\pi}{L}\sqrt{\tfrac{2}{g}}\widehat{N}_- - c_-^2\left(\tfrac{1}{a}-\tfrac{\pi}{L}\right)\right]$$

$$U_- \; = \; e^{ic_-\Phi_-}$$

Exactly solvable for arbitrary g

Impurity scattering in Luttinger liquids – 4 : Refermionization

Spinless Luttinger liquids + Impurity scattering at x=0

2. Backward scattering

$$
\begin{aligned}
H_B &= \frac{v\lambda_B}{2\pi}\left[e^{i\theta_B}\tilde{\psi}_L^\dagger(0)\tilde{\psi}_R(0)+e^{-i\theta_B}\tilde{\psi}_R^\dagger(0)\tilde{\psi}_L(0)\right] \\
&= \frac{v\lambda_B}{2\pi a}\left[F_L^\dagger F_R e^{i(\tilde{\phi}_L(0)-\tilde{\phi}_R(0)+\theta_B)}+F_R^\dagger F_L e^{i(\tilde{\phi}_R(0)-\tilde{\phi}_L(0)-\theta_B)}\right] \\
&= \frac{v\lambda_B}{2\pi a}\left[F_L^\dagger F_R e^{i(\sqrt{2g}\Phi_+ +\theta_B)}+F_R^\dagger F_L e^{-i(\sqrt{2g}\Phi_+ +\theta_B)}\right] ,
\end{aligned}
$$

$$\Phi_\pm \equiv \Phi_\pm(0) = \tfrac{1}{\sqrt{2}} g^{\mp\frac{1}{2}}\left(\tilde{\phi}_L(0)\mp\tilde{\phi}_R(0)\right)$$

$$
\begin{aligned}
H_+(\Phi_+) = {}& v\left[\int_{-L/2}^{L/2}\frac{dx}{2\pi}\frac{1}{2}\,{}^*_*(\partial_x\Phi_+(x))^2\,{}^*_*\; + \;(\tfrac{2\pi}{L})\,g\widehat{N}_+^2\right] \\
&+ \frac{v\lambda_B}{2\pi a}\left[\boxed{F_L^\dagger F_R e^{i(\sqrt{2g}\Phi_+ +\theta_B)}}+F_R^\dagger F_L e^{-i(\sqrt{2g}\Phi_+ +\theta_B)}\right]
\end{aligned}
$$

Difficult to solve this, except for g = 1/2

$$\Psi_+(x) \;\equiv\; F_+\frac{1}{\sqrt{a}}e^{-i(\hat{N}_+-\frac{1}{2})\frac{2\pi x}{L}}e^{-i\Phi_+(x)}$$

Impurity scattering in Luttinger liquids – 5 : Refermionization

Backward scattering: exactly solvable only at g = ½, via refermionization

$$H_+(\Phi_+) = v\left[\int_{-L/2}^{L/2}\frac{dx}{2\pi}\frac{1}{2}\,{}^{*}_{*}(\partial_x\Phi_+(x))^2\,{}^{*}_{*} + \left(\frac{2\pi}{L}\right)g\widehat{N}_+^2\right]$$

$$+\ \frac{v\lambda_B}{2\pi a}\left[F_L^\dagger F_R e^{i(\sqrt{2g}\Phi_+ + \theta_B)} + F_R^\dagger F_L e^{-i(\sqrt{2g}\Phi_+ + \theta_B)}\right]$$

At g = ½,

1. "Gluing" two Klein factors into one $F_+ \equiv F_R^\dagger F_L$

→ *F₊* satisfies the conditions for a Klein factor.

$$\{F_+, F_+^\dagger\} = 2$$

$$\widehat{N}_+ = \tfrac{1}{2}(\widehat{N}_L - \widehat{N}_R)$$

$$[\widehat{N}_+, F_+^\dagger] = F_+^\dagger$$

$$[\widehat{N}_-, F_+^\dagger] = [\Phi_\pm(x), F_+^\dagger] = 0$$

2. Introducing an auxiliary fermion
(it differs from the original fermions $c_{k\pm}$)

$$\Psi_+(x)\ \equiv\ F_+\frac{1}{\sqrt{a}}e^{-i(\widehat{N}_+ - \frac{1}{2})\frac{2\pi x}{L}}e^{-i\Phi_+(x)}$$

$$N_+\ \in\ \mathbb{Z} + P/2 \qquad P\ =\ 0\ \text{or}\ 1$$

$$\widehat{\mathcal{N}} \equiv \sum_{\bar{k}}{}^{*}_{*}c_{\bar{k}}^\dagger c_{\bar{k}}{}^{*}_{*} = \widehat{N}_+ - P/2 \qquad \mathcal{N}\in\mathbb{Z}$$

***P=0, 1*: two decoupled subspaces**

$$c_{\bar{k}} \equiv \int_{-L/2}^{L/2}\frac{dx}{(2\pi L)^{1/2}}\,e^{ikx}\,\Psi_+(x)\ = \int_{-L/2}^{L/2}\frac{dx}{(2\pi L)^{1/2}}\,e^{ikx}\,F_+\frac{1}{\sqrt{a}}e^{-i(\widehat{N}_+ - \frac{1}{2})\frac{2\pi x}{L}}e^{-i\Phi_+(x)}$$

Impurity scattering in Luttinger liquids – 6 : Refermionization

3. Writing H_+ in terms of the auxiliary fermion

$$H_+ = H_{0+} + H_B = \sum_{\bar{k}} \varepsilon_{\bar{k}*}^{\;*} c^\dagger_{\bar{k}} c_{\bar{k}*}^{\;*} + \Delta_L \frac{P}{8} + \sqrt{\Delta_L \Gamma} \sum_{\bar{k}} \left(c^\dagger_{\bar{k}} e^{i\theta_B} + c_{\bar{k}} e^{-i\theta_B} \right)$$

$$\Delta_L \equiv v\frac{2\pi}{L}\,, \quad \varepsilon_{\bar{k}} \equiv v\bar{k} = \Delta_L \left(n_{\bar{k}} - \frac{1-P}{2} \right)$$

4. Making $H+$ quadratic in c

$$\Gamma \equiv \frac{v\lambda_B^2}{a(4\pi)^2}$$

$$U_+ \equiv e^{i(\frac{\pi}{2}\widehat{\mathcal{N}}^2 - \theta_B \widehat{\mathcal{N}})}$$

$$H'_+ \equiv U_+ H_+ U_+^{-1} = \Delta_L \frac{P}{8} + \sum_{\bar{k}} \left[\varepsilon_{\bar{k}*}^{\;*} c^\dagger_{\bar{k}} c_{\bar{k}*}^{\;*} + \sqrt{\Delta_L \Gamma} \left(c^\dagger_{\bar{k}} + c_{\bar{k}} \right) \left(i\sqrt{2}\,\alpha_d \right) \right]$$

Majorana fermion $$\alpha_d \equiv \frac{1}{\sqrt{2}} e^{i\pi\widehat{\mathcal{N}}}$$

$$\alpha_d^\dagger = \alpha_d \qquad \{\alpha_d, \alpha_d\} = 1$$

$$\{F_+, \alpha_d\} = \{F_+^\dagger, \alpha_d\} = \{c_{\bar{k}}, \alpha_d\} = \{c^\dagger_{\bar{k}}, \alpha_d\} = 0$$

$$U_+ \left(F_+^\dagger e^{i\theta_B} \right) U_+^{-1} = F_+^\dagger\, e^{i\pi(\widehat{\mathcal{N}} + \frac{1}{2})} = F_+^\dagger (i\sqrt{2}\,\alpha_d)$$

$$U_+ \left(c^\dagger_{\bar{k}} e^{i\theta_B} \right) U_+^{-1} = c^\dagger_{\bar{k}} (i\sqrt{2}\,\alpha_d)$$

Impurity scattering in Luttinger liquids – 7 : Refermionization

5. Diagonalization

$$H'_+ \equiv U_+ H_+ U_+^{-1} = \Delta_L \frac{P}{8} + \sum_{\bar{k}} \left[\varepsilon_{\bar{k}}{}^{*}_{*} c^\dagger_{\bar{k}} c_{\bar{k}}{}^{*}_{*} + \sqrt{\Delta_L \Gamma} \left(c^\dagger_{\bar{k}} + c_{\bar{k}} \right) \left(i\sqrt{2}\, \alpha_d \right) \right]$$

$$\downarrow$$

$$\begin{pmatrix} \alpha_{\bar{k}} \\ \beta_{\bar{k}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} c_{\bar{k}} \\ c^\dagger_{-\bar{k}} \end{pmatrix}$$

$$\{\alpha_n, \alpha_{-n'}\} = \delta_{nn'}, \quad \{\beta_{\bar{k}}, \beta_{-\bar{k}'}\} = \delta_{\bar{k}\bar{k}'}, \quad \{\alpha_n, \beta_{\bar{k}'}\} = 0$$

$$\alpha^\dagger_n = \alpha_{-n}, \quad \beta^\dagger_{\bar{k}} = \beta_{-\bar{k}}$$

$$H'_+ = \Delta_L \tfrac{P}{8} + \sum_{\bar{k}>0} \varepsilon_{\bar{k}} \left(\alpha^\dagger_{\bar{k}} \alpha_{\bar{k}} + \beta^\dagger_{\bar{k}} \beta_{\bar{k}} \right) + i\, 2\sqrt{\Delta_L \Gamma} \sum_{\bar{k}>0} \left(\alpha^\dagger_{\bar{k}} + \alpha_{\bar{k}} \right) \alpha_d$$

$$\varepsilon_{\bar{k}} \equiv v\bar{k}$$

$$\downarrow$$

$$\tilde{\alpha}_\varepsilon = \sum_{n=d,\bar{k}} A^\dagger_{\varepsilon n}\, \alpha_n$$

$$H'_+ \equiv \sum_{\varepsilon>0} \varepsilon\, \tilde{\alpha}^\dagger_\varepsilon \tilde{\alpha}_\varepsilon \; + \; \sum_{\bar{k}>0} \varepsilon_{\bar{k}}\, \beta^\dagger_{\bar{k}} \beta_{\bar{k}} \; + \; E'_G$$

$$A_{\bar{k}\varepsilon} \;=\; (A^\dagger_{\varepsilon\bar{k}})^* = \frac{i2\sqrt{\Delta_L \Gamma}\, A_{d\varepsilon}}{\varepsilon - \varepsilon_{\bar{k}}}$$

$$A_{d\varepsilon} \;=\; (A^\dagger_{\varepsilon d})^* = -i\, \mathrm{sgn}(\varepsilon) \left[\frac{4\Delta_L \Gamma}{4\Delta_L\Gamma + \varepsilon^2 + (4\pi\Gamma)^2} \right]^{1/2}$$

$$\mathrm{sgn}(\varepsilon = 0) \equiv i$$

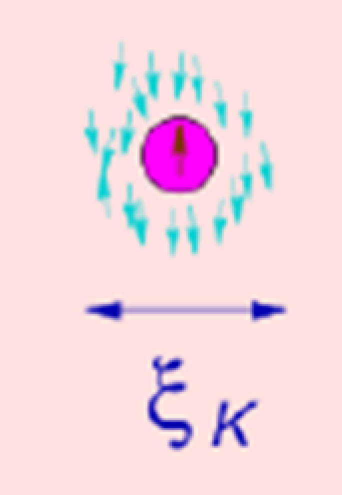
$$\tilde{\alpha}_0 = \tilde{\alpha}^\dagger_0 \quad \longrightarrow \quad \alpha_d \text{ as } \Gamma \to 0$$

Ground state

$$\langle G'_B | \beta_{\bar{k}} \beta^\dagger_{\bar{k}'} | G'_B \rangle = \delta_{\bar{k}\bar{k}'} \theta(\varepsilon_{\bar{k}})$$

$$\langle G'_B | \tilde{\alpha}_\varepsilon \tilde{\alpha}^\dagger_{\varepsilon'} | G'_B \rangle = \delta_{\varepsilon\varepsilon'} \theta(\varepsilon)$$

Quantum impurity problem: Kondo physics



$$H = \sum_{\vec{k}\sigma} \Psi^\dagger_{\vec{k}\sigma} \Psi_{\vec{k}\sigma} \epsilon_k + J \vec{S}_{\rm imp} \cdot \vec{S}_{\rm el}(r=0)$$

$$\vec{S}_{\rm el}(\vec{r}) \equiv \Psi^\dagger_\alpha \frac{\vec{\sigma}_{\alpha\beta}}{2} \Psi_\beta(\vec{r})$$

