SUSY and Localization

SUNGJAY LEE

UNIVERSITY OF CHICAGO

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Part I

Equivariant Volume & Localization

Why do we care ?

[1] Nekrasov Instanton Partition Functions and Seiberg-Witten Theories

[2] Volume of Sasaki-Einstein Manifolds and AdS/CFT correspondence

Long-Standing Problem

Question

What measures # of degrees of freedom that decreases monotonically along the renormalization group flow ?

Renormalization 101

- Describe a system by microscopic d.o.f and their interactions
- Zoom out, or coarse grain: average out **"heavy modes"** that are irrelevant to long-distance physics
- At each typical energy scale, the d.o.f describing the same system may look very different to each other
- This procedure of zooming out and ignoring the small irrelevant details is known as Renormalization Group (RG) flow

A Little History

We know the answer in 2 dimensions due to the work of Zamoldchikov [86]

- One can define a real number **c** for any 2d QFTs, even for strongly interacting system, from 2-point function of energy-momentum tensor
- He has shown that this number always decrease monotonically along RG flow,

$$\boxed{C_{\mathrm{UV}} > C_{\mathrm{IR}}}$$

- It counts # of d.o.f, generalizing the notion of counting in free theory

$\Rightarrow$ **" Irreversibility of Renormalization Group Flow ! "**

Two Questions Arise

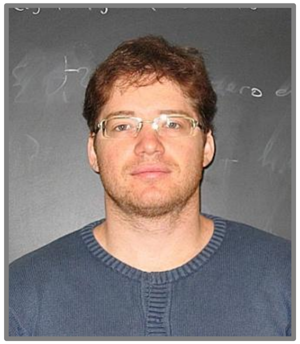
[1] Similar measure in higher dimensions ?

[2] Counterpart in the bulk geometry via AdS/CFT correspondence

Higher Dimensions

In 4D, it has been conjecture by John Cardy soon after Zamoldchikov's work

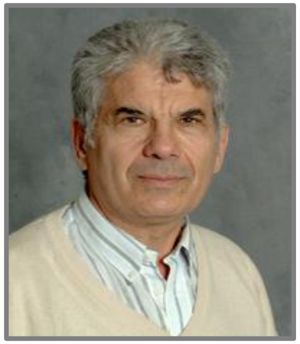
- One can easily generalize the definition of c-function in 2D to 4 dimensions, known as a-function



- Conjecture

$$\boxed{a_{\mathrm{UV}} > a_{\mathrm{IR}}}$$

A proof is recently proposed by Komargodski and Schwimmer [11]



What about three dimensions ?

Not even clue until very recently. This is because the definition of c- and a-function cannot be generalized to any odd dimensions

Higher Dimensions

Answer turns out to be S^3 partition function

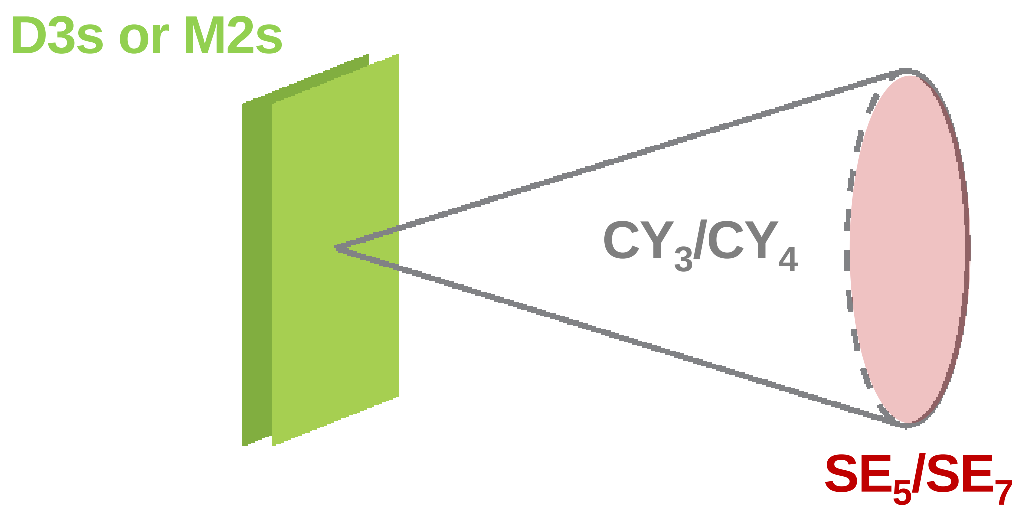
More precisely, defining $F_{S^3} = -\log Z_{S^3}$,

[Jafferis]
[Jafferis,Klebanov,Pufu,Safdi]
[Closset,Dumitrescu,Festuccia, Komargodski,Seiberg]

$$\boxed{F_{\mathrm{UV}} > F_{\mathrm{IR}}}$$

I may have a chance to introduce this story a little bit on Saturday…

AdS/CFT & Volume Minimization



NB **$AdS_{5(4)}/CFT_{4(3)}$** : **a(F)**-maximization vs **Vol[$SE_{5(7)}$]**-minimization

Equivariant Volume & Localization Method play a key role

SYMPLECTIC MANIFOLD (X, ω)

- $\dim[X] = 2n$
- $\omega = \frac{1}{2}\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu$, symplectic 2-form satisfying "$d\omega = 0$"

SYMPLECTIC VOLUME

- Volume form is defined as $\frac{1}{n!}\omega^n$

$$\mathrm{Vol}(X) = \int_X \frac{1}{n!}\omega^n = \int_X e^\omega$$

SYMPLECTIC VOLUME

i.e., 'symplectic mfd.'

- **Example**: Kähler manifold (= a complex mfd. w/ a closed 2-form)

  - metric $ds^2 = g_{a\underline{b}}\, dz^a d\bar{z}^{\underline{b}}$

  - Kähler 2-form $\omega = \frac{i}{2} g_{a\underline{b}}\, dz^a \wedge d\bar{z}^{\underline{b}}$ & $d\omega = 0$

  - symplectic volume

$$\mathrm{Vol}(X) = \int_X \frac{1}{n!} \left(\frac{i}{2}\right)^n g_{a_1\underline{b_1}} \cdots g_{a_n\underline{b_n}} \left[\epsilon^{a_1 \cdots a_n} \epsilon^{\underline{b_1} \cdots \underline{b_n}}\right] dz^1 d\bar{z}^{\underline{1}} \cdots dz^n d\bar{z}^{\underline{n}}$$

$$= \int_X \sqrt{\det G} \qquad G : \text{metric on } X$$

NB SYMPLECTIC VOLUME = CANONICAL VOLUME for Kähler mfd.

SUPERSPACE & SUPERSYMMETRY

- Superspace $T[1]X = (x^\mu, \psi^\mu)$ where x^μ : coordinates of X
  ψ^μ : Grassmanian Variable

- function $f(x,\psi)$ on $T[1]X$ = space of differential form $\Omega^*(X)$

$$f(x,\psi) = f^{(0)}(x) + f^{(1)}_\mu(x)\psi^\mu + \cdots + \frac{1}{n!} f^{(n)}_{\mu_1\cdots\mu_n}(x)\psi^{\mu_1}\cdots\psi^{\mu_n}$$

- INTEGRATION

'top-form'

$$\int_{T[1]X} [dx\,d\psi] f(x,\psi) = \int_X [dx^\mu] \frac{1}{n!} f^{(n)}_{\mu_1\cdots\mu_n}(x)\epsilon^{\mu_1\cdots\mu_n} = \int_X f^{(n)}(x)$$

- $\omega = \frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu$ is invariant under

$$\delta X^\mu = \psi^\mu \qquad \delta\psi^\mu = 0$$

- $\delta^2 * = 0$ ... nilpotent!

- supercharge $Q \doteq \psi^\mu\partial_\mu \doteq dx^\mu\partial_\mu = d$ (exterior derivative)

- $[dX^\mu][d\psi^\mu]$ is obviously an invariant measure under SUSY transf. rule

$\Rightarrow \int_{T[1]X}[dX^\mu][d\psi^\mu]\,\delta\# = 0$! (nothing but. <u>Stokes thm</u>)

- $\delta\omega = \frac{1}{2}\partial_\rho\omega_{\mu\nu}\delta X^\rho\psi^\mu\psi^\nu + \omega_{\mu\nu}\delta\psi^\mu\psi^\nu = \frac{1}{2}\partial_\rho\omega_{\mu\nu}\psi^\rho\psi^\mu\psi^\nu = 0$! $(\because d\omega = 0)$

- Adding a Q-exact term doesn't affect the volume integral

$$I[t] = \int [dx^\mu][d\psi^\mu]\, e^{\omega + t\delta V} \qquad I[0] = \mathrm{vol}(X)$$

<u>PROOF</u>:

$$\frac{d}{dt} I[t] = \int [dx^\mu][d\psi^\mu]\, \delta V\, e^{\omega + t\delta V} = \int [dx^\mu][d\psi^\mu]\, \delta\left[V \cdot e^{\omega + t\delta V} \right]$$

$$= 0 \,! \qquad \blacksquare \qquad \because \quad \delta\omega = 0 \ \& \ \delta^2 V = 0$$

- This implies that $I[\infty] = \mathrm{Vol}(X)$

BUT, it is not helpful! $\because$ δV doesn't contain any bosonic terms

EQUIVARANT VOLUME

- Hamiltonian flow : $(\underset{\text{function}}{H}, \underset{\text{vector field on } X}{V^\mu})$ s.t. $i_V \omega = dH$

• $\mathcal{L}_V \omega = [d\, i_V + i_V d]\omega = d^2 H = 0 \quad \blacksquare$

NB In many physical systems (e.g. QM with target space X)

$$V^\mu \cong \text{global symmetry}$$

- DEFORM THE SUSY !

$$\delta_\epsilon X^\mu = \psi^\mu \qquad \delta_\epsilon \psi^\mu = \epsilon V^\mu(\vec{x})$$