Kondo screening cloud

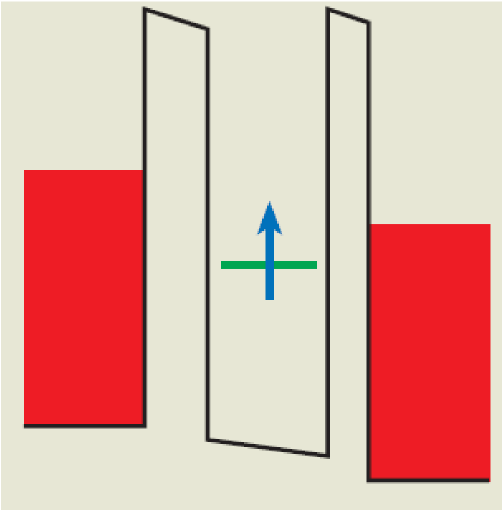
$$\xi_K = \hbar v_F/(k_B T_K)$$

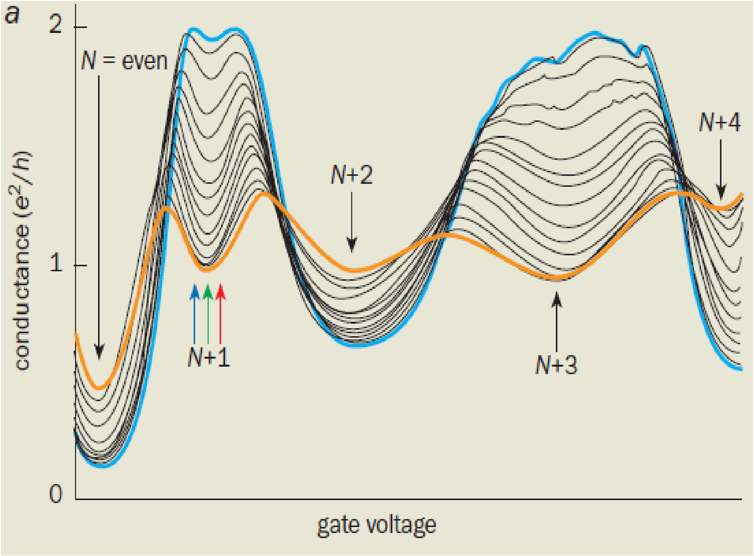
$$T_K \sim 1 \text{ K} \qquad v_F \sim 10^5\text{–}10^6 \text{ m/s} \longrightarrow \xi_K \sim 1 \ \mu\text{m}$$

$$|G\rangle = \frac{1}{\sqrt{2}} \left(|\Uparrow\rangle |g_{N_s=-1/2}\rangle + |\Downarrow\rangle |g_{N_s=1/2}\rangle \right)$$

Entanglement in Kondo: Lee, Park, Sim, preprint (2014)
How to detect Kondo cloud: Park, Lee, Oreg, Sim, PRL (2013)

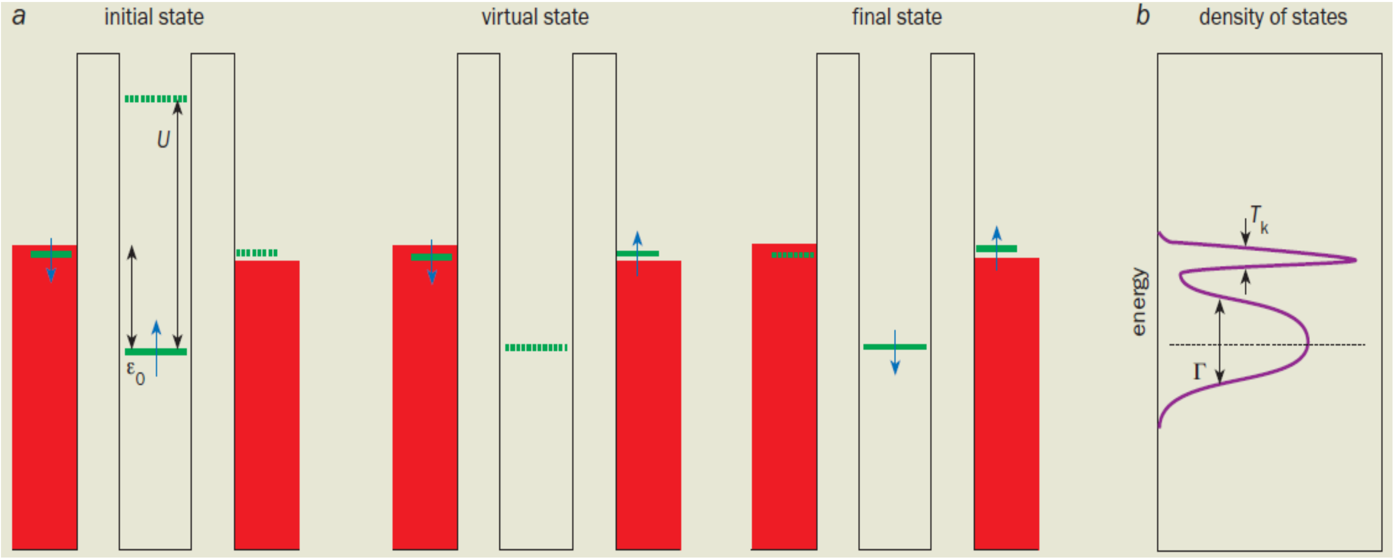
Kondo effect in a quantum dot





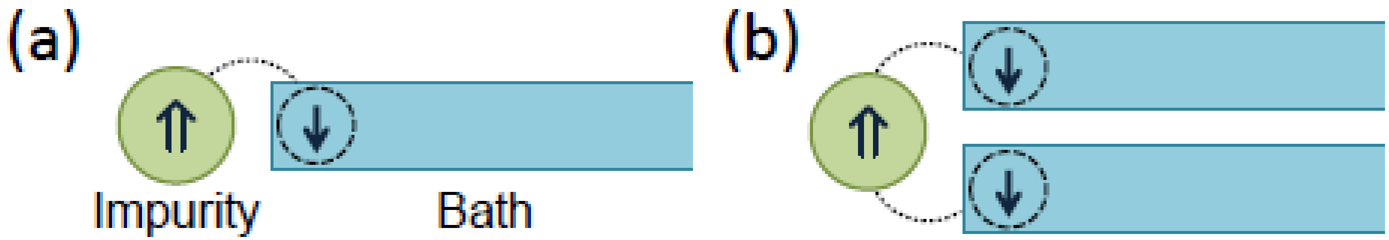
Kouwenhoven & Glazman (2001)

Goldhaber-Gordon et al. (1998)
Cronenwett et al. (1998))
Van der Wiel et al. (2000)



Scaling collapse
(single energy scale)
"Kondo temperature"
T_K

Two-channel Kondo effect



Two channel Kondo

$$H_{2CK} = J_1 \mathbf{s}_1 \cdot \mathbf{S} + J_2 \mathbf{s}_2 \cdot \mathbf{S} + H_{\text{reservoirs}}$$

- **At T=0 :**

The emergence of Majorana fermions (non-Fermi liquid behavior), by the competition of spin screening between the two channels

Majorana fermions

$$\Gamma_0 = \Gamma_0^\dagger$$

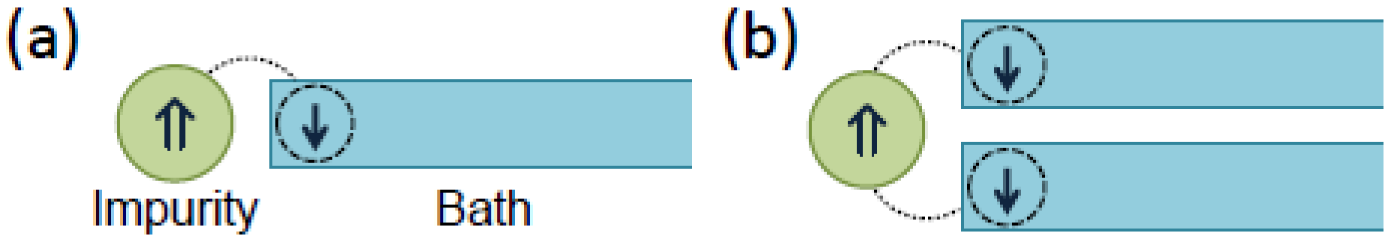
antiparticle = particle

$$\gamma_1 = \Psi + \Psi^\dagger \qquad \Psi = \gamma_1 + i\gamma_2 \qquad \gamma_i^2 = 1$$

$$\gamma_2 = -i(\Psi - \Psi^\dagger) \qquad \Psi^\dagger = \gamma_1 - i\gamma_2 \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

Two Majoranas = one electron

Kondo effect: Bosonization - 1



Impurity spin + non-interacting reservoir electrons + interaction between the impurity spin and the reservoir electrons

→ A nontrivial problem not exactly solvable.

Steps:
(i) Bosonize the baths
(ii) Identify symmetries and conserved quantities
(iii) Convert the impurity spin into a pseudo-fermion (similar to Wigner-Jordan), with certain (Emery-Kivelson) transformation.
(iv) Refermionize the total Hamiltonian

Kondo effect: Bosonization – 2 (Single-channel case)

(i) Bosonize the baths

$$\mathcal{H}_{\mathrm{K}} = 2J_z S_z s_z + J_\perp (S_+ s_- + \mathrm{H.c.}) + \sum_{k\sigma} \tilde{\epsilon}_{k\sigma} f^\dagger_{k\sigma} f_{k\sigma}$$

$$H_0 = \sum_{k\alpha j} k : c^\dagger_{k\alpha j} c_{k\alpha j} : \; = \sum_{\alpha j} \frac{\Delta_L}{2} \hat{N}_{\alpha j}(\hat{N}_{\alpha j} + 1 - P_0) + \sum_{\substack{\alpha j \\ q>0}} q \, b^\dagger_{q\alpha j} b_{q\alpha j}$$

$$\alpha = (\uparrow, \downarrow) = (+, -)$$

$$j = (1, 2) = (+, -)$$

(ii) Identify conserved quantities

$$\frac{1}{2\pi} : \psi^\dagger_{\alpha j}(x) \psi_{\alpha j}(x) := \frac{1}{2\pi} \partial_x \phi_{\alpha j}(x) + \hat{N}_{\alpha j}/L$$

$$\mathcal{N}_c \qquad\qquad S_T = \mathcal{N}_s + S_z$$

Total charge **Total spin (incl. impurity)**

$$\mathcal{S}_{\mathrm{phys}}(S_T, \mathcal{N}_c) \equiv \{ |\mathcal{N}_c, S_T - \tfrac{1}{2}; \Uparrow\rangle \oplus |\mathcal{N}_c, S_T + \tfrac{1}{2}; \Downarrow\rangle \}$$

$$\longrightarrow$$

$$H_0 = \Delta_L \left[\hat{\mathcal{N}}_c (1 - P_0) + \hat{\mathcal{N}}_c^2 + \hat{\mathcal{N}}_s^2 \right] \; + \sum_{q>0} q \, (b^\dagger_{qc} b_{qc} + b^\dagger_{qs} b_{qs})$$

$$H_z = \lambda_z \left[\partial_x \varphi_s(0)/\sqrt{2} + \Delta_L \hat{\mathcal{N}}_s \right] S_z \qquad\qquad H_\perp = \frac{\lambda_\perp}{2a} \left[\boxed{e^{-i\sqrt{2}\varphi_s(0)} S_+ F^\dagger_\downarrow F_\uparrow} + \mathrm{h.c.} \right]$$

Kondo effect: Bosonization – 3 (Single-channel case)

(iii) Emery-Kivelson transformation

$$H_0 = \Delta_L\left[\hat{\mathcal{N}}_c(1-P_0)+\hat{\mathcal{N}}_c^2+\hat{\mathcal{N}}_s^2\right] + \sum_{q>0} q\,(b^\dagger_{qc}b_{qc}+b^\dagger_{qs}b_{qs})$$

$$H_z = \lambda_z\left[\partial_x\varphi_s(0)/\sqrt{2}+\Delta_L\hat{\mathcal{N}}_s\right] S_z$$

$$H_\perp = \frac{\lambda_\perp}{2a}\left[\boxed{e^{-i\sqrt{2}\varphi_s(0)}S_+F^\dagger_\downarrow F_\uparrow} + \text{h.c.}\right]$$

EK

$$H \rightarrow H' = UHU^\dagger$$

$$U \equiv e^{i\gamma S_z\varphi_s(0)}\ .$$

$$\gamma \equiv \lambda_z/\sqrt{2}$$

$$S_\pm \rightarrow US_\pm U^\dagger = e^{\pm i\gamma\varphi_s(0)}S_\pm$$

$$U(H_0+H_z)U^{-1} = H_0 + (\lambda_z/\sqrt{2}-\gamma)\partial_x\varphi_s(0)S_z$$
$$+\lambda_z\Delta_L\hat{\mathcal{N}}_s\ S_z + \text{const}$$

$$H'(\lambda_\perp=0) = \Delta_L\left[\hat{\mathcal{N}}_c(\mathcal{N}_c+1-P_0)+\hat{\mathcal{N}}_s^2+\lambda_z\hat{\mathcal{N}}_s S_z\right] + \sum_{q>0} q\,(b^\dagger_{qc}b_{qc}+b^\dagger_{qs}b_{qs}) + \text{const}$$

$$H_\perp = \frac{\lambda_\perp}{2a}\left[\ e^{-i(\sqrt{2}-\lambda_z/\sqrt{2})\varphi_s(0)}\ S_+F^\dagger_\downarrow F_\uparrow + \text{h.c.}\right]$$

Kondo effect: Bosonization – 4 (Single-channel case)

$$H'(\lambda_\perp = 0) = \Delta_L \left[\hat{\mathcal{N}}_c(\mathcal{N}_c + 1 - P_0) + \hat{\mathcal{N}}_s^2 + \lambda_z \hat{\mathcal{N}}_s S_z \right] + \sum_{q>0} q\,(b_{qc}^\dagger b_{qc} + b_{qs}^\dagger b_{qs}) \ + \text{const}$$

$$H_\perp = \frac{\lambda_\perp}{2a} \left[e^{-i(\sqrt{2} - \lambda_z/\sqrt{2})\varphi_s(0)}\, S_+ F_\downarrow^\dagger F_\uparrow + \text{h.c.} \right]$$

Toulouse point $\lambda_z^* \equiv 2 - \sqrt{2}\,, \qquad \gamma^* \equiv \sqrt{2} - 1$

$$\longrightarrow \qquad H'_\perp = \frac{\lambda_\perp}{2a} (S_+ F_\downarrow^\dagger F_\uparrow \boxed{e^{-i\varphi_s(0)}} + \text{h.c.})$$

Refermionizable

Another transformation for correct statistics between pseudofermions

$$U_2 H' \, U_2^{-1} \qquad U_2 = e^{i\pi \hat{\mathcal{N}}_s \, S_z}$$

$$c_d^\dagger \equiv S_+ e^{i\pi(\hat{\mathcal{N}}_s - S_z)}$$

$$c_d^\dagger c_d = S_z + 1/2$$

$$\psi_s(x) \equiv \frac{F_\downarrow^\dagger F_\uparrow}{a^{1/2}} e^{-i(\hat{\mathcal{N}}_s - \text{sgn}(S_T)[1+P]/2)2\pi x/L - i\varphi_s(x)}$$

$$\equiv \sqrt{2\pi/L} \sum_{\bar{k}} c_{\bar{k}s} e^{-i\bar{k}x} \,,$$

$$\{c_{\bar{k}s}, c_{\bar{k}'s}^\dagger\} = \delta_{\bar{k}\bar{k}'} \,,$$

$$\{c_d, c_d^\dagger\} = 1 \,,$$

$$\{c_d, c_{\bar{k}s}^\dagger\} = \{c_d, c_{\bar{k}s}\} = 0$$

$$[c_d, \hat{\mathcal{N}}_s] = c_d \,.$$

(iv) Refermionize

$$U_2 H' U_2^{-1} = \sum_q q\, b_{qc}^\dagger b_{qc} \ + : \sum_{\bar{k}} \bar{k} :c_{\bar{k}s}^\dagger c_{\bar{k}s}: \ + \varepsilon_d :c_d^\dagger c_d: \ + \sqrt{\Delta_L \Gamma} \sum_{\bar{k}} (c_{\bar{k}s}^\dagger c_d + c_d^\dagger c_{\bar{k}s})$$

Diagonalizable

(v) Study the resulting energy spectrum, with changing $\Gamma \equiv \lambda_\perp^2/4a$ **toward the low-energy fixed point**

Kondo effect: Bosonization – 5 (Two-channel case)

$$\mathcal{H}_{\rm K} = 2J_z S_z s_z + J_\perp (S_+ s_- + \text{H.c.}) + \sum_{k\sigma} \tilde{\epsilon}_{k\sigma} f^\dagger_{k\sigma} f_{k\sigma}$$

(i) Bosonize the baths

$$H_0 = \sum_{k\alpha j} k : c^\dagger_{k\alpha j} c_{k\alpha j} : \; = \sum_{\alpha j} \frac{\Delta_L}{2} \hat{N}_{\alpha j}(\hat{N}_{\alpha j} + 1 - P_0) + \sum_{\substack{\alpha j \\ q>0}} q \, b^\dagger_{q\alpha j} b_{q\alpha j}$$

(ii) Identify conserved quantities

$$\begin{pmatrix} \hat{\mathcal{N}}_c \\ \hat{\mathcal{N}}_s \\ \hat{\mathcal{N}}_f \\ \hat{\mathcal{N}}_x \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{N}_{\uparrow 1} \\ \hat{N}_{\downarrow 1} \\ \hat{N}_{\uparrow 2} \\ \hat{N}_{\downarrow 2} \end{pmatrix}$$

$\mathcal{N}_c$ $\quad$ $\ddot{\mathcal{N}}_f$ $\quad$ $S_T = \mathcal{N}_s + S_z$

Total charge **Total flavor** **Total spin (incl. impurity)**

$$\mathcal{S}_{\rm phys}(S_T, \mathcal{N}_c, \mathcal{N}_f) = \boxed{\sum_{\oplus' \mathcal{N}_x}} \Big\{ |\mathcal{N}_c, S_T - 1/2, \mathcal{N}_f, \mathcal{N}_x; \Uparrow\rangle \oplus |\mathcal{N}_c, S_T + 1/2, \mathcal{N}_f, \mathcal{N}_x + 1; \Downarrow\rangle \Big\}$$

Cf. 1CK $\mathcal{S}_{\rm phys}(S_T, \mathcal{N}_c) \equiv \{ |\mathcal{N}_c, S_T - \frac{1}{2}; \Uparrow\rangle \oplus |\mathcal{N}_c, S_T + \frac{1}{2}; \Downarrow\rangle \}$

Kondo effect: Bosonization – 6 (Two-channel case)

(iii) After appropriate EK transformation,

$$H'(\lambda_\perp=0) = \lambda_z \Delta_L \hat{\mathcal{N}}_s S_z + \sum_y \Delta_L \hat{\mathcal{N}}_y^2/2 \;+ \sum_{y,\,q>0} q\, b^\dagger_{qy} b_{qy} + H_h + \text{const}$$

$$H'_\perp = \frac{\lambda_\perp}{2a}\left[S_+\left(F^\dagger_{\downarrow 1}F_{\uparrow 1}e^{-i\varphi_x(0)} + F^\dagger_{\downarrow 2}F_{\uparrow 2}e^{i\varphi_x(0)}\right) + \text{h.c.}\right]$$

$$c_d \equiv F_s^\dagger S_-$$

$$c_d^\dagger c_d = S_z + 1/2$$

(iv) After refermionization,

$$H' = \sum_{c,s,f}\sum_{q>0} q\, b^\dagger_{qy} b_{qy} \;+ \sum_{\bar k} \bar k :c^\dagger_{\bar k x} c_{\bar k x}: \;+ \sqrt{\Delta_L\Gamma}\sum_{\bar k}(c^\dagger_{\bar k x} + c_{\bar k x})\underset{\uparrow}{(c_d - c_d^\dagger)}$$

Majorana fermion from the impurity

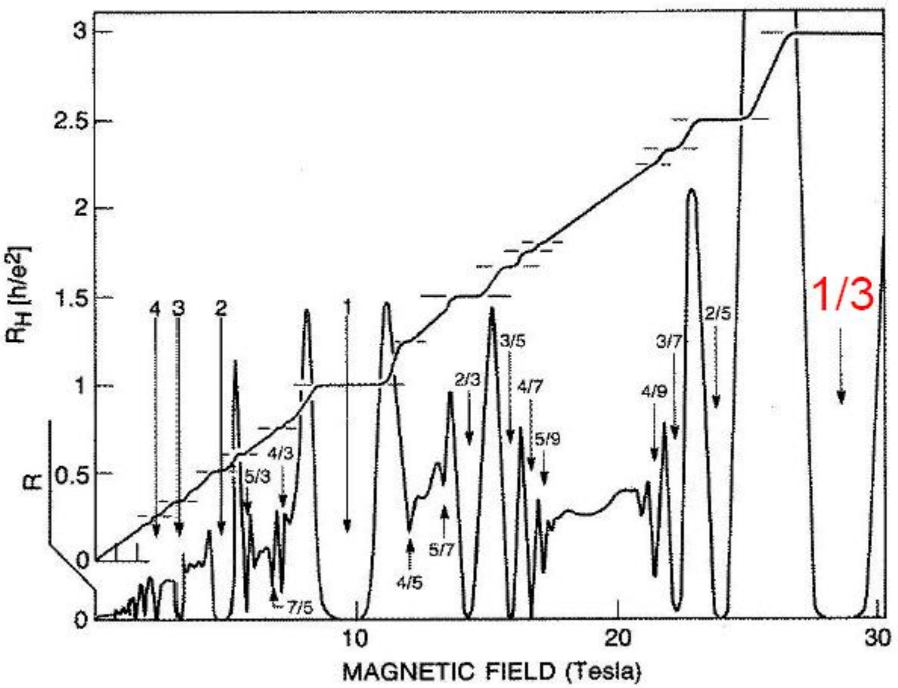
Another Majorana fermion $c_d + c_d^\dagger$ **is decoupled from the baths!**

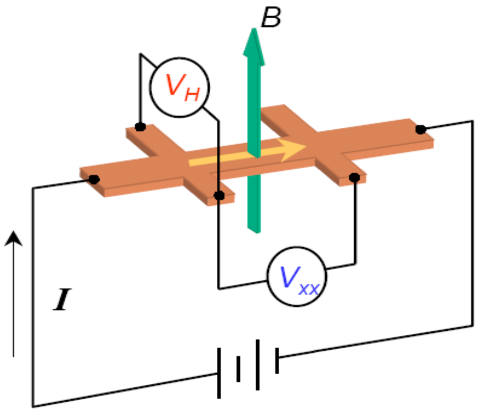
→ Non-Fermi liquid

Cf. 1CK $$U_2 H' U_2^{-1} = \sum_q q\, b^\dagger_{qc} b_{qc} \;+ {:}\sum_{\bar k} \bar k :c^\dagger_{\bar k s} c_{\bar k s}: \;+ \varepsilon_d :c_d^\dagger c_d: \;+ \sqrt{\Delta_L\Gamma}\sum_{\bar k}(c^\dagger_{\bar k s} c_d + \underset{\nearrow}{c_d^\dagger} c_{\bar k s})$$

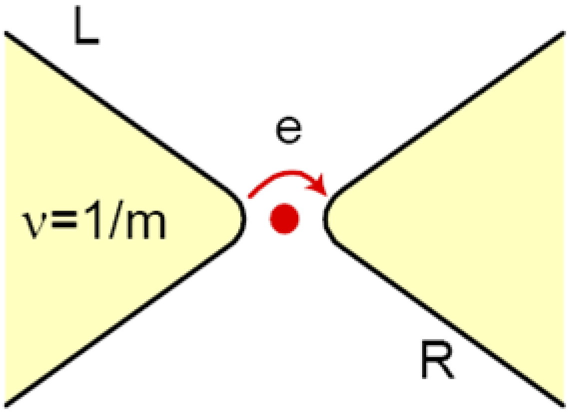
Fermi liquid **complex fermion**

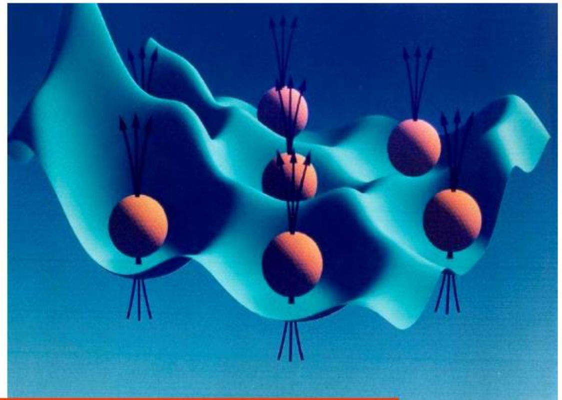
Quantum impurity problem:
Tunneling between fractional quantum Hall edge states