• Then, one can show that $\delta_\epsilon^2 X^\mu = \epsilon V^\mu \quad \delta_\epsilon^2 \psi^\mu = \epsilon \partial_\nu V^\mu \psi^\nu$

$$\delta_\epsilon^2 = \mathcal{L}_V \left(= d i_V + i_V d\right)$$

- Equivariant Volume

$$\mathrm{Vol}_\epsilon[X] = \int e^{\omega - \epsilon H}, \quad \text{where} \quad dH = i_V \omega$$

Note that $\underline{\delta_\epsilon[\omega - \epsilon H] = 0}$!

$$\left(\begin{array}{c} \delta_\epsilon \omega = \epsilon \omega_{\mu\nu} V^\mu \psi^\nu = \partial_\nu H \cdot \psi^\nu \\ \qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\quad \delta_\epsilon H \end{array} \right)$$

Example S^2

$$\omega = d\theta \wedge \sin\theta\, d\phi \qquad V = \partial_\phi \qquad i_V\omega = +d\cos\theta \rightarrow H = \cos\theta$$

Then,

$$Vol_\epsilon[S^2] = \int_{\Pi[1]S^2} e^{\omega - \epsilon H} \qquad (\omega = \sin\theta\, \psi^\theta \psi^\phi)$$

$$= \int d\theta\, d\phi\, \sin\theta\, e^{-\epsilon\cos\theta}$$

$$= \int_{-1}^{1} dx \cdot 2\pi \cdot e^{-\epsilon x} = \frac{2\pi}{\epsilon}\left[e^{\epsilon} - e^{-\epsilon}\right] = 4\pi \frac{\sinh\epsilon}{\epsilon}$$

$$\lim_{\epsilon\to 0} Vol_\epsilon[S^2] = 4\pi\,!$$

Example $\mathbb{R}^2$

$\omega = dx \wedge dy \qquad V = -x\partial_y + y\partial_x \qquad i_V\omega = +\frac{1}{2} d(x^2+y^2)$

$\Rightarrow H = +\frac{1}{2}(x^2+y^2)$

$$\mathrm{Vol}_\epsilon[\mathbb{R}^2] = \int_{\Pi[1]\mathbb{R}^2} e^{\omega - \epsilon H} \qquad \text{with} \quad \omega = \psi^x \psi^y$$

$$= \int_{\mathbb{R}^2} e^{-\frac{\epsilon}{2}(x^2+y^2)} = \left(\sqrt{\frac{2\pi}{\epsilon}}\right)^2 = \frac{2\pi}{\epsilon}$$

Equivariant volume of a non-compact space can be finite !

What are they good for? [DH-formula]

- can use the localization techique

$$I_\epsilon[t] = \int_{T[1]X} e^{\omega - \epsilon H - t\delta\mathcal{V}}$$

IF $\mathcal{V}(x,\psi)$ IS INVARIANT UNDER $\mathcal{L}_V$, $I_\epsilon[t]$ IS INDEP. OF t

- Choose $\mathcal{V} = g_{\mu\nu}\psi^\mu V^\nu$ $\quad \delta\Psi^\mu = V^\mu$:

$$S_{def} = \delta\mathcal{V} = \underbrace{\epsilon g_{\mu\nu} V^\mu V^\nu}_{\to \text{positive def.}} + \partial_\rho g_{\mu\nu}\psi^\rho\psi^\mu V^\nu + g_{\mu\nu}\psi^\mu \partial_\rho V^\nu \psi^\rho$$

• TAKE A LIMIT $t \to \infty$,

$$\mathrm{vol}_\epsilon[X] = \int_{T[1]X} e^{\omega - \epsilon H - t[\underbrace{\epsilon g_{\mu\nu} V^\mu V^\nu + \cdots}_{= S_{def}}]}$$

[1] THE INTEGRAL LOCALIZES NEAR $\|V\| = 0$! $\quad V^\mu(x_*) = 0$

[2] SADDLE-POINT APPROXITION BECOMES EXACT $\quad \delta\Psi^\mu = V^\mu = 0$

SUSY condition

$$V^\mu \cong V^\mu_\rho(x_*)\, x^\rho + \cdots$$

$$S_{def} = \epsilon\, g_{\mu\nu}(x_*) V^\mu_\alpha V^\nu_\beta\, x^\alpha x^\beta + g_{\mu\nu}(x_*)\, \psi^\mu \psi^\rho\, V^\nu_\rho(x_*)$$

$$\therefore \quad \mathrm{vol}_\epsilon[X] = \sum_{x^*} e^{-\epsilon H(x_*)} \left[\frac{2\pi}{\epsilon}\right]^n \frac{\mathrm{Pf}\left[g_{[\mu\nu}(x_*) V^\nu_{\rho]}\right]}{\det^{1/2}\left[g_{\mu\nu}(x_*) V^\mu_\alpha V^\nu_\beta\right]}$$

EXAMPLE S^2 REVISITED! $\quad \mathrm{Vol}(S^2) = \int_{T[1]S^2} e^{\omega - \epsilon H - S_{def}}$

$$S_{def} = \delta_\epsilon [g_{ab}\psi^a V^b] = \delta_\epsilon[\sin^2\theta\, \psi^\phi] = \epsilon \sin^2\theta + 2\sin\theta\cos\theta\, \psi^\theta \psi^\phi$$

In the limit $t\to\infty$, the volume integral is localized onto $\theta=0$ (N) & $\theta=\pi$ (S)

① Near $\theta=0$ Around N-pole, $S^2 \sim R^2$

$x \equiv \theta\cos\phi,\ y \equiv \theta\sin\phi$

$\psi^x = \cos\phi\,\psi^\theta + y\,\psi^\phi$

$\psi^y = \sin\phi\,\psi^\theta - x\,\psi^\phi$

$$S_{def} = -t[\epsilon(x^2+y^2) - 2\psi^x\psi^y]$$

$$\Rightarrow -e^{-\epsilon}\int dx\,dy\,d\psi^x d\psi^y\, e^{-\epsilon(x^2+y^2)+2\psi^x\psi^y}$$

$$= -\frac{2\pi}{\epsilon}e^{-\epsilon}$$

② Near $\theta=\pi$, one can obtain $\frac{2\pi}{\epsilon}e^{+\epsilon}$ $\quad \therefore \mathrm{Vol}(S^2) = \frac{4\pi}{\epsilon}\sinh\epsilon$

HARISH-CHANDRA-ITZYKSON-ZUBER [HCIZ] INTEGRAL

$$I(A,B) = \int dU \, e^{-S[A,B;U]}$$ with $$S[A,B;U] = \mathrm{tr}[AUBU^{+}]$$

U : unitary

$A = \mathrm{diag}(a_1, \cdots, a_n)$

$B = \mathrm{diag}(b_1, \cdots, b_n)$

Use the localization technique to show

$$I(A,B) \propto \frac{\det(e^{-a_i b_j})}{\prod_{i<j} (a_i - a_j)(b_i - b_j)}$$

This integral arises from Brownian motions

Note that an integration domain $X = U(n)/U(1)^n$ they don't rotate B

Mathematical Facts

I. X is a Kähler manifold

II. Coordinates $(z^1, \cdots, z^N)$ $N = n(n-1)/2$

Kähler 2-form $\omega = \frac{i}{2} g_{a\underline{b}} dz^a \wedge d\bar{z}^{\underline{b}}$

Then, one can rewrite the HCIZ integral as follow

$$I(A,B) = \int_{T[1]X} e^{-S[A,B;z] + \omega}$$

Supersymmetry

- SUSY transf. rules

$$\delta z^a = \psi^a \qquad \delta\psi^a = -2i g^{a\underline{b}} \partial_{\underline{b}} S$$

$$\delta \bar{z}^{\underline{b}} = \bar{\psi}^{\underline{b}} \qquad \delta\bar{\psi}^{\underline{b}} = +2i g^{a\underline{b}} \partial_a S$$

- Under the above transf. rules, one can show $S[A,B] - w$ is invariant

$$\delta S[A,B] = \partial_a S\, \delta z^a + \partial_{\underline{b}} S\, \delta\bar{z}^{\underline{b}} = \partial_a S\, \psi^a + \partial_{\underline{b}} S\, \bar{\psi}^{\underline{b}}$$

they vanish $\because$ X = Kähler

$$\delta w = \frac{i}{2} g_{a\underline{b}}\, \delta\psi^a \bar{\psi}^{\underline{b}} - \frac{i}{2} g_{a\underline{b}}\, \psi^a \delta\bar{\psi}^{\underline{b}} + \frac{i}{2} \partial_c g_{a\underline{b}}\, \delta z^c \psi^a \bar{\psi}^{\underline{b}} + \frac{i}{2} \partial_{\underline{c}} g_{a\underline{b}}\, \delta\bar{z}^{\underline{c}} \psi^a \bar{\psi}^{\underline{b}}$$

$$= \partial_{\underline{b}} S\, \bar{\psi}^{\underline{b}} + \partial_a S\, \psi^a$$

$$\Rightarrow \delta[S - w] = 0\,!$$

- Q-exact terms : We choose $S_{def} = QV(z, \bar{z}, \psi, \bar{\psi})$

$$\begin{cases} V = i\partial_a S\, \psi^a - i\partial_{\underline{b}} S\, \bar{\psi}^{\underline{b}} \\ S_{def} = 4 g^{a\underline{b}} \partial_a S\, \partial_{\underline{b}} S - 2i\, \partial_a \partial_{\underline{b}} S\, \psi^a \bar{\psi}^{\underline{b}} \end{cases}$$

→ 'positive def. bosonic terms'

- Now, it's time to identify the saddle pts.

<u>SOLVE</u> $\partial_a S = \partial_{\underline{b}} S = 0$!

Suppose that $U_0 \in X = U(n)/U(1)^n$ satisfies the saddle pt. eqn., i.e.,

$$\delta S[U_0] = S[e^{ih} U_0] - S[U_0] = \mathrm{tr}[A\, ih\, U_0 B U_0^+] - \mathrm{tr}[A\, U_0 B U_0^+\, ih]$$

→ "linear part"

$$= \mathrm{tr}[\, ih\, \underbrace{[U_0 B U_0^+, A]}\,] = 0 \, !$$

$U_0 B U_0^+$ & A commute to each other !

saddle pts : $U_0 B U_0^{+} = \mathrm{diag}(b_{p(1)}, \cdots, b_{p(N)})$ where $p \in S_N$, permutation group.

- ONE-LOOP DETERMINANT

Fix a saddle pt. $U_0 = id_N$

quadratic piece of S_{def} around U_0 ;

$$U = e^{ih} \quad \text{where } h = \begin{bmatrix} 0 & & y_{ij} \\ & \ddots & \\ \bar{y} & & 0 \end{bmatrix} \quad i<j$$

[a] $\delta^{(2)} S = tr[AhBh] - \frac{1}{2} tr[Ah^2 B] - \frac{1}{2} tr[ABh^2]$

$$= \sum_{i<j} \underbrace{[a_i b_j + a_j b_i - a_i b_i - a_j b_j]}_{= -(a_i - a_j)(b_i - b_j)} y_{ij} \bar{y}_{ij}$$

[b] $g^{a\underline{b}}\partial_a S\,\partial_{\underline{b}} S = \sum_{i<j} (a_i-a_j)^2 (b_i-b_j)^2\, y_{ij}\,\bar{y}_{ij}$ $\qquad [\partial_a \doteq \partial_{y_{ij}},\ \partial_{\underline{b}} \doteq \partial_{\bar{y}_{ij}}]$

$$\partial_a \partial_{\underline{b}} S \cdot \psi^a \bar{\psi}^{\underline{b}} = -\sum_{i<j} (a_i-a_j)(b_i-b_j)\,\psi_{ij}\,\bar{\psi}_{ij}$$

[c] $S[\mathrm{id}] = \sum_i a_i b_i$

$$\det{}_f / \det{}_b \Big|_{u_0=\mathrm{id}.} = \prod_{i<j} \frac{(a_i-a_j)(b_i-b_j)}{(a_i-a_j)^2 (b_i-b_j)^2} = \prod_{i<j} \frac{1}{(a_i-a_j)(b_i-b_j)}$$

[d] Sum over other saddle pts.

$$\Rightarrow \qquad I(AB) \propto \sum_{p\in S_N} \frac{\prod_i e^{-a_i b_{p(i)}}}{\prod_{i<j} (a_i-a_j)(b_{p(i)}-b_{p(j)})}$$

$$= \frac{\det(e^{-a_i b_j})}{\prod_{i<j} (a_i-a_j)(b_i-b_j)} \qquad \blacksquare$$

In general, it is not easy to compute $\mathrm{Val}_\epsilon[X]$

① Suppose that the space X is embedded into a (flat) ambient space M in a 'nice' way

② Can we compute $\mathrm{Val}_\epsilon[X]$ in the ambient space M ?

SYMPLECTIC QUOTIENT $X = M /\!/ U(1)$

On M, G generate $U(1)$ rotation $\Rightarrow$ Hamiltonian flow = moment map

$$i_G \omega = d\mu$$

[1] Define a level surface $\mu^{-1}(\xi) = \{x \in M \mid \mu(x) = \xi\}$

[2] $\mu^{-1}(\xi)$ is invariant under $U(1)$

$$\mu(\vec{x}+\vec{G}) - G^\mu \partial_\mu \mu = G^\mu G^\nu \omega_{\mu\nu} = 0 \,!$$

[3] $X = \mu^{-1}(\xi)/U(1)$ (well-defined)

IN GLSM, $U(1)$ = gauge symmetry & μ = D-term

FORMULA $X = \mu^{-1}(\xi)/U(1)$

$$\mathrm{Vol}_\epsilon[X] = \frac{1}{2\pi\,\mathrm{Vol}[U(1)]}\int_{T[1]M} [dx^M][d\psi^M][d\phi]\; e^{\omega + i\phi[\mu(X^M)-\xi]}$$

- introduce an auxiliary variable ϕ (Lagrangian multiplier)

- SUSY transf. rules

generate $U(1)$!

$$\delta X^M = \psi^M \qquad \delta\psi^M = -i\phi V^M \qquad \delta\phi = 0$$

Then, one can show

$$\delta[\omega + i\phi\mu] = -i\phi\,\omega_{MN}V^M\psi^N + i\phi\,\partial_N\mu\,\psi^N = 0$$

$\Rightarrow \mathrm{Vol}_\epsilon[X]$ is invariant under SUSY!

Proof of formula

- Introduce a coordinate system $\{x^i, x^v, x^n\} \equiv \{x^M\}$

x^v: $V = \frac{\partial}{\partial x^v}$

x^i: $X = M /\!/ U(1)$

x^n: normal to $\mu^{-1}(\xi)$

$d\mu = i_V \omega$ implies that $\omega_{vi} = \frac{\partial}{\partial x^i}\mu$ & $\omega_{vn} = \frac{\partial}{\partial x^n}\mu$

- $\mathrm{Vol}(X) = \int_{\Pi[1]X} [dx^i][d\psi^i]\, e^{\omega_X}$ with $\omega_X = \frac{1}{2}\omega_{ij}\psi^i\psi^j$

$$= \frac{1}{\mathrm{Vol}[U(1)]}\int [dx^v][dx^i][d\psi^i]\, e^{\omega_X} \quad (\because \mu^{-1}(\xi) \text{ is inv. under } U(1)_V)$$

$$= \frac{1}{\mathrm{Vol}[U(1)]}\int [dx^n][dx^v][dx^i][d\psi^i]\, e^{\omega_X}\, \delta(\mu-\xi)\cdot\frac{\partial\mu}{\partial x^n}$$

Note that $\delta(\mu-\xi) = \frac{1}{2\pi}\int d\phi\, e^{i\phi(\mu-\xi)}$ & $\frac{\partial\mu}{\partial x^n} = \int d\psi^v d\psi^n\, e^{\frac{\partial\mu}{\partial x^n}\psi^v\psi^n}$

$$= \frac{1}{2\pi\,\mathrm{Vol}[U(1)]}\int_{\Pi[1]M} e^{\omega_M} \quad \text{with } \omega_M = \frac{1}{2}\omega_{ij}\psi^i\psi^j + \omega_{vn}\psi^v\psi^n + \omega_{vi}\psi^v\psi^i$$

don't affect the result!

- It is convenient to introduce more auxiliary variables

$$\bar{\phi} : B \qquad \eta : F$$

$$h : B \qquad \chi : F$$

SUSY transf. $\delta \bar{\phi} = \eta \qquad \delta \eta = 0 \qquad \delta \chi = h \qquad \delta h = 0$

- Q-exact term $S_{def} = t\,\delta \mathcal{V}$ s.t. $\mathcal{V}$ is invariant under $U(1)$

$$\mathcal{V} = \chi\left(\mu(x^M) - \xi - \tfrac{1}{2} h\right) + \bar{\phi} f_M(x) \psi^M \rightarrow \text{gauge charge "0"}$$

$$S_{def} = t\left[h(\mu(x) - \xi) - \tfrac{1}{2} h^2 + -i\phi\bar{\phi} f_M(x) V^M - \chi \partial_M \mu \psi^M + \bar{\eta} \psi^M f_M(x)\right]$$

$$\mathrm{Vol}_\epsilon[X] = \int [dx^M][d\psi^M][d\phi][d\bar{\phi}][dh][d\chi][d\eta]\, e^{\omega + i\phi\mu(x) + t\delta\mathcal{V}}$$

Localization symmetry of $X \Rightarrow$ symmetry of M, generated by V

$$\delta_\epsilon X^M = \psi^M \qquad \delta_\epsilon \psi^M = -i\phi G^M(x) + \varepsilon V^M(x) \qquad \delta_\epsilon \phi = 0$$

NB $[V, G] = 0$!

Follow the same steps

$$\mathrm{Vol}_\epsilon[X] = \frac{1}{2\pi \,\mathrm{Vol}[U(1)]} \int [dX^M][d\psi^M][d\phi]\, e^{\omega + i\phi\mu(x) - \varepsilon H(x) + \delta_\epsilon V}$$

Example $\mathbb{P}^1 \equiv S^2$

$\mathbb{C}^2 /\!/ U(1)$ $\quad (z_1, z_2) \xrightarrow{U(1)} e^{i\alpha}(z_1, z_2) \quad \mathbf{V} = z_i \partial_i - \bar{z}_i \partial_{\underline{i}}$

[1] isometry $\mathbf{R} = z_1\partial_1 - z_2\partial_2 - \bar{z}_1\bar{\partial}_1 + \bar{z}_2\bar{\partial}_2 \; : \; (z_1, z_2) \longrightarrow (e^{i\lambda}z_1, e^{-i\lambda}z_2)$

[2] Q_ϵ-exact terms $\quad S_{def} = t[\delta_\epsilon \mathcal{V}]$

$$\epsilon\mathcal{V} = -g_{a\underline{b}}\,\delta_\epsilon\psi^a\bar{\psi}^{\underline{b}} - g_{a\underline{b}}\,\psi^a\delta_\epsilon\bar{\psi}^{\underline{b}} + \chi\left(\mu(z,\bar{z}) - 1 - \frac{h}{2}\right) + \bar{\phi}\left[\partial_a\mu\,\psi^a - \partial_{\underline{b}}\mu\,\bar{\psi}^{\underline{b}}\right]$$

$$S^b_{def} = +2\left[\phi V^a + i\epsilon R^a\right]\left[\phi V^{\underline{a}} + i\epsilon R^{\underline{a}}\right] g_{a\underline{a}} + h\left(\mu - 1 - \frac{h}{2}\right)$$
$$+\bar{\phi}\left[-i\phi V^a + \epsilon R^a\right]\partial_a\mu - \bar{\phi}\left[-i\phi V^{\underline{a}} + \epsilon R^{\underline{a}}\right]\partial_{\underline{a}}\mu$$

$$S^f_{def} = \left[i\phi V^a{}_b - \epsilon R^a{}_b\right]\psi^b\bar{\psi}^{\underline{b}} g_{a\underline{b}} - \left[i\phi V^{\underline{a}}{}_{\underline{b}} - \epsilon R^{\underline{a}}{}_{\underline{b}}\right]\psi^a\bar{\psi}^{\underline{b}} g_{a\underline{a}}$$
$$+(\eta - \chi)\,\psi^a\partial_a\mu + (\eta + \chi)\,\bar{\psi}^{\underline{b}}\partial_{\underline{b}}\mu\Big]$$

[3] saddle points $\quad \delta_\epsilon \psi^a = \delta_\epsilon \bar{\psi}^{\underline{b}} = \delta_\epsilon \chi = 0$ & $\delta S_{def}[\Phi_*] = 0$

① $\phi V^a + i\epsilon R^a = \phi(z^1, z^2) + i\epsilon(z^1, -z^2) = 0$!