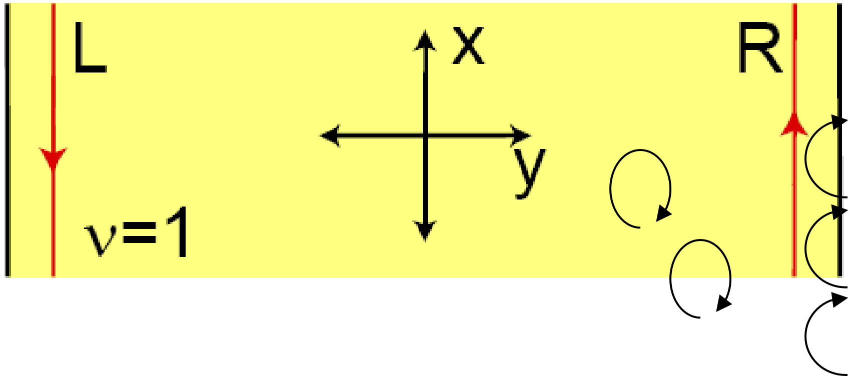
$R_H = V_H / I = B/ne$

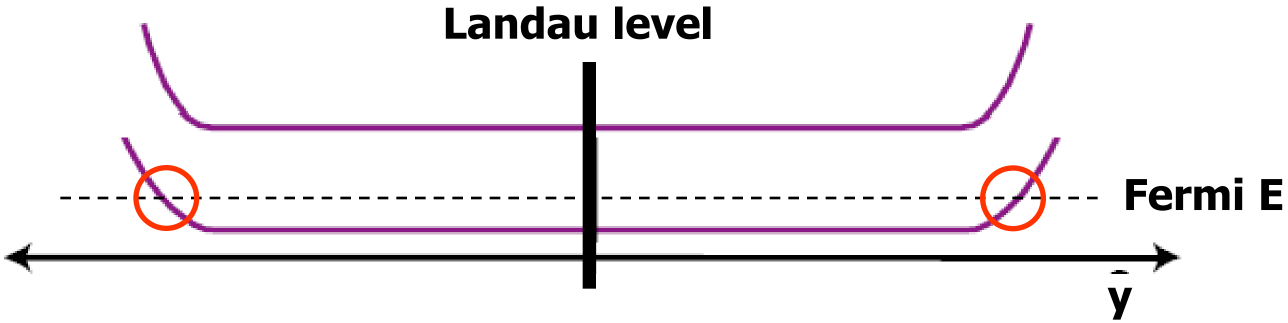




A bizarre fluid state of many electrons having *fractionally charged excitations.*

Integer Quantum Hall edge states





$$E_n = \hbar\omega_c(n + \frac{1}{2})$$

Fermi liquid behavior : $g = 1$

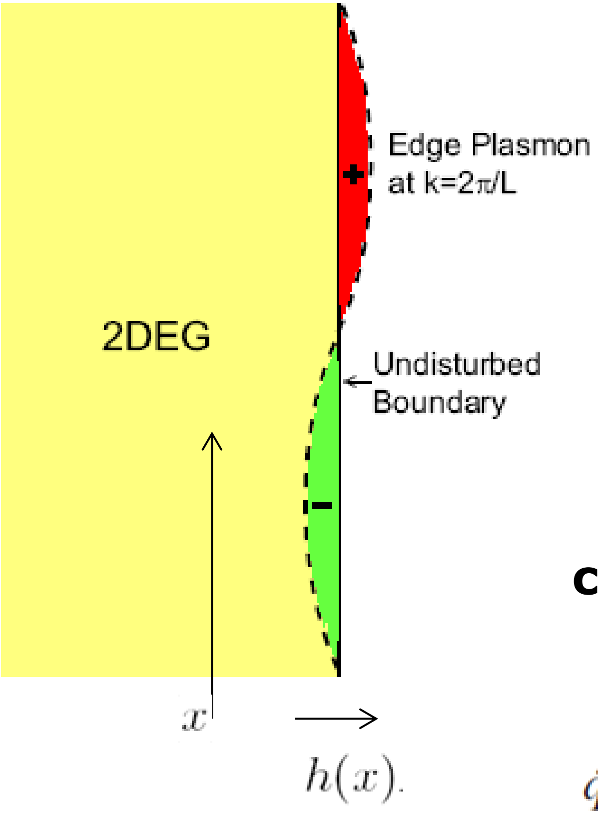
even with interactions

(short-range interactions renormalize only Fermi velocity.)

Fractional Quantum Hall edge: Chiral LL

Fractional quantum Hall edge : Chiral Luttinger liquid (Laughlin quasiptls)

Ref.: X.-G. Wen, PRL 1990



$$H = \int dx \frac{1}{2} e\rho(x)h(x)E \quad = \pi\hbar\frac{v}{\nu}\int dx[\rho(x)]^2$$

$$= 2\pi\hbar\frac{v}{\nu}\sum_{q>0}\rho_q\rho_{-q}$$

$$h(x) = \frac{\rho(x)}{n}$$

$$v = (E/B)$$

canonical quantization

$$q_q = \rho_q$$

$$p_q = \frac{ih}{\nu q}\rho_{-q}$$

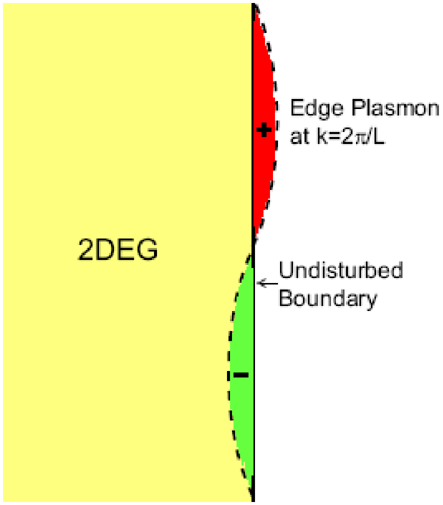
$$\dot{q}_q = \frac{\partial H}{\partial p_q}, \quad \dot{p}_q = -\frac{\partial H}{\partial q_q}$$

$$[q_q, p_{q'}] = i\hbar\,\delta_{q,q'}$$

$$[\rho_q, \rho_{q'}] = \frac{\nu}{2\pi} q\delta_{q,-q'}$$

$$\boxed{g = \nu = \frac{1}{m}}$$

Fractional Quantum Hall edge: Chiral LL -2



$$[\rho_q, \rho_{q'}] = \frac{\nu}{2\pi} q \delta_{q,-q'}$$

$$[\rho(x), \rho(x')] = i\sigma_H \partial_x \delta(x-x')$$

$$\partial_x \phi = 2\pi\hbar\rho$$

$$\longrightarrow \qquad [\phi(x), \phi(x')] = -i\pi\nu\, \mathrm{sgn}(x-x')$$

$$[\rho(x), \psi(x')] = -\delta(x-x')\psi(x)$$

electron operator

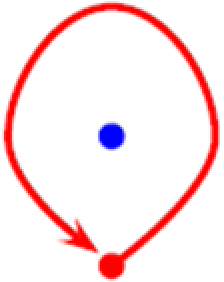
$$\longrightarrow \quad \psi(x) \propto e^{i(1/\nu)\,\phi}$$

$\psi(x)\psi(x') = (-1)^{1/\nu}\psi(x')\psi(x) \longrightarrow$ $1/\nu$ to be an odd integer

electron operator
(bunching of quasiparticles) $\psi_e \sim e^{im\phi}$

Laughlin quasiparticle operator $\psi_{QP} \sim e^{i\phi}$

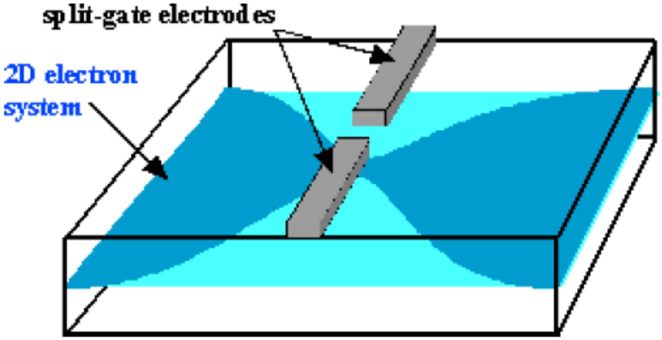
$\longrightarrow$ **Fractional statistics**



$$e^{i\theta^*}$$

$$\theta^* = 2\pi\nu$$

Tunneling between fractional edges via QPC



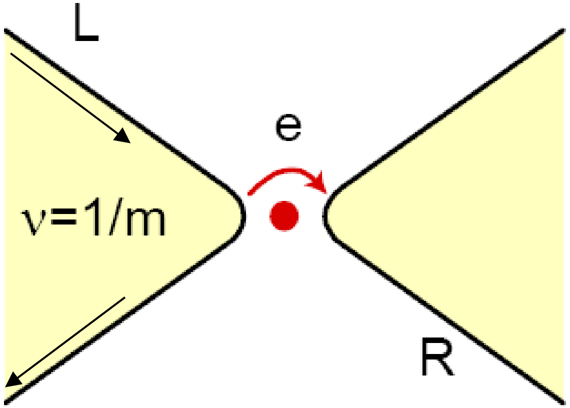
(1) Weak electron tunneling

electron operator $\psi_e^\dagger \sim e^{im\phi}$

$$H_T = t\psi_R^\dagger \psi_L \sim te^{im(\phi_R - \phi_L)}$$

$$G(T) \sim t^2 T^{2m-2}$$

$$\sim t^2 T^4 \text{ for } \nu = 1/3$$



*** Any finite system should have an integer # of electrons.**

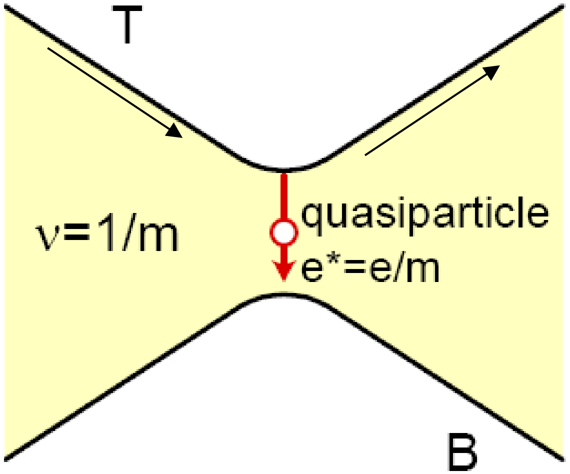
(2) Weak quasiparticle backscattering

quasiparticle operator $\psi_{QP} \sim e^{i\phi}$

$$H_T = \mathrm{v}\psi_{Q.P.,B}^\dagger \psi_{Q.P.,T} \sim te^{i(\phi_B - \phi_T)}$$

$$G(T) \sim \mathrm{v}^2 T^{2/m-2}$$

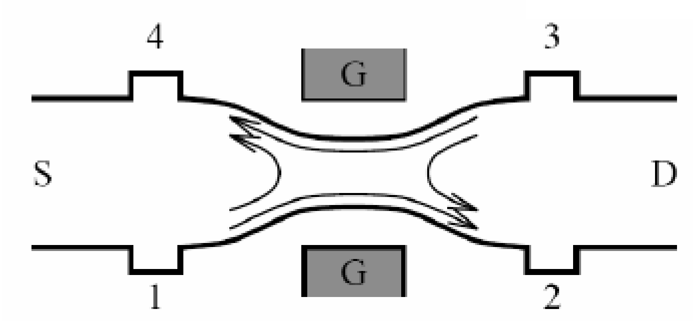
$$\sim \mathrm{v}^2 T^{-4/3} \text{ for } \nu = 1/3$$

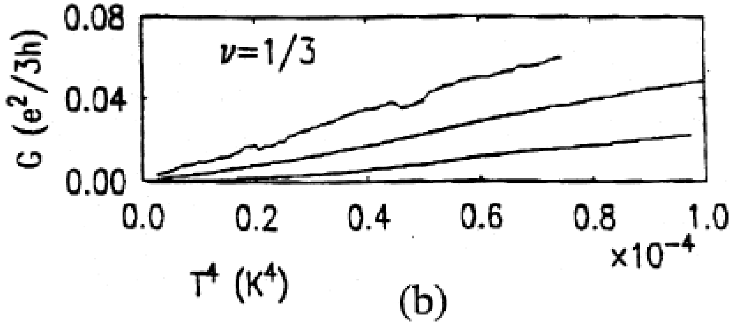


Tunneling exponent: Experiments

Weak electron tunneling $G \sim T^4$

Ref.: Miliken et al., PRL 1994

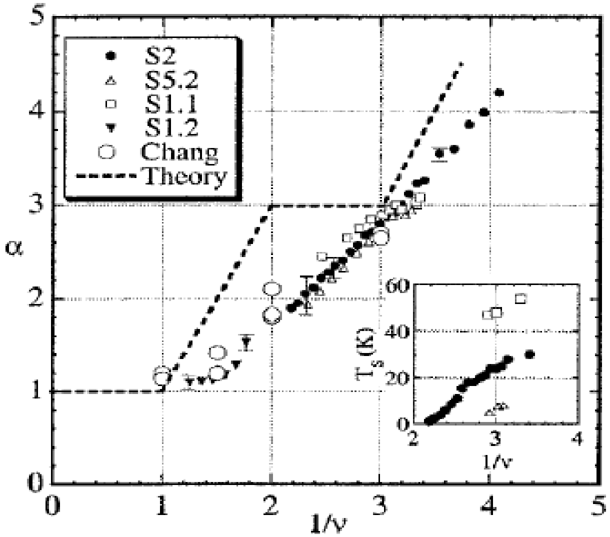




Filling-factor dependence:

Experimental data disagree with theoretical prediction.