$\Rightarrow$ two solutions : $(\phi = -i\epsilon,\ z^2 = 0), (\phi = +i\epsilon,\ z^1 = 0)$

② $h = 0$

③ $\dfrac{\delta S_{def}}{\delta h} = 0 \longrightarrow h = \mu(z, \bar{z}) - 1$

$\left.\right\}$ $h = 0$ & $\mu = 1$

④ $\dfrac{\delta S_{def}}{\delta \phi} = 0 \longrightarrow \bar{\phi}[V^a \partial_a \mu - V^{\underline{b}} \partial_{\underline{b}} \mu] \propto \bar{\phi}\mu = 0 \quad \therefore \bar{\phi} = 0$!

$\therefore \exists$ two saddle pts $(z^1 = 1, z^2 = 0, \phi = -i\epsilon, \bar{\phi} = 0, h = 0)$ N

$(z^1 = 0, z^2 = 1, \phi = +i\epsilon, \bar{\phi} = 0, h = 0)$ S

HW: work out the one-loop determinant around these two fixed points and see if one can get the same answer

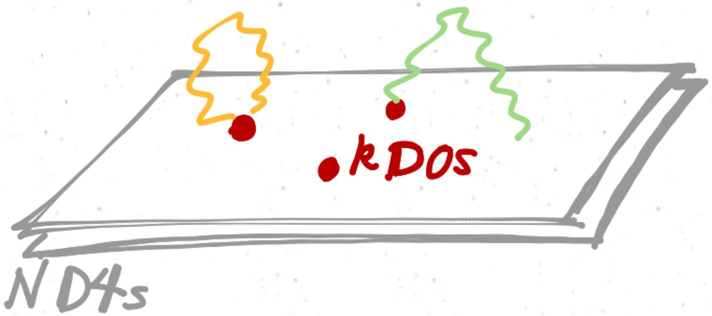
NB Quotient with Non-abelian group

$$\delta\varphi^M = \psi^M \qquad \delta\psi^M = [\phi, \varphi^M] \qquad \delta\bar{\phi} = \eta \qquad \delta\eta = [\phi, \bar{\phi}]$$

$$\delta\chi = h \qquad \delta h = [\phi, h] \qquad \delta^2 = \mathrm{Gauge}(\phi)$$

Adjoint Representation

NEKRASOV INSTANTON PARTION FUCTION



= EQUIVARIANT VOLUME OF INSTANTON MODULI SPACE

	0	1	2	3	4	5	6	7	8	9
D4	x	x	x	x	x					
D0	x									

Global symmetry

$$G = SO(4) \times Sp(4) \times SU(N)_F$$

Low-energy dynamics on D0 branes can be described as "$U(k)$ GAUGED QM with 8 supercharges", involving

(i) Vector $(A_0, \varphi^I, \bar{\lambda}^{i}_{\dot\alpha})$

(ii) adjoint hyper $(a_m, \lambda^{i}_{\alpha})$

(iii) fund. hyper $(q_{\dot\alpha}{}^{A}, \psi^{iA})$

$I = 1,2,\cdots,5$ $SO(5)$ indices

$i = 1,2,\cdots,4$ $Sp(4)$ indices

SUSY Action

$$
\begin{aligned}
L_{SYM} \;=\; & \mathrm{tr}_k \Big(\frac{1}{2} D_t\varphi_I D_t\varphi_I + \frac{1}{2} D_t a_m D_t a_m + \frac{1}{4}[\varphi_I, \varphi_J]^2 + \frac{1}{2}[a_m, \varphi_I]^2 + \frac{1}{4}[a_m, a_n]^2 \\
& + \frac{i}{2}(\bar\lambda^{i\dot\alpha})^\dagger D_t \bar\lambda^{i\dot\alpha} + \frac{1}{2}(\bar\lambda^{i\dot\alpha})^\dagger (\gamma^I)^i{}_j [\varphi_I, \bar\lambda^{j\dot\alpha}] + \frac{i}{2}(\lambda^i_\alpha)^\dagger D_t \lambda^i_\alpha - \frac{1}{2}(\lambda^i_\alpha)^\dagger (\gamma^I)^i{}_j [\varphi_I, \lambda^j_\alpha] \\
& - \frac{i}{2}(\lambda^i_\alpha)^\dagger (\sigma^m)_{\alpha\dot\beta} [a_m, \bar\lambda^{i\dot\beta}] + \frac{i}{2}(\bar\lambda^{i\dot\alpha})^\dagger (\bar\sigma^m)^{\dot\alpha\beta} [a_m, \lambda^i_\beta] \Big) \; .
\end{aligned}
$$

$$
\begin{aligned}
L_f \;=\; & D_t q_{\dot\alpha} D_t \bar q^{\dot\alpha} - (\varphi_I \bar q^{\dot\alpha} - \bar q^{\dot\alpha} v_I)(q_{\dot\alpha}\varphi_I - v_I q_{\dot\alpha}) + i(\psi^i)^\dagger D_t \psi^i + (\psi^i)^\dagger (\gamma^I)^i{}_j \;\; \psi^j \varphi_I \\
& + \sqrt{2} i \left((\bar\lambda^{i\dot\alpha})^\dagger \bar q^{\dot\alpha} \psi^i - (\psi^i)^\dagger q_{\dot\alpha} \bar\lambda^{i\dot\alpha} \right) \; ,
\end{aligned}
$$

SUSY Transformation Rules

$$\bar{Q}^{i\dot{\alpha}} A_t \;=\; i\bar{\lambda}^{i\dot{\alpha}} \;, \quad \bar{Q}^{i\dot{\alpha}} \varphi^I = -i(\gamma^I)^i{}_j \bar{\lambda}^{j\dot{\alpha}}$$

$$\bar{Q}^{i\dot{\alpha}} \bar{\lambda}^{j\dot{\beta}} \;=\; \epsilon^{\dot{\alpha}\dot{\beta}} (\gamma^I \omega)^{ij} D_0 \varphi^I - \frac{i}{2} \epsilon^{\dot{\alpha}\dot{\beta}} (\gamma^{IJ} \omega)^{ij} [\varphi^I, \varphi^J] - 2i\omega^{ij} D^{\dot{\alpha}}{}_{\dot{\gamma}} \epsilon^{\dot{\gamma}\dot{\beta}}$$

$$\bar{Q}^{i\dot{\alpha}} a^m = (\bar{\sigma}^m)^{\dot{\alpha}\beta} \lambda^i_\beta$$

$$\bar{Q}^{i\dot{\alpha}} \lambda^j_\beta = (\bar{\sigma}^m)^{\dot{\alpha}\gamma} \epsilon_{\gamma\beta} \left(i\omega^{ij} D_t a_m + (\gamma^I \omega)^{ij} [\varphi_I, a_m] \right)$$

$$\bar{Q}^{i\dot{\alpha}} q_{\dot{\beta}} = \sqrt{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \psi^i$$

$$\bar{Q}^{i\dot{\alpha}} \psi^j = \sqrt{2} \left[i\omega^{ij} D_t q_{\dot{\beta}} - (\gamma^I \omega)^{ij} q_{\dot{\beta}} \varphi_I \right] \epsilon^{\dot{\beta}\dot{\alpha}}$$

Why do we study D0-D4 QM (or dim'l reduction to matrix model)?

<u>SW theory</u> can determine the low-energy effective theory in the Coulomb branch of 4d $\mathcal{N}=2$ SUSY gauge theories

Question derivation from first principles ?