Edge reconstruction?
Not low-energy regime?
Neutral mode?



Ref.: Grayson et al. PRL 1998

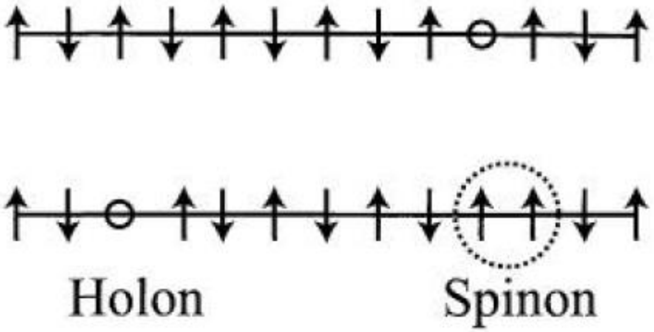
IV. Fractionalization

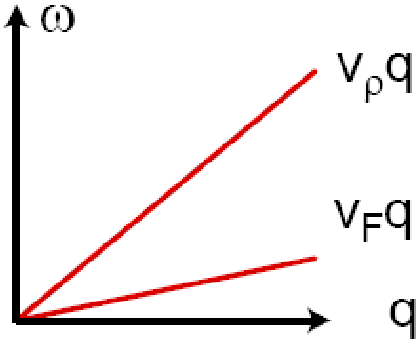
Spin-charge separation

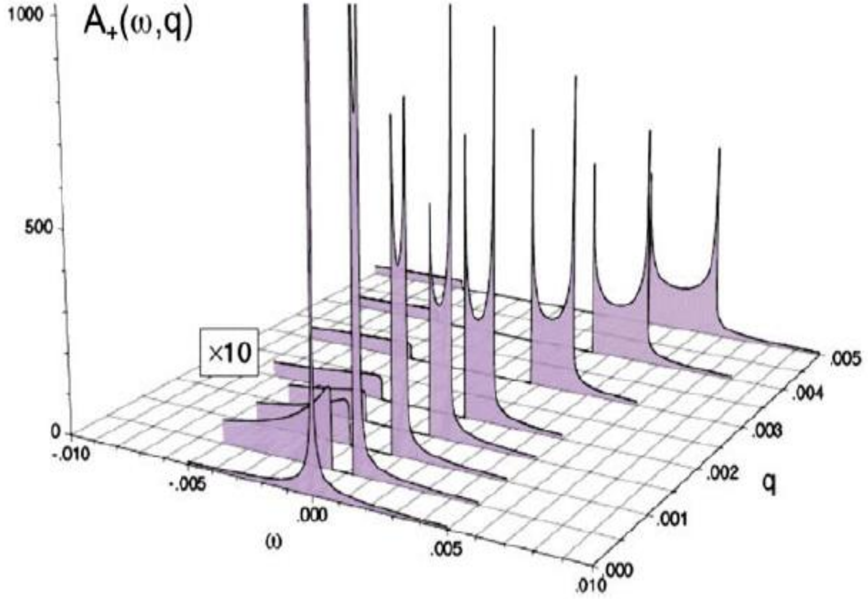
Charge fractionalization

Fractional statistics

Spin-charge separation - 1







- **double-peak structure: spin, charge modes**

Momentum- and energy-resolved spectroscopy is necessary!

(ex) ARPES

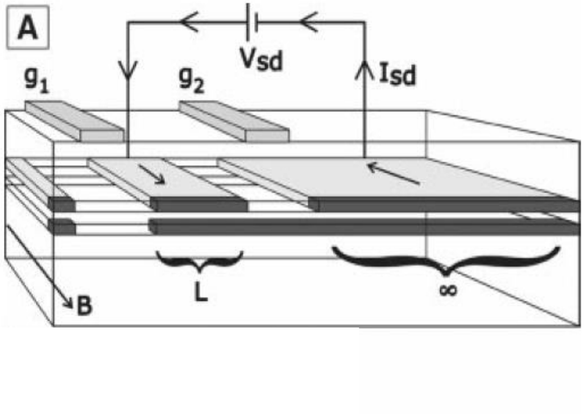
(cf) ARPES studies on Au-Si: P. Segovia et al. Nature 1999; Losio et al. PRL 2001.

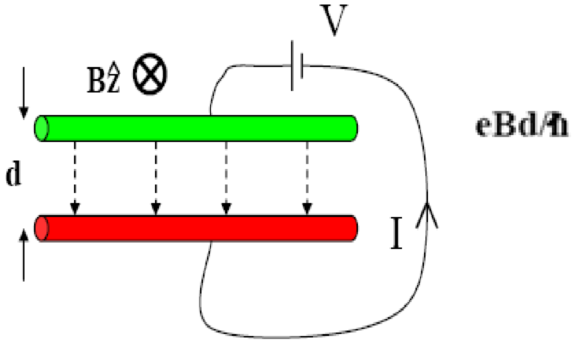
Spin-charge separation - 2

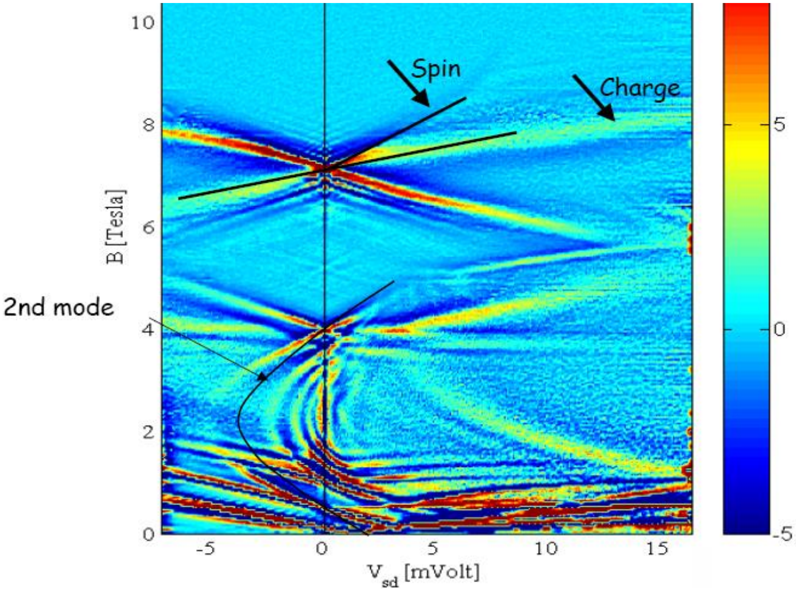
1D LL to 1D LL tunneling

Ref.: Auslaender, Yacoby et al., Science 2002; Science 2005

$$\frac{dI}{dV} \propto A(k,\epsilon) \star A(k,\epsilon)$$







$$V_c \approx \frac{V_F}{0.7}$$

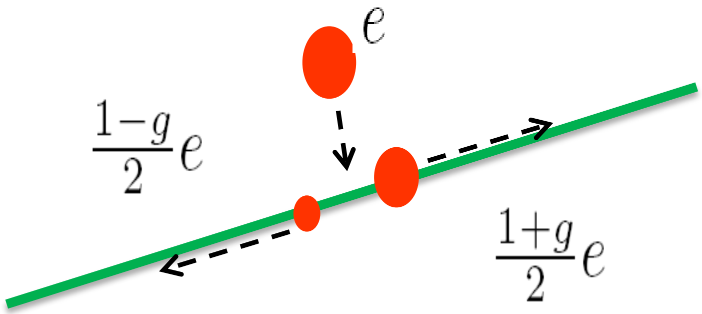
$$V_\sigma \approx V_F$$

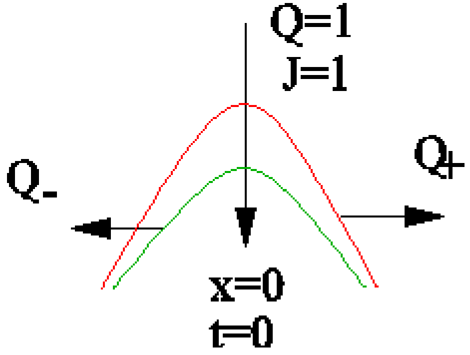
2D FL to 1D LL tunneling

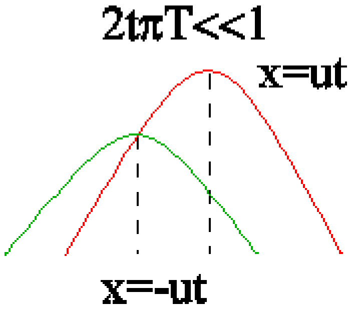
Ref.: Jompol, Chris Ford et al., Science 2009

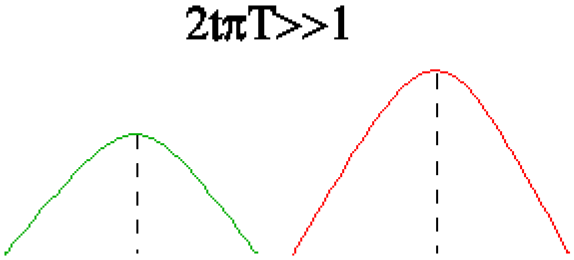
Charge fractionalization - 1

Fractionalization









$$\psi_+(x,t) = \frac{e^{i(k_F+\frac{\pi}{2L})x}}{\sqrt{2\pi a}} e^{i\phi_0(x,t)} e^{i[c_+\varphi(x-vt)+c_-\varphi(-x-vt)]}$$

Fractionalized charges (plasmonic) are good quasiparticles!

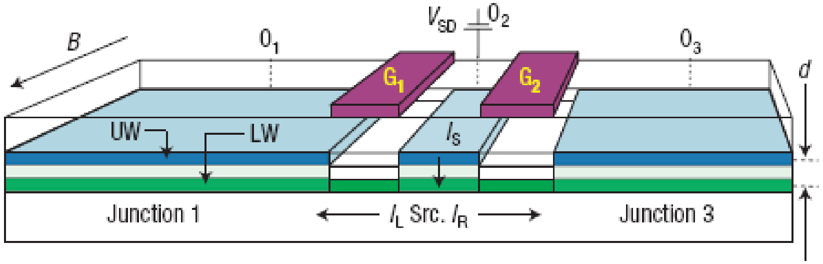
Ref.: KV Pham et al. PRB 2000

Charge fractionalization - 2

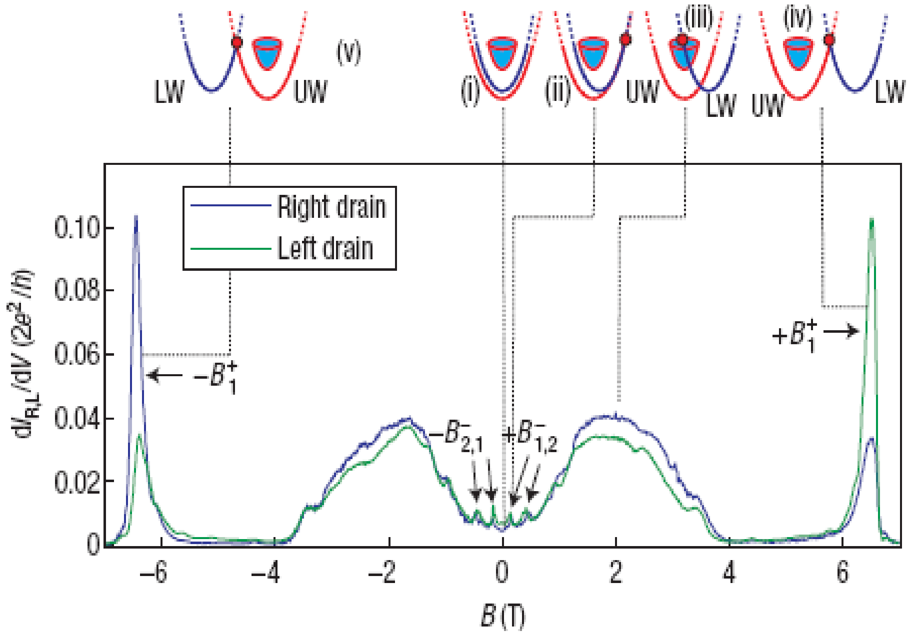
Experimental detection of charge fractionalization

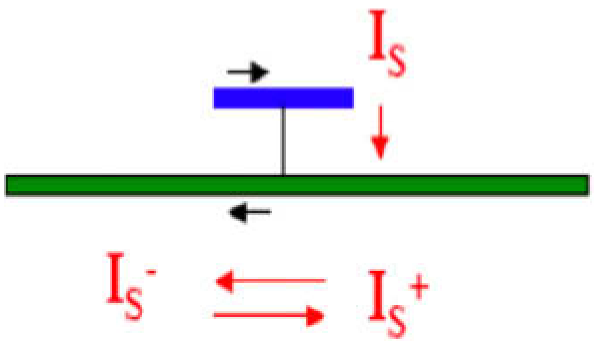
Ref.: H Steinberg, A Yacoby et al. Nat. Phys. 2008

a



b





$$N_+ = (1-f) = (1-g)/2$$

$$N_- = f = (1+g)/2$$

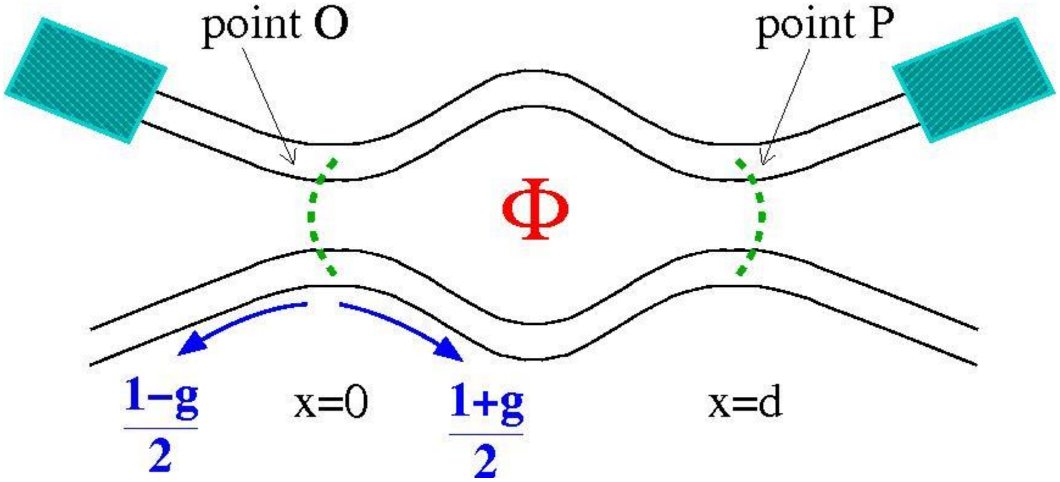
$$\frac{I_S^- - I_S^+}{I_S} = (2f-1) = g$$

$$0.7 < f < 0.75$$

Charge fractionalization - 3

Influence of charge fractionalization on electron coherence: exponential decay

Ref.: K Le Hur PRL 2005



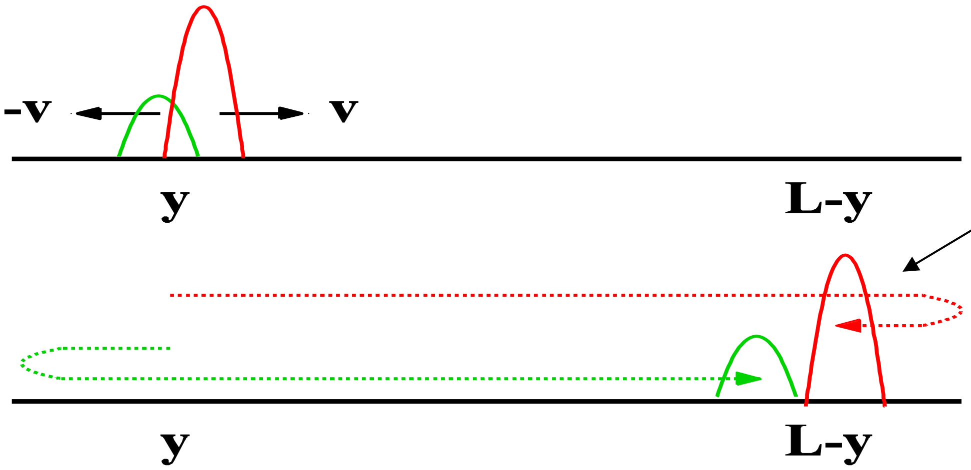
$$T \approx 1\text{-}(T_0 + T_L) + 2\sqrt{T_0 T_L}\cos(2\pi\varphi)\boxed{\exp\left(-2L/l_\phi\right)}$$

$$l_\varphi^{-1} = \frac{\pi}{2L_T}\sum_{j=1,2}\left(\frac{g_j + g_j^{-1}}{2} - 1\right), \quad \text{with} \quad L_T = \hbar v / kT$$

Charge fractionalization - 4

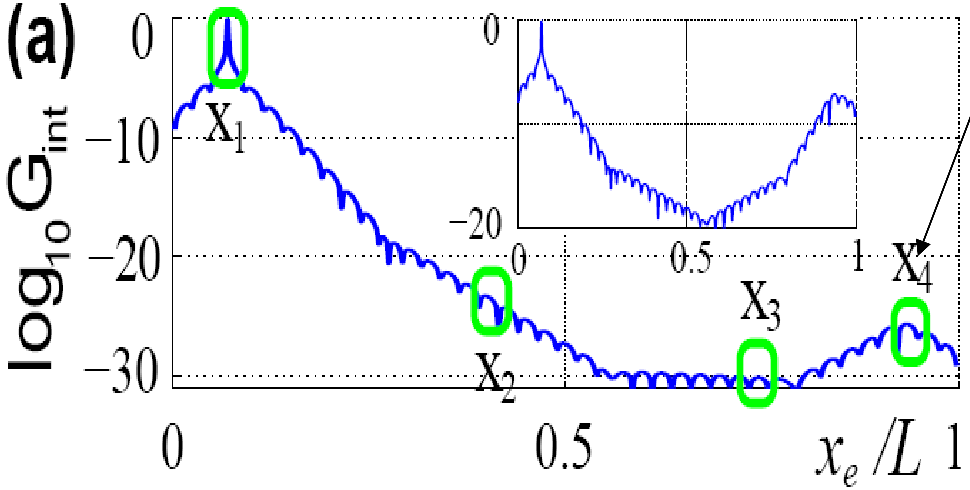
charge fractionalization and recombination

Ref.: Kim, Lee, Lee, Sim PRL 2009

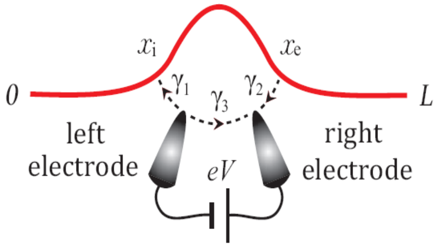


recombination

Coherence revival

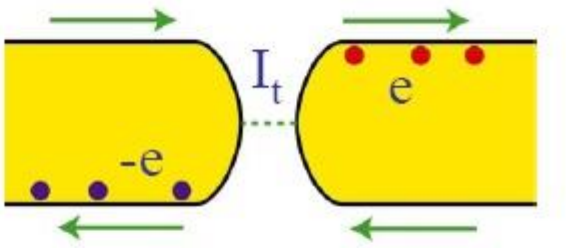


$$G_{\rm int} \sim e^{-|x_{\rm e} - x_{\rm i}|/\ell_\phi(n)}$$

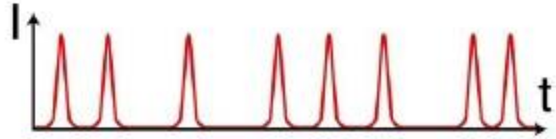


Fractional charge in the quantum Hall regime

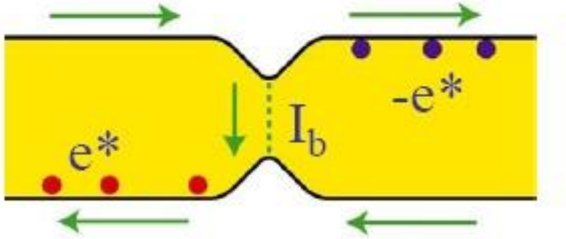
Direct measurement of fractional charge using shot noise



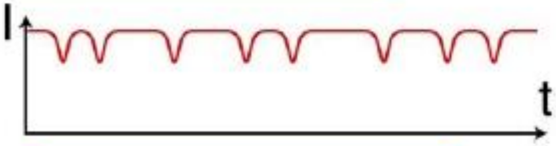
Strong pinch-off
electron tunneling



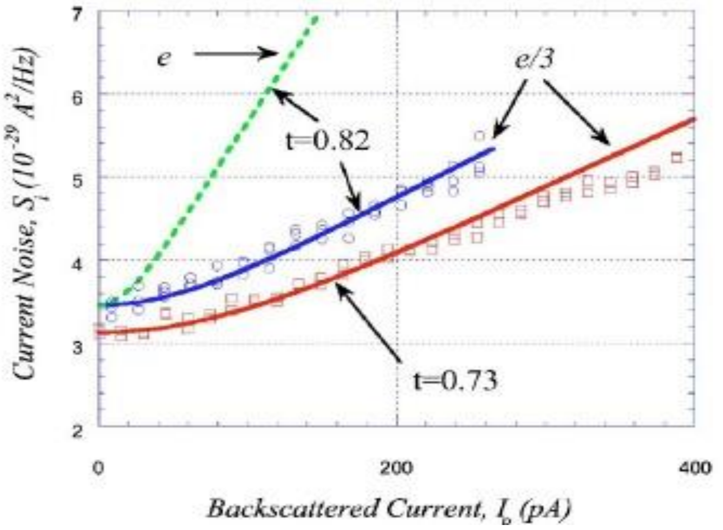
$$\boxed{S = 2e\, I_t}$$



Weak pinch-off
quasiptl tunneling



$$\boxed{S = 2e^* \, I_b}$$



$$e^* = e/3$$

Ref.: De Picciotto et al. Nature 1997
Saminadayer et al. PRL 1997

Fractional charge in the quantum Hall regime - 2

fractional charge of 5/2 FQHE
$e^* = e/4$ (= ½ filling per h/e * ½ from SC h/2e vortex)

Ref.: M. Dolev et al. Nature 2008
M. Dolev et al. PRB 2010

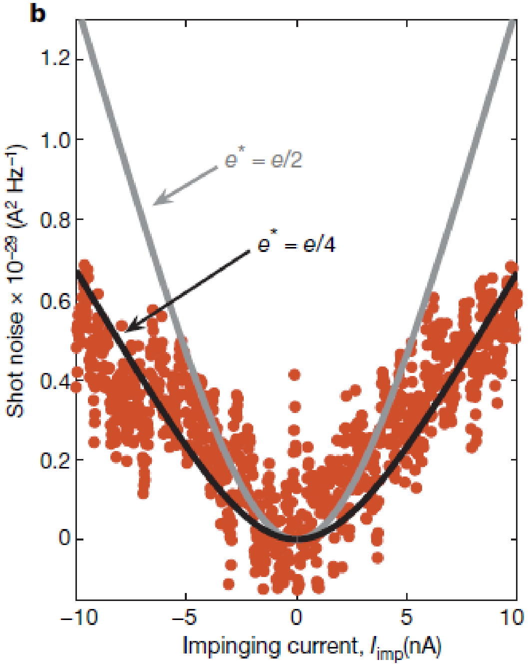
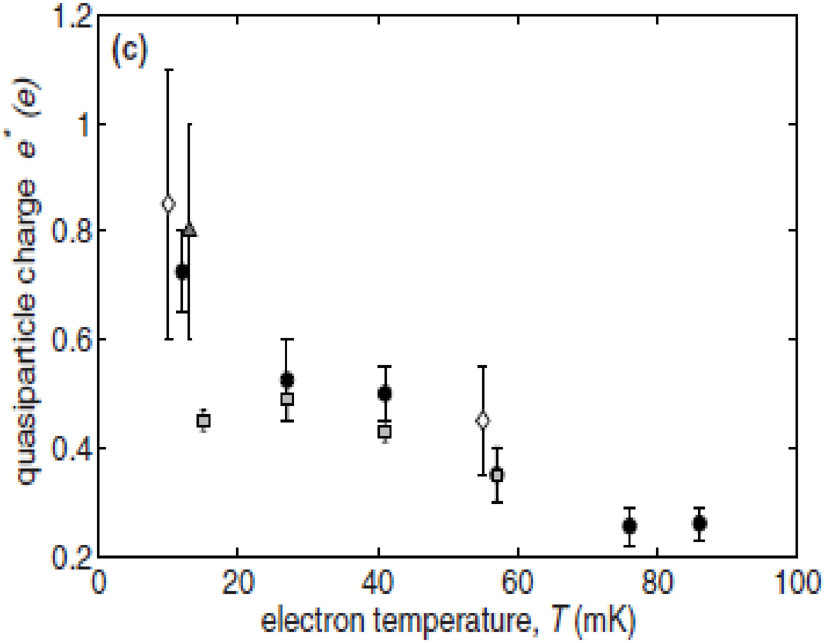