$\Rightarrow$ equivariant Volume of $\mathcal{M}^{k,N}_{inst}$ moduli space of YM instantons !

partition function of D0-D4 QM can be identified as the volume of $\mathcal{M}^{k,N}_{inst}$

$SU(2)_\ell \times SU(2)_\lambda \times SU(2)_L \times SU(2)_R \subset$ global sym. G

BRST charge

superchange $\bar{Q}^i_{\dot\alpha}$: $(1,2,2,1) \oplus (1,2,1,2)$

Vector multiplet

$A_0 + i\varphi_5 \equiv \phi$: $(1,1,1,1)$

$A_0 - i\varphi_5 \equiv \bar\phi$: $(1,1,1,1)$

φ_m : $(1,1,2,2)$

λ^i_α : $(2,1,2,1) \oplus (2,1,1,2)$

adj. hyper

a_m : $(2,2,1,1)$

$\bar\lambda^i_{\dot\alpha}$: $(1,2,2,1) \oplus (1,2,1,2)$

fund. hyper

$q_{\dot\alpha}$: $(1,2,1,1)$

ψ^i : $(1,1,2,1) \oplus (1,1,1,2)$

$$SU(2)_l \times \widehat{SU(2)}_r \times SU(2)_L \subset SU(2)_l \times SU(2)_r \times SU(2)_L \times SU(2)_R$$

supercharge $\bar{Q}^i_{\dot\alpha}$: $(1,2,2,1)\oplus(1,2,1,2) \to \cdots \oplus (1,1,1) \oplus (1,3,1)$

Vector multiplet $A_0 + i\varphi_5 \equiv \phi$: $(1,1,1,1) \to (1,1,1)$

$A_0 - i\varphi_5 \equiv \bar{\phi}$: $(1,1,1,1) \to (1,1,1)$

φ_m : $(1,1,2,2) \to (1,2,2)$

λ^i_α : $(2,1,2,1)\oplus(2,1,1,2) \to \underset{\vec{\chi}}{(2,1,2)} \oplus \underset{\Psi_m}{(2,2,1)}$

adj. hyper a_m : $(2,2,1,1) \to (2,2,1)$

$\bar{\lambda}^i_{\dot\alpha}$: $(1,2,2,1)\oplus(1,2,1,2) \to \underset{\Psi_{m+4}}{(1,2,2)} \oplus \underset{\eta}{(1,1,1)} \oplus \underset{\vec{\chi}}{(1,3,1)}$

fund. hyper $q_{\dot\alpha}$: $(1,2,1,1) \to (1,2,1)$

ψ^i : $(1,1,2,1)\oplus(1,1,1,2) \to \underset{\xi_a}{(1,1,2)} \oplus \underset{\psi_{\dot\alpha}}{(1,2,1)}$

SUSY transf. = SUSY transf. in the case of symplectic quotient.

$$\delta\phi = 0, \quad \delta\bar\phi = \eta, \quad \delta\eta = [\phi, \bar\phi], \quad \delta a_m = \Psi_m, \quad \delta\varphi_m = \Psi_{m+4}$$

$$\delta\Psi_m = [\phi, a_m], \quad \delta\Psi_{m+4} = [\phi, \varphi_m], \quad \delta q_{\dot\alpha} = \psi_{\dot\alpha}, \quad \delta\psi_{\dot\alpha} = \phi\, q_{\dot\alpha}$$

$$\delta\vec\chi = \underbrace{\overset{7}{\vec H}, \quad \delta\vec H = [\phi, \vec\chi], \quad \delta\xi_a = \overset{2}{h_a}}_{\text{auxiliary variables!}}, \quad \delta h_a = \phi\,\xi_a$$

$$\Rightarrow \delta^2 = \text{gauge}_{U(k)}[\phi]$$

Then, the SUSY Lagrangian in '0'-dimensions [dim'l red. of QM] is Q-exact!!

SUSY ACTION

$$\vec{\mathcal{E}} = (\mathcal{E}_R^A, \mathcal{E}_L^A, \mathcal{E})$$
$$\equiv \left(\frac{i}{2}\bar{\eta}^A_{mn}\left([\varphi_m,\varphi_n]-[a_m,a_n]\right) - \bar{x}^{\dot\alpha}x_{\dot\beta}(\tau^A)^{\dot\beta}{}_{\dot\alpha} + \zeta^A, \frac{i}{2}\eta^A_{mn}\left([\varphi_m,a_n]+[a_m,\varphi_n]\right), -i[\varphi_m,a_m] \right)$$

$$i\mathcal{F}_a \equiv -(\sigma^m)_{a\dot\alpha}\,\epsilon^{\dot\alpha\dot\beta}x_{\dot\beta}\varphi_m$$

$$\underset{\delta\mathcal{V}_0}{\overset{\parallel}{S}} = \delta\Big[\frac{1}{8}\eta[\phi,\bar\phi] - \frac{1}{2}\Psi_m[\bar\phi,a_m] - \frac{1}{2}\Psi_{m+4}[\bar\phi,\varphi_m] + \vec\chi\cdot\left(\frac{1}{2}\vec H - i\vec{\mathcal{E}}\right)$$
$$+\xi_a\left(\frac{1}{2}h_a^\dagger - i\mathcal{F}_a^\dagger\right) + \xi_a^\dagger\left(\frac{1}{2}h_a - i\mathcal{F}_a\right) + \bar\psi^{\dot\alpha}\bar\phi\, q_{\dot\alpha} + \bar q^{\dot\alpha}\bar\phi\,\psi_{\dot\alpha}\Big]$$

$$\Rightarrow\ Z_{D\text{-}1} = \int [da_m\, dq\, d\bar q\, d\Psi_m\, d\psi\, d\bar\psi][d\varphi_m\, d\Psi_{m+4}][d\bar\phi\, d\eta\, d\vec\chi\, d\vec H\, d\xi\, d\bar\xi\, d\mathcal{F}\, d\bar{\mathcal{F}}]$$
$$[d\phi]\ \mathrm{Exp}[\omega + i\phi\mu + \delta\mathcal{V}_0]$$

Note also that, in flat ambient space. $\omega + i\phi\mu = \delta(\#)$

Equivariant Volume of $\mathcal{M}^{k,N}_{\text{inst}}$ = "Path"-integral of D-1 matrix theory

$$Z_{D\text{-}1} = \int [da_m\, dq\, d\bar{q}\, d\Psi_m\, d\psi\, d\bar{\psi}]\,[d\varphi_m\, d\Psi_{m+4}]\,[d\bar{\Phi}\, d\eta\, d\vec{\chi}\, d\vec{H}\, d\xi\, d\bar{\xi}\, dF\, d\bar{F}]\,[d\phi]\ \mathrm{Exp}[\omega + i\phi\mu + \delta\mathcal{V}_0]$$

- Volume of the instanton moduli space [$U(k)$-quotient], $\mathcal{M}^{k,N}_{\text{inst}}$

- $\mathcal{M}^{k,N}_{\text{inst}}$ is non-compact

- Equivariant parameters $\quad \underbrace{SU(2)_\ell \times \widehat{SU(2)}_r}_{\text{Nekrasov's } \Omega\text{-parameters } [\epsilon_1, \epsilon_2]} \times \overbrace{SU(N)_F}^{\text{Coulomb branch parameter } a}$

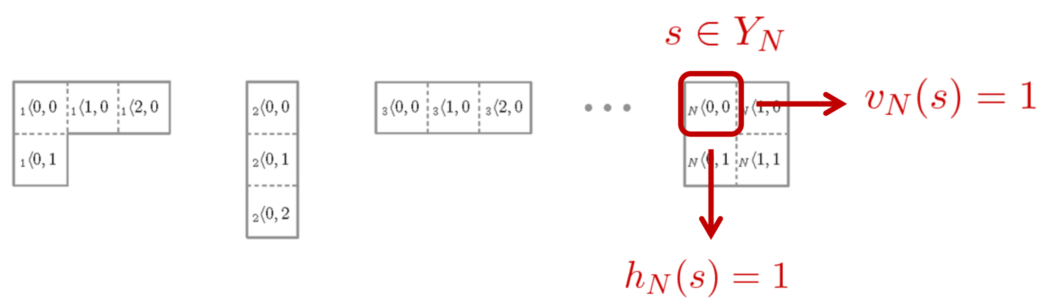
- Turning on FI parameter, $\quad Z_{D\text{-}1} = Z^{k}_{\text{Nek}}[\vec{a}, \epsilon_1, \epsilon_2]$

Nekrasov Partition Function

Result: [Nekrasov]

. Volume can be computed by Gaussian path-integrals over a set of saddle-points (solutions of deformed ADHM), characterized by N-colored Young diagrams

$$\vec{Y} = \{Y_1, Y_2, ..Y_N\} \quad :$$



$$\Rightarrow \qquad I(\vec{\mu}, \gamma, q) = \sum_{\vec{Y}=\{Y_1,Y_2,..Y_N\}} q^{|\vec{Y}|} I_{\{Y_1,Y_2,..Y_N\}}$$

where

$$I_{\{Y_1,Y_2,\cdots,Y_N\}} = \prod_{i,j=1}^{N} \prod_{s\in Y_i} \frac{\sinh \frac{E_{ij}-i(\gamma_2+\gamma_R)}{2} \sinh \frac{E_{ij}+i(\gamma_2-\gamma_R)}{2}}{\sinh \frac{E_{ij}}{2} \sinh \frac{E_{ij}-2i\gamma_R}{2}}$$

$$E_{ij} = \mu_i - \mu_j + i(\gamma_1 - \gamma_R)h_j(s) + i(\gamma_1 + \gamma_R)(v_i(s) + 1)$$

Part II

Exact S^2 Partition Function

Hard Problems in Theoretical Physics

Particles (or strings) that do strongly interact with each other

- Many intriguing phenomena in quantum field theory and string theory

e.g. **-** Quark confinement in QCD

- M-theory: unified framework for quantum gravity

inherently strongly interacting

string theories in weak-coupling limits

- Non-fermi liquid, High T_c superconductor, and so on

- These systems are **far beyond the perturbative regime**

How to Study Strong Dynamics

In theoretical physics, there are three different methods

1. DUALITY

Gauge/Gravity duality (a.k.a. holography) leads to

a number of novel predictions on strong dynamics

Strong Gauge Dynamics = Classical Einstein Gravity

2. SUPERSYMMETRY

Complementary strategy to attack strong coupling problem

3. INTEGRABILITY

So powerful! But rare in quantum field theories

How to Study Strong Dynamics

A New Technique: Exact Result

Supersymmetry:

- Exactly computable observables in SUSY theories in the past ['80-'90]

- They know the vacuum dynamics only, or are relevant to math

e.g.
- Witten Index : Vacuum
- Topological Invariants: Math
- **A**ffleck-**D**ine-**S**eiberg superpotential: Vacuum
- Seiberg-Witten solution: limited to 4d N=2 theories

How to Study Strong Dynamics

A New Technique: Exact Result

Question:

“ Can we find an exactly computable observable in SUSY theories? ”

- Wide application to any SUSY theories
- Relevance to physics beyond the vacuum dynamics
- Remarkable progress recently (including my own), using SUSY localization techniques

Localization

Start with a following path-integral

$$Z[t] = \int \mathcal{D}\Phi \; e^{-S[\Phi] - tQ.V[\Phi]} \qquad \begin{aligned} Q.S[\phi] &= 0 \\ Q^2 &= J \end{aligned}$$

- $S[\Phi]$: action of a theory we want to study
- The term V is invariant under J, $J.V[\Phi] = 0$

Supersymmetry tells us

$Z[0]$	=	$Z[\infty]$
$S[\Phi]$		$S_{\text{def}} = Q.V[\Phi]$
Quantum		**Semi-Classical**
Hard to evaluate		Easy to evaluate (**Gaussian Integral**)

Localization

RESULT

$$\boxed{Z[0] = \sum_{\phi_*} e^{-S[\phi_*]} \left(\frac{\det \Delta_f^{\text{def}}}{\det \Delta_b^{\text{def}}} \right) \Bigg|_{\phi_*}}$$

where $\boldsymbol{\phi}_*$ satisfy (1) equation of motion of the deformed theory $\left.\frac{\delta S_{\text{def}}}{\delta \phi}\right|_{\phi=\phi_*} = 0$

(2) supersymmetric condition

For BPS operators $Q.\mathcal{O}_{\text{BPS}} = 0$,

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \sum_{\phi_*} e^{-S[\phi_*]} \left(\frac{\det \Delta_f^{\text{def}}}{\det \Delta_b^{\text{def}}} \right) \Bigg|_{\phi_*} \mathcal{O}_{\text{BPS}}(\phi_*)$$

Old vs New

80s-90s: constant spinor

$$\nabla_\mu \epsilon = 0$$

- Nilpotent supercharge $\mathcal{Q}^2 = 0$
- Topological twisting needed
- Full Hilbert space is projected down to the subspace "topological Hilbert space"

Recent : Killing spinor

$$\nabla_\mu \epsilon = \gamma_\mu \tilde{\epsilon}$$

- Not Nilpotent supercharge, but $\mathcal{Q}^2 = J$
- **J** generates an isometry of (compact) space
- No topological twisting needed !
- $\exists$ new observable telling us about something **beyond the vacuum**

Recent Developments

New Physical Observables:

Sphere-Partition Function	Superconformal Index
S^4 : [Pestun]	**S^3 x S^1**: [Romelsberger]
S^3 : [Kapustin,Willet,Yaakov]	**S^2 x S^1**: [Kim]
[Jafferis]	[Imamura,Yokohama]
[Hama,Hosomichi,**S.L**]	

We learned AGT correspondence, F-theorem, Test of Dualities and so on

S^2 : **This is what I want to discuss today !**

New Exact Results in 2d SUSY Theories

- SUSY Lagrangian on Two-Sphere
- Exact Two-Sphere Partition Function

N=(2,2) SUSY on S^2

SUSY on Two-Sphere: **SU(2|1)**

- Subalgebra of N=(2,2) SCA:

$$\{S_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \qquad [J_m, S^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} S_\beta \qquad [R, S_\alpha] = +S_\alpha$$

$$\{\bar{S}_\alpha, \bar{Q}_\beta\} = -\gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \qquad [J_m, Q^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} Q_\beta \qquad [R, Q_\alpha] = -Q_\alpha$$

$$\{Q_\alpha, \bar{Q}_\beta\} = \gamma^m_{\alpha\beta} K_m + \frac{1}{2} C_{\alpha\beta} \mathcal{A} \qquad [J_m, \bar{Q}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{Q}_\beta \qquad [R, \bar{Q}_\alpha] = +\bar{Q}_\alpha$$

$$\{S_\alpha, \bar{S}_\beta\} = \gamma^m_{\alpha\beta} K_m - \frac{1}{2} C_{\alpha\beta} \mathcal{A} \qquad [J_m, \bar{S}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{S}_\beta \qquad [R, \bar{S}_\alpha] = -\bar{S}_\alpha$$

$$[J_m, J_n] = i\epsilon_{mnp} J^p \qquad [K_m, S^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{Q}_\beta \qquad [\mathcal{A}, S_\alpha] = \bar{Q}_\alpha$$

$$[K_m, K_n] = -i\epsilon_{mnp} J^p \qquad [K_m, Q^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{S}_\beta \qquad [\mathcal{A}, Q_\alpha] = -\bar{S}_\alpha$$

$$[J_m, K_n] = i\epsilon_{mnp} K^p \qquad [K_m, \bar{Q}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} S_\beta \qquad [\mathcal{A}, \bar{Q}_\alpha] = -S_\alpha$$

$$[K_m, \bar{S}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} Q_\beta \qquad [\mathcal{A}, \bar{S}_\alpha] = Q_\alpha \, .$$

N=(2,2) SUSY on S^2

SUSY on Two-Sphere: **SU(2|1)**

- Subalgebra of N=(2,2) SCA

- Bosonic subalgebra:

  - SU(2): rotational symmetry of S^2

  - U(1): vector U(1) R-symmetry

NB: axial U(1) R-symmetry is broken unless the theory is conformal

- Parametrized by Killing spinors $(\epsilon, \bar{\epsilon})$ satisfying

$$\nabla_i \epsilon = +\frac{1}{2l}\gamma_i \gamma^3 \epsilon \qquad \nabla_i \bar{\epsilon} = -\frac{1}{2l}\gamma_i \gamma^3 \bar{\epsilon}$$

N=(2,2) SUSY on S^2

SUSY on Two-Sphere: **SU(2|1)**

- Parametrized by Killing spinors $(\epsilon, \bar{\epsilon})$ satisfying

$$\nabla_i \epsilon = +\frac{1}{2l}\gamma_i \gamma^3 \epsilon \qquad \nabla_i \bar{\epsilon} = -\frac{1}{2l}\gamma_i \gamma^3 \bar{\epsilon}$$

Solutions of Killing spinor equations are given by

two-component constant spinors

2 $$\epsilon = \exp\left(-\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\epsilon_\circ$$

2 $$\bar{\epsilon} = \exp\left(+\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\bar{\epsilon}_\circ$$

N=(2,2) SUSY on S^2

SUSY Rep. in Field Theories: $\exists\ 1/l$ - correction in SUSY transf. rules,

- Vector multiplet, $(A_i, \sigma_1, \sigma_2, \lambda, D)$

$$\delta\lambda = \cdots + i\Big(F_{12} + i[\sigma_1, \sigma_2] + \frac{1}{l}\sigma_1\Big)\gamma^3\epsilon$$

- (charged) Chiral multiplet of $U(1)_R$ charge **q**, (ϕ, ψ, F)

$$\delta\psi = \cdots + i\frac{q}{2l}\phi\gamma_3\epsilon$$

- Twisted chiral multiplet, (Y, χ, G)

$$\delta\chi = \cdots - \frac{1+\gamma^3}{2}\epsilon\left(\bar{G} + \frac{\Delta}{l}\overline{Y}\right) - \frac{1-\gamma^3}{2}\epsilon\left(G + \frac{\Delta}{l}Y\right)$$

$\mathbf{\Delta}$: free parameter

N=(2,2) SUSY on S^2

SUSY Rep. in Field Theories:

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \delta_{SU(2)}(\xi) + \delta_R(\alpha) + \delta_G(\Lambda) \qquad [\delta_\epsilon, \delta_\epsilon] = 0 \qquad [\delta_{\bar{\epsilon}}, \delta_{\bar{\epsilon}}] = 0$$

[1] SU(2) rotation $\quad \mathcal{L}_\xi \equiv \xi^i \nabla_i + \underline{\frac{1}{4}\nabla_i \xi_j \gamma^{ij}} \qquad \xi^i = -i\bar{\epsilon}\gamma^i\epsilon$

Lorentz transf. acting on fermions

[2] $U(1)_R$ rotation $\quad \alpha = -\frac{1}{2r}\bar{\epsilon}\gamma^{\hat{3}}\epsilon$

[3] Gauge rotation $\quad \Lambda = (\bar{\epsilon}\epsilon)\sigma_1 - i(\bar{\epsilon}\gamma^{\hat{3}}\epsilon)\sigma_2 + \xi^i A_i$

N=(2,2) SUSY on S^2

SUSY Lagrangian on S^2: up to $(1/l)^2$ - corrections

- Kinetic Lagrangians for

[1] Vector multiplet :

$$\mathcal{L}_{\text{v.m.}} = \frac{1}{2g^2}\mathrm{Tr}\left[\left(F_{12}+\frac{\sigma_1}{l}\right)^2 + (D_i\sigma_1)^2 + (D_i\sigma_2)^2 - [\sigma_1,\sigma_2]^2 + \mathrm{D}^2 + i\bar{\lambda}\gamma^i D_i\lambda + i\bar{\lambda}[\sigma_1,\lambda] + \bar{\lambda}\gamma^3[\sigma_2,\lambda]\right]$$

[2] Chiral multiplet of $U(1)_R$ charge q :

$$\mathcal{L}_{\text{c.m.}} = D_i\bar{\phi}D^i\phi + \bar{\phi}\left[\sigma_1^2 + \sigma_2^2 + i\frac{q-1}{l}\sigma_2 - \frac{q^2}{4l^2} + \frac{q}{4}\mathcal{R}\right]\phi + \bar{F}F + i\bar{\phi}\mathrm{D}\phi$$

$$- i\bar{\psi}\gamma^i D_i\psi + \bar{\psi}\left[i\sigma_1 - \left(\sigma_2 + i\frac{q}{2l}\right)\gamma^3\right]\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi$$

[3] Twisted chiral multiplet :

$$\mathcal{L}_{t c.m.} = D^i\overline{Y}D_iY + i\bar{\chi}\gamma^i D_i\chi + \left(\bar{G}+\frac{\Delta}{l}\overline{Y}\right)\left(G+\frac{\Delta}{l}Y\right)$$

N=(2,2) SUSY on S^2

SUSY Lagrangian on S^2: up to $(1/l)^2$ - corrections

- Superpotential $\mathcal{W}(\phi)$: possible if **q[W] = 2** (q : $U(1)_R$ charge)

$$\mathcal{L}_{\mathcal{W}} = \frac{\partial \mathcal{W}}{\partial \phi^i} F^i - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial \phi^i \partial \phi^j} \psi^i \psi^j + \text{c.c.}$$