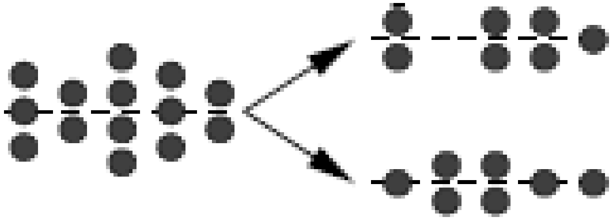
Figure 3 | Conductance and shot noise measurements of partitioned particles at the 5/2 state. a, b, For a filling factor in the bulk $\nu_B = 3$ and

However,… open issues (also in other simple fractions):

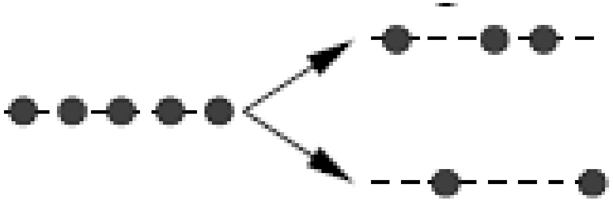


Quantum statistics

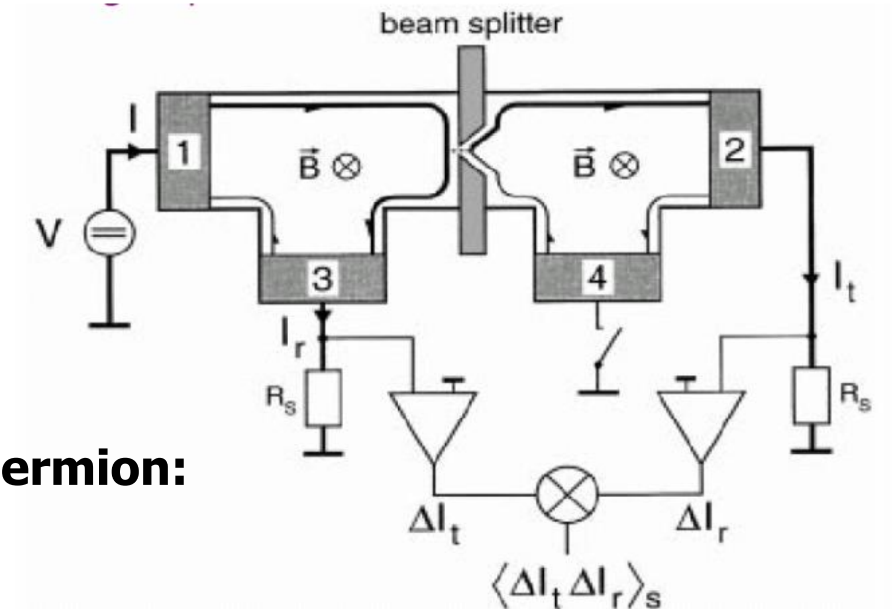
Bunching, bosons
Positive or negative correlation

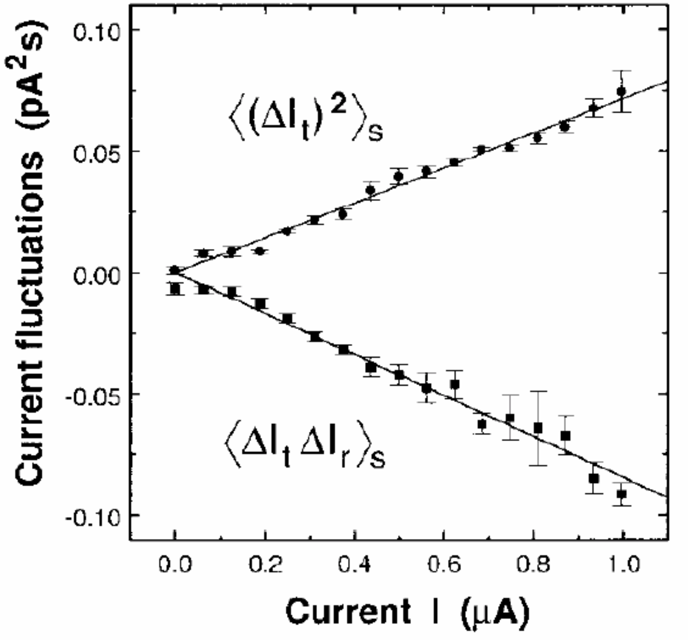


Antibunching, fermions
Negative correlation



Fermion:

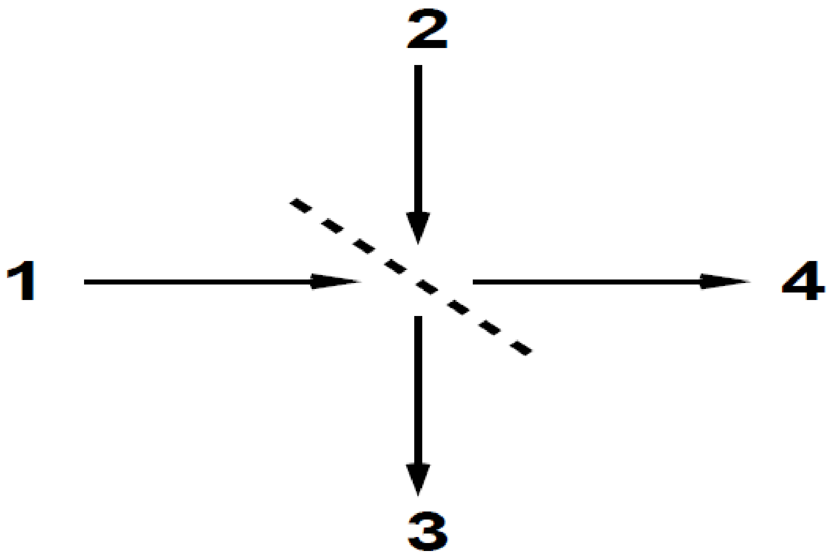




Henny et al., Science (1999); Oliver et al., Science (1999).

Quantum statistics

$$|r|^2 = |t|^2 = 1/2$$



One particle from 1,
and another from 2:

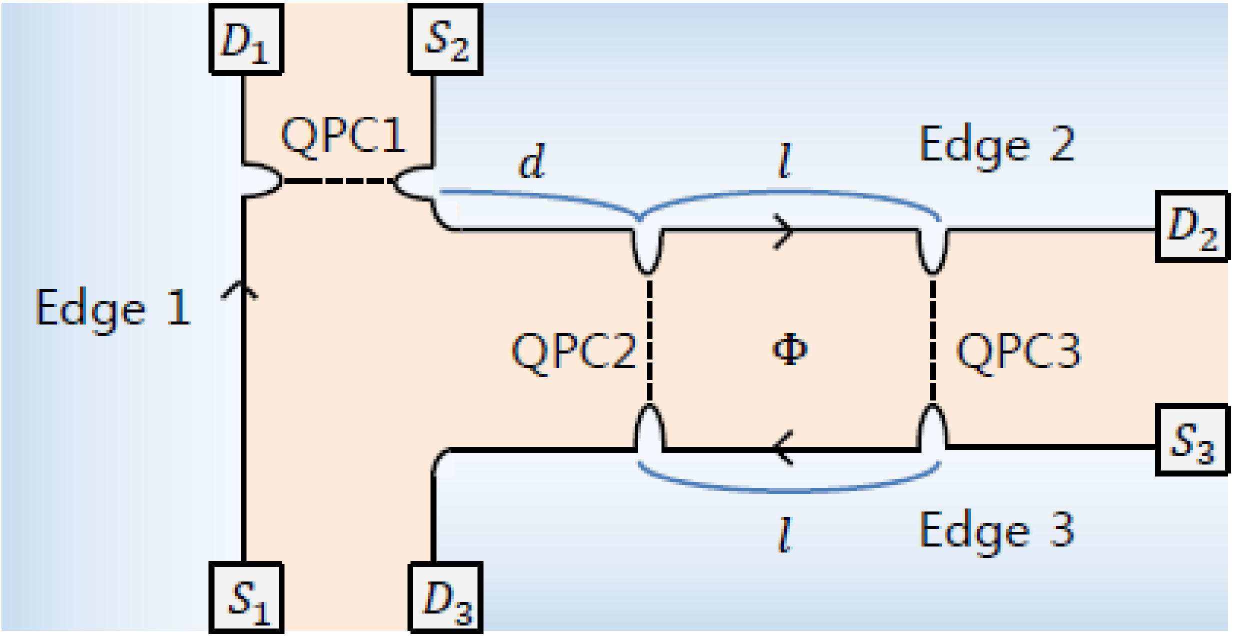
Electrons:
P(1,1) = 1
P(2,0) = P(0,2) = 0

Bosons:
P(1,1) = 0
P(2,0) = P(0,2) = 1/2

Classical particles:
P(1,1) = 1/2
P(2,0) = P(0,2) = 1/4

How about anyons?

Anyon interferometry



Anyon interferometer is well described by bosonizations.

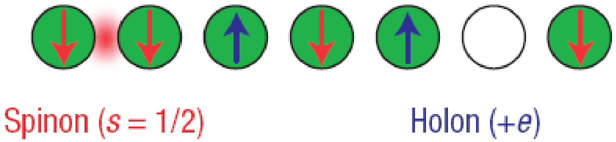
Anyonic statistics between different edges is adjusted by commutators between Klein factors.

Han, Park, Gefen, Sim (2014)

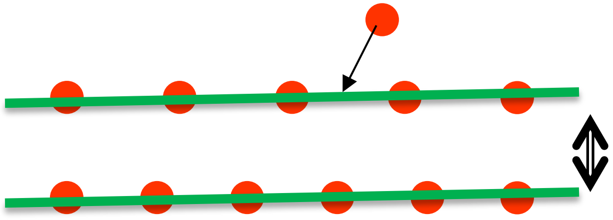
Summary

Bosonization (plasmons)

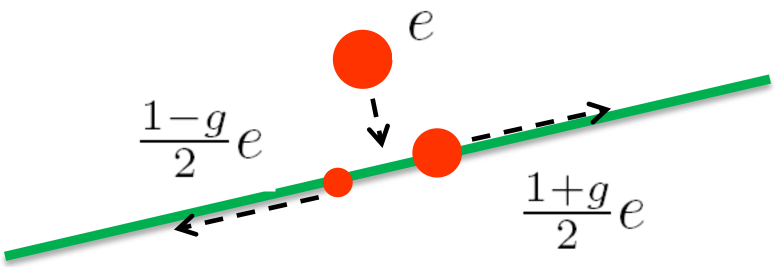
Spin-charge separation



Orthogonality catastrophe (tunneling exponent)



Charge fractionalization



Next?