NB: SUSY on S^2 should contain a conserved $U(1)_R$ charge

- Twisted Superpotential $W(Y)$ (twisted chiral multiplet (Y, χ, G))

$$\mathcal{L}_W = -iW'(Y)G - W''(Y)\bar{\chi}\left(\frac{1-\gamma^3}{2}\chi\right) + \frac{i}{l} W(Y)$$

e.g. $\mathcal{L}_{\mathrm{FI},\theta} = -i\mathrm{Tr}\left[\xi\left(\mathrm{D} - \frac{\sigma_2}{l}\right) + \frac{\theta}{2\pi} F\right]$

Localize the Path-Integral

Strategy: Localization Principle

- Choice of Supercharge Q and Q-exact Deformation Terms

- SUSY Saddle Point Configurations

- Integration along the Directions Transverse to Saddle Point Locus, Gaussian Approximation Becomes Exact

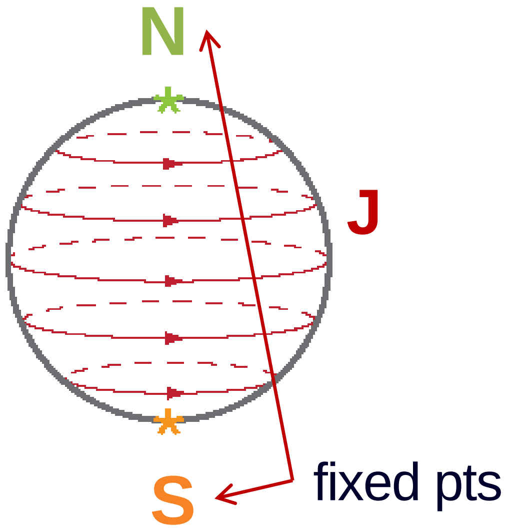
Exact Partition Function

Localization Scheme

- Choice of supercharge : $\mathcal{Q}^2 = J + \frac{R}{2}$

$$\epsilon = \exp\left(-\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\begin{pmatrix}1\\0\end{pmatrix} = e^{+i\varphi/2}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

$$\bar{\epsilon} = \exp\left(+\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\begin{pmatrix}0\\1\end{pmatrix} = e^{-i\varphi/2}\begin{pmatrix}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$



$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \delta_{SU(2)}(\xi) + \delta_R(\alpha) + \delta_G(\Lambda)$$

$$\frac{i}{r}\partial_\varphi \qquad \alpha = \frac{1}{2r} \quad \Lambda = \cos\theta\sigma_1 - i\sigma_2 - A_\varphi$$

Exact Partition Function

Localization Scheme

- *Q*-exact deformation : Given the above choice,

$$\underbrace{\mathcal{L}_{\text{v.m.}} = \mathcal{Q}V_{\text{v.m.}} \quad \mathcal{L}_{\text{c.m.}} = \mathcal{Q}V_{\text{c.m.}} \quad \mathcal{L}_{\text{t.c.m.}} = \mathcal{Q}V_{\text{t.c.m.}}}_{\text{Kinetic Lagrangians: } Q\text{-exact deformations}} \qquad \underbrace{\mathcal{L}_{\mathcal{W}} = \mathcal{Q}V_{\mathcal{W}}}_{\text{Superpotential}}$$

e.g.

$$g^2\,\mathcal{L}_{\text{v.m.}} = \delta_{\mathcal{Q}}\delta_{\bar{\epsilon}_{\mathcal{Q}}}\,\mathrm{Tr}\left(\frac{1}{2}\bar{\lambda}\gamma^{\hat{3}}\lambda - 2i\mathrm{D}\sigma_2 + \frac{i}{r}\sigma_2^2\right) \qquad \mathbf{V}_{\text{v.m.}}$$

$$(\mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}}) = \delta_{\mathcal{Q}}\delta_{\bar{\epsilon}_{\mathcal{Q}}}\,\mathrm{Tr}\left(\bar{\psi}\gamma^{\hat{3}}\psi - 2\bar{\phi}\left(\sigma_2 + m + i\frac{q}{2r}\right)\phi + \frac{i}{r}\bar{\phi}\phi\right) \qquad \mathbf{V}_{\text{c.m.}}$$

Exact Partition Function

Localization Scheme

- Q-exact deformation : Given the above choice,

$$\underbrace{\mathcal{L}_{\rm v.m.} = QV_{\rm v.m.} \quad \mathcal{L}_{\rm c.m.} = QV_{\rm c.m.} \quad \mathcal{L}_{\rm t.c.m.} = QV_{\rm t.c.m.}}_{\text{Kinetic Lagrangians: } Q\text{-exact deformations}} \qquad \underbrace{\mathcal{L}_{\mathcal{W}} = QV_{\mathcal{W}}}_{\text{Superpotential}}$$

Decoupling Theorem: S^2 partition function is independent of

(1) Gauge coupling constant : tell us about **the physics at infrared**

(2) Parameters in superpotential $\mathcal{W}(\phi)$

Exact Partition Function

Gauged Linear Sigma Model (GLSM)

N=(2,2) gauge theory with gauge group G, coupled to chiral multiplets of $U(1)_R$ charge q in rep. **R**

- SUSY saddle point configurations = **Columb branch vacua**

$$\sigma_2 = \sigma = const. \qquad F_{12} + \frac{\sigma_1}{l} = 0$$

$$[\sigma_1, \sigma_2] = 0 \qquad \int_{S^2} F_{12} = 2\pi \underbrace{B}_{\text{GNO Quantized}}$$

and all other fields vanish

Exact Partition Function

Gauged Linear Sigma Model (GLSM)

- **One-loop determinant**

  - Chiral multiplet in rep. **R** : Expand $\mathbf{S}_{\mathbf{def}}$ to quadratic order in the fluctuation fields

$$\Delta_b = \Big[-D_i^2 + (\sigma_1 r)^2 + (\sigma_2 r + i\frac{q}{2})^2 - i(\sigma_2 r + i\frac{q}{2}) \Big]$$

$$\Delta_f = \Big[-i\gamma^i \tilde{D}_i + i\sigma_1 r - (\sigma_2 r + i\frac{q}{2})\gamma^3 \Big]$$

Monopole background

$$A = \frac{B}{2}(\kappa - \cos\theta)\,\mathrm{d}\varphi$$

Exact Partition Function

Gauged Linear Sigma Model (GLSM)

- **One-loop determinant**

  - Chiral multiplet in rep. **R** :

$$\det\Delta_b = \prod_{\rho\in\mathbf{R}}\prod_{J=0}^{\infty}\left[\left(J+\frac{|\rho\cdot B|+q}{2}-il\rho\cdot\sigma\right)\cdot\left(J+1+\frac{|\rho\cdot B|-q}{2}+il\rho\cdot\sigma\right)\right]^{2J+|\rho\cdot B|+1}$$

$$\det\Delta_f = \prod_{\rho\in\mathbf{R}}(-1)^{\frac{|\rho\cdot B|-\rho\cdot B}{2}}\prod_{J=0}^{\infty}\left[\left(J+\frac{|\rho\cdot B|+q}{2}-il\rho\cdot\sigma\right)^{2J+|\rho\cdot B|}\times\left(J+1+\frac{|\rho\cdot B|-q}{2}+il\rho\cdot\sigma\right)^{2J+|\rho\cdot B|+2}\right]$$

$\Rightarrow$ **Huge Cancellation between B and F Eigenvalues Occurs !**

Exact Partition Function

Gauged Linear Sigma Model (GLSM)

- **One-loop determinant**

  - Chiral multiplet of weight ρ :

$$Z_{\text{1-loop}} = \prod_{J=0}^{\infty} \frac{J + 1 - \frac{q}{2} + il\rho \cdot \sigma - \frac{\rho \cdot B}{2}}{J + \frac{q}{2} - il\rho \cdot \sigma - \frac{\rho \cdot B}{2}}$$

[1] Still left with infinite number of unpaired eigen modes

[2] Index Theorem for Transversally Elliptic Operators

[3] It is obviously divergent !

Exact Partition Function

Gauged Linear Sigma Model (GLSM)

- UV divergence in sphere-partition function

Λ : cut-off scale

$$-\log Z_{S^2} \equiv F_{S^2} = (l\Lambda)^2 + \cdots + \boxed{\frac{c}{3} \cdot \log(l\Lambda)} + \text{finite terms}$$

“ universal ”

- Power divergence: cured by local counter terms

- Log divergence: reflects trace anomaly and c denotes central charge.

Exact Partition Function

Gauged Linear Sigma Model (GLSM)

- **One-loop determinant**

  - Chiral multiplet of weight $\boldsymbol{\rho}$:

$$Z_{\text{1-loop}} = \prod_{J=0}^{\infty} \frac{J + 1 - \frac{q}{2} + il\rho\cdot\sigma - \frac{\rho\cdot B}{2}}{J + \frac{q}{2} - il\rho\cdot\sigma - \frac{\rho\cdot B}{2}}$$

$$\log Z_{\text{1-loop}} = (\underbrace{1-q}_{[1]} + \underbrace{2il\rho\cdot\sigma}_{[2]})\log[l\Lambda] + \cdots$$

Λ : cut-off scale

[1] Central charge

[2] One-loop correction to FI parameter

Exact Partition Function

Gauge Linear Sigma Model (GLSM)

ξ : FI parameter θ : theta angle

W : Weyl group **r** : rank of G

- **Result:**

$$\boxed{Z_{S^2}^{\text{GLSM}} = \frac{(l\Lambda)^{c/3}}{|W|} \sum_B \int_{\mathfrak{t}} d^r\sigma \;\; e^{-4\pi i \xi_{\text{ren}}\sigma + i\theta B} \times Z_{\text{1-loop}}^{\text{reg}}(\sigma)}$$

- (regularized) One-loop determinant :

$$Z_{\text{v.m.}} = \prod_{\alpha\in\Delta^+} \left[\left(\frac{\alpha\cdot B}{2} \right)^2 + (\alpha\cdot l\sigma)^2 \right] \qquad Z_{\text{c.m.}} = \prod_{\rho\in\mathbf{R}} \frac{\Gamma\left(\frac{q}{2} - il(\rho\cdot\sigma + m) - \frac{\rho\cdot B}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + il(\rho\cdot\sigma + m) - \frac{\rho\cdot B}{2}\right)}$$

- Central charge :

$$\frac{c}{3} = \sum_i \underbrace{\dim[\mathbf{R}_i](1-q_i)}_{\text{chiral multiplets}} - \underbrace{\dim[G]}_{\text{vector multiplet}} = \text{Tr}_f[R]$$

Index Theorem

Let's compute the one-loop determinant in a " fancy " way !