Recent direction:

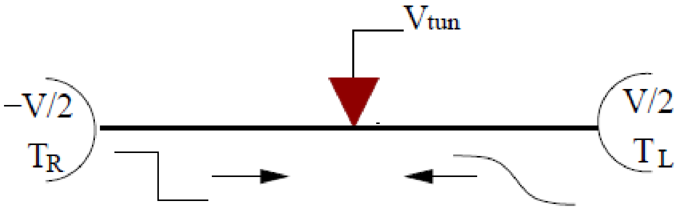
(1) nonlinear Luttinger liquid

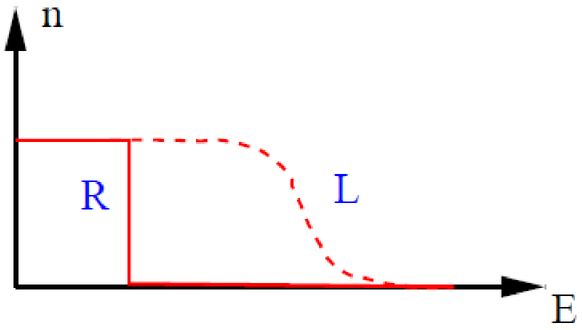
- **Parabolic dispersion relation in the noninteracting limit**

Ref.: Imambekkov and Glazman, Science (2009)

(2) Toward non-equilibrium

Ref.: Mirlin et al





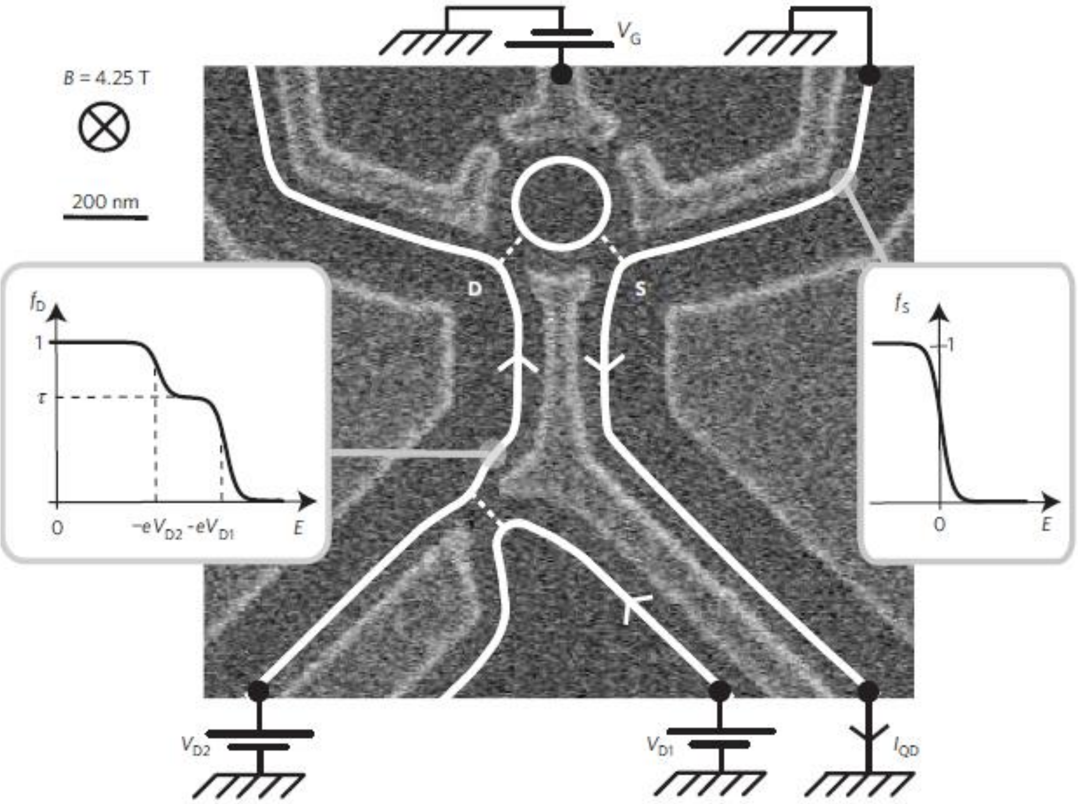
Ref.: Teo and Kane

(3) Toward two dimension
(how to construct topological order from multiple wires)

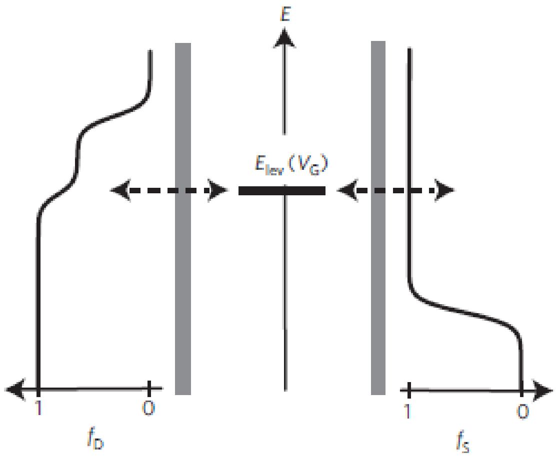
Recent direction:

Ref.: Altimiras, Pierre et al., Nat. Phys. (2010)

(4) Nonequilibrium spectroscopy of quantum Hall edges



$$I_{QD} = I_{QD}^{max}(f_S(E_{lev}) - f_D(E_{lev}))$$



$$f_D(E) = \tau f_{D1}(E) + (1-\tau) f_{D2}(E)$$

Bosonization (+ refermionization): Usefulness

Bosonization offers:

- **Exact solutions for 1D interacting electrons (under certain conditions)**
    - **Luttinger liquids**
- **Exact solutions for quantum impurity problems:**
    - **(non-Fermi, quantum critical, topological effects)**
    - **Impurity scattering + electron interaction,**
    - **multi-channel Kondo,**
    - **Y-junctions, etc**
- **Tool for describing chiral edge channel along quantum Hall edges:**
    - **Electron interferometry**
    - **Anyons**
    - **Topological quantum computation**

Bosonization is compatible with

Numerical tools (NRG, DMRG, ED,...)

Supplement: Another derivation (field theoretical/less rigorous) -2

Kac-Moody algebra

$$
\begin{aligned}
[\rho_R(q), \rho_R(-q)] &= \sum_{k,k'>0} [c_k^\dagger c_{k+q}, c_{k'}^\dagger c_{k'-q}] \\
&= \sum_{kk'>0} c_k^\dagger c_{k'-q}\delta_{k+q,k'} - c_{k'}^\dagger c_{k+q}\delta_{k,k'-q} \\
&= \sum_{k>0} \Theta(k+q)(c_k^\dagger c_k - c_{k+q}^\dagger c_{k+q}) \\
&= \mathrm{sgn}(q) \sum_{|q|>k>0} c_k^\dagger c_k \\
&\simeq \mathrm{sgn}(q) \frac{L}{2\pi} \int_0^{|q|} dk = \frac{Lq}{2\pi}
\end{aligned}
$$

*

$n_k = 1$ for $k \ll k_F$

$|q| \ll k_F$

* $[AB, C] = ABC - CAB = ABC + ACB - ACB - CAB = A\{B, C\} - \{A, C\}B$

Exercise - 1

Time-ordered Green's function of free electrons in a fermion language

$$-G^{>}_{\eta\eta'}(\tau,x) \equiv \langle \psi_\eta(\tau,x)\psi^\dagger_{\eta'}(0,0)\rangle$$

$$G^{<}_{\eta\eta'}(\tau,x) \equiv \langle \psi^\dagger_{\eta'}(0,0)\psi_\eta(\tau,x)\rangle$$

$\langle \mathcal{T}\psi_\eta\psi^\dagger_\eta\rangle$ **at zero temperature**

$$\begin{aligned}
-G_{\eta\eta'}(\tau,x) &\equiv \theta(\tau)G^{<}_{\eta\eta'}(\tau,x)+\theta(-\tau)G^{<}_{\eta\eta'}(\tau,x)\\
&= \delta_{\eta\eta'}\left[\theta(\tau)\tfrac{2\pi}{L}\sum_{k>0}e^{-k(\tau+ix+\sigma a)}-\theta(-\tau)\tfrac{2\pi}{L}\sum_{k<0}e^{-k(\tau+ix+\sigma a)}\right]\\
&= \delta_{\eta\eta'}\tfrac{2\pi}{L}\sigma y^{-\sigma\delta_b/2}\sum_{n=1}^{\infty}y^n = \delta_{\eta\eta'}\tfrac{2\pi}{L}\sigma\frac{y^{-(\sigma\delta_b+1)/2}}{y^{-1/2}-y^{1/2}}\\
&= \frac{\delta_{\eta\eta'}e^{\frac{\pi}{L}(\delta_b+\sigma)(\tau+ix)}}{\frac{L}{\pi}\sinh[\frac{\pi}{L}(\tau+ix+\sigma a)]}\ \xrightarrow{L\to\infty}\ \frac{\delta_{\eta\eta'}}{\tau+ix+\sigma a}\ .
\end{aligned}$$

$$c^\dagger_{k\eta}(\tau) = e^{k\tau}c^\dagger_{k\eta}$$

$$\langle c^\dagger_{k\eta}c_{k'\eta'}\rangle = \frac{\delta_{\eta\eta'}\delta_{kk'}}{e^{\beta k}+1}$$

$$T = 0$$

$$(e^{\beta k}+1)^{-1} = \theta(-k)$$

$$\sigma \equiv \mathrm{sgn}(\tau) \qquad y \equiv e^{-\frac{2\pi}{L}(\sigma\tau+\sigma ix+a)}) \qquad k = \tfrac{2\pi}{L}(n_k-\tfrac{1}{2}\delta_b)$$

$T \neq 0$

$$\langle \mathcal{T}\psi_\eta(z)\psi^\dagger_{\eta'}(0)\rangle = \frac{\delta_{\eta\eta'}}{\frac{\beta}{\pi}\sin[\frac{\pi}{\beta}(z+\sigma a)]}$$

Exercise -2

Time-ordered Green's function of free electrons in a fermion language

$$T \;=\; 0 \qquad \langle \mathcal{T}\psi_\eta(z)\psi^\dagger_{\eta'}(0)\rangle_{T=0} \;=\; \frac{\delta_{\eta\eta'}e^{\frac{\pi}{L}(\delta_b+\sigma)z}}{\frac{L}{\pi}\sinh[\frac{\pi}{L}(z+\sigma a)]}$$

$$\langle e^{\lambda\hat{B}}\rangle = e^{\langle\hat{B}^2\rangle\lambda^2/2}$$

$$\hat{B} = \textstyle\sum_{q>0}(\lambda_q b^\dagger_q + \tilde{\lambda}_q b_q)$$

Same in a boson language

$$\begin{aligned}\langle \mathcal{T}\psi_\eta(z)\psi^\dagger_{\eta'}(0)\rangle \;&=\; a^{-1}\Big[\theta(\tau)\langle F_\eta e^{-\frac{2\pi}{L}(\widehat{N}_\eta-\frac{1}{2}\delta_b)z}e^{-i\phi_\eta(z)}e^{i\phi_{\eta'}(0)}F^\dagger_{\eta'}\rangle \\ &\qquad -\theta(-\tau)\langle e^{i\phi_{\eta'}(0)}F^\dagger_{\eta'}F_\eta e^{-\frac{2\pi}{L}(\widehat{N}_\eta-\frac{1}{2}\delta_b)z}e^{-i\phi_\eta(z)}\rangle\Big] \\ &=\; \delta_{\eta\eta'}\,\sigma a^{-1}e^{\frac{\pi}{L}\delta_b z}e^{\langle\mathcal{T}\phi_\eta(z)\phi_\eta(0)-\phi_\eta(0)\phi_\eta(0)\rangle_{T=0}}\end{aligned}$$

$$\langle \mathcal{T}\phi_\eta\phi_\eta\rangle$$

$$\begin{aligned}-\mathcal{G}_{\eta\eta'}(\tau,x) \equiv \theta(\tau)\mathcal{G}^{>}_{\eta\eta'}(\tau,x) \;+\; \theta(-\tau)\mathcal{G}^{<}_{\eta\eta'}(\tau,x) \;&=\; \mathcal{G}^{>}_{\eta\eta'}(\sigma\tau,\sigma x) \\ =\; \delta_{\eta\eta'}\sum_{q>0}^{\infty}\frac{1}{n_q}e^{-q(\sigma\tau+\sigma ix+a)} = \delta_{\eta\eta'}\sum_{n_q=1}^{\infty}\frac{1}{n_q}y^{n_q} &= -\delta_{\eta\eta'}\ln(1-y) \\ =\; -\delta_{\eta\eta'}\ln\left(1-e^{-\frac{2\pi}{L}(\sigma\tau+\sigma ix+a)}\right) &\xrightarrow{L\to\infty} -\delta_{\eta\eta'}\ln\left[\tfrac{2\pi}{L}(\sigma\tau+\sigma ix+a)\right]\end{aligned}$$

$$-\mathcal{G}^{>}_{\eta\eta'}(\tau,x) \;\equiv\; \langle\phi_\eta(\tau,x)\phi_{\eta'}(0,0)\rangle$$

$$-\mathcal{G}^{<}_{\eta\eta'}(\tau,x) \;\equiv\; \langle\phi_{\eta'}(0,0)\phi_\eta(\tau,x)\rangle$$

$$\mathcal{G}^{<}_{\eta\eta'}(\tau,x) = \mathcal{G}^{>}_{\eta'\eta}(-\tau,-x)$$

$$\phi_\eta(x) = -\sum_{q>0}\tfrac{1}{\sqrt{n_q}}\left(e^{-iqx}b_{q\eta}+e^{iqx}b^\dagger_{q\eta}\right)e^{-aq/2}$$

$$b_{q\eta}(\tau) = e^{-q\tau}b_{q\eta}$$

$$\langle b^\dagger_{q\eta}b_{q'\eta'}\rangle \;=\; \frac{\delta_{\eta\eta'}\delta_{qq'}}{e^{\beta q}-1}$$

$$T \;=\; 0 \qquad (e^{\beta q}-1)^{-1} \;=\; -\theta(-q)$$