Cohomological Basis :

e.g. chiral multiplet (ϕ, ψ, F)

2 **4** **2**

X **: 2 B**

$\phi \ , \bar{\phi}$

Ψ **: 2 F**

$\epsilon\gamma^3\psi \ , \ \bar{\epsilon}\gamma^3\bar{\psi}$

$\hat{Q}\Psi$ **: 2 B**

$\hat{Q}X$ **: 2 F**

$\bar{\epsilon}\psi \ , \ \epsilon\bar{\psi}$

Index Theorem

Cohomological Basis : X **: B** $\quad\quad \Psi$ **: F**

$\hat{Q}\Psi$ **: B** $\quad\quad \hat{Q}X$ **: F**

In the above basis, the deformed Lagrangian can be written as follows :

$$\mathcal{L}_{\mathrm{def}} = \hat{Q}\mathcal{V} = \mathcal{L}_b + \mathcal{L}_f \qquad \mathcal{V} = (\hat{Q}X \quad \Psi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} X \\ \hat{Q}\Psi \end{pmatrix} ,$$

$$\mathcal{L}_{\mathrm{b}} = (X \quad \hat{Q}\Psi)\begin{pmatrix} H & \\ & 1 \end{pmatrix}\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} X \\ \hat{Q}\Psi \end{pmatrix}$$

$$\hat{Q}^2 \equiv H$$

$$\mathcal{L}_{\mathrm{f}} = (\hat{Q}X \quad \Psi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} 1 & \\ & H \end{pmatrix}\begin{pmatrix} \hat{Q}X \\ \Psi \end{pmatrix}$$

Index Theorem

One-loop determinant can be written as

$$\left(\frac{\det K_f}{\det' K_b}\right)^2 = \frac{\det_\Psi H}{\det'_X H} = \frac{\det_{\mathrm{Coker} D_{10}} H}{\det'_{\mathrm{Ker} D_{10}} H}$$

Can read off from the index of the differential operator D_{10}

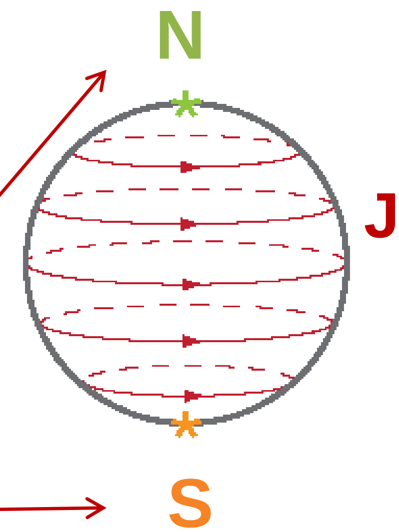
$$\mathrm{ind}\, D_{10} = \mathrm{Tr}_{\mathrm{Ker} D_{10}} \left[e^{-Ht}\right] - \mathrm{Tr}_{\mathrm{Coker} D_{10}} \left[e^{-Ht}\right]$$

NB The differential operator D_{10} is a **transversally elliptic**

Index Theorem

Atiyah-Bott Localization Formula

$$\mathrm{ind}[D_{10}] = \sum_{x_*:\text{fixed points}} \frac{\mathrm{Tr}_X[e^{-Ht}] - \mathrm{Tr}_\Psi[e^{-Ht}]}{\det(1-\partial\tilde{x}/\partial x)|_{x_*}}$$



- Fixed points X_* : $\tilde{x}_* = x_* \quad (\tilde{x} = e^{-Ht}x)$

- Operator H : $[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \delta_{SU(2)}(\xi) + \delta_R(\alpha) + \delta_G(\Lambda)$ **H**

$$H|_{x_*} = \partial_\varphi - \frac{i}{2}Q_R + \left[\sigma - i\frac{B}{2}\cos(\theta_*) + A_\varphi(\theta_*)\right]$$

$$\sigma_2 = \sigma$$

$$A = \frac{B}{2}(\kappa - \cos\theta)\, d\varphi$$

$$\sigma_1 = -\frac{B}{2}$$

Index Theorem

Atiyah-Bott Localization Formula

$$\mathrm{ind}[D_{10}] = \sum_{x_*:\text{fixed points}} \frac{\mathrm{Tr}_X[e^{-Ht}] - \mathrm{Tr}_\Psi[e^{-Ht}]}{\det(1-\partial\tilde{x}/\partial x)|_{x_*}}$$

$$H|_{x_*} = \partial_\varphi - \frac{i}{2}Q_R + \left[\sigma - i\frac{B}{2}\cos(\theta_*) + A_\varphi(\theta_*)\right]$$

Near N pole, $\left(z \simeq \theta e^{i\varphi}\ ,\ \bar{z} \simeq \theta e^{-i\varphi}\right)$

[1] $\det(1-\partial\tilde{x}/\partial x) = (1-y)(1-y^{-1})$ $\qquad y = e^{-it}$

Index Theorem

Atiyah-Bott Localization Formula

$$\mathrm{ind}[D_{10}] = \sum_{x_*:\text{fixed points}} \frac{\mathrm{Tr}_X[e^{-Ht}] - \mathrm{Tr}_\Psi[e^{-Ht}]}{\det(1-\partial\tilde{x}/\partial x)|_{x_*}}$$

$$H|_{\mathrm{N}} = \partial_\varphi - \frac{i}{2}Q_R + \left[\sigma - i\frac{B}{2}\cos(\theta=0) + A_\varphi(\theta=0)\right]$$

Near N pole,

[2] $$\mathrm{Tr}_X\left[e^{-tH}\right] = y^{q/2-B/2-i\sigma} + y^{-q/2+B/2+i\sigma}$$

$$\phi \qquad \bar{\phi}$$

$$Q_R[\phi] = -q \quad Q_R[\bar{\phi}] = +q$$

Index Theorem

Atiyah-Bott Localization Formula

$$\mathrm{ind}[D_{10}] = \sum_{x_*:\text{fixed points}} \frac{\mathrm{Tr}_X[e^{-Ht}] - \mathrm{Tr}_\Psi[e^{-Ht}]}{\det(1-\partial\tilde{x}/\partial x)|_{x_*}}$$

$$H|_{\mathrm{N}} = \partial_\varphi - \frac{i}{2}Q_R + \left[\sigma - i\frac{B}{2}\cos(\theta=0) + A_\varphi(\theta=0)\right]$$

Near N pole,

[3] $\mathrm{Tr}_\Psi\left[e^{-Ht}\right] = \underbrace{y^{q/2-1-B/2-i\sigma}}_{\epsilon\gamma^3\psi} + \underbrace{y^{-q/2+1+B/2+i\sigma}}_{\bar{\epsilon}\gamma^3\bar{\psi}}$

$$Q_R[\epsilon\gamma^3\psi] = 1 + (1-q)$$

Index Theorem

Atiyah-Bott Localization Formula

$$\mathrm{ind}\, D_{10} = \underset{X}{\mathrm{Tr}_{\mathrm{Ker} D_{10}}} \left[e^{-Ht}\right] - \underset{\Psi}{\mathrm{Tr}_{\mathrm{Coker} D_{10}}} \left[e^{-Ht}\right]$$

Near N pole, let's collecting all the results,

$$\mathrm{ind}[D_{10}]|_{\mathrm{N}} = \frac{y^{q/2-B/2-i\sigma}}{1-y} + \frac{y^{-q/2+B/2+i\sigma}}{1-y^{-1}}$$

$$= \sum_{J=0} \left[e^{-t}\right]^{iJ+\sigma+iq/2-iB/2} + \left[e^{-t}\right]^{-iJ-\sigma-iq/2+iB/2}$$

Bosonic Eigenvalue of H

Index Theorem

Atiyah-Bott Localization Formula

$$\mathrm{ind}\, D_{10} = \underset{X}{\mathrm{Tr}_{\mathrm{Ker} D_{10}}} \left[e^{-Ht}\right] - \underset{\Psi}{\mathrm{Tr}_{\mathrm{Coker} D_{10}}} \left[e^{-Ht}\right]$$

Near S pole, one can obtain

$$\mathrm{ind}[D_{10}]|_{\mathrm{S}} = -\frac{y^{q/2-1+B/2-i\sigma}}{1-y^{-1}} - \frac{y^{-q/2+1-B/2+i\sigma}}{1-y}$$

$$= -\sum_{J=0} \left(\left[e^{-t}\right]^{-i(J+1)+iq/2+\sigma+iB/2} + \left[e^{-t}\right]^{i(J+1)-iq/2-\sigma-iB/2} \right)$$

Fermionic Eigenvalue of H

Index Theorem

One-Loop Determinant

$$\left(\frac{\det K_f}{\det' K_b}\right)^2 = \frac{\det_\Psi H}{\det'_X H} = \frac{\det_{\mathrm{Coker}\,D_{10}} H}{\det'_{\mathrm{Ker}\,D_{10}} H}$$

$$Z_{\text{1-loop}} = \frac{\det K_f}{\det K_b} = \prod_{J=0} \frac{J+1+i\sigma-q/2-B/2}{J-i\sigma+q/2-B/2}$$

Higgs Branch Representation

Factorization of S^2 partition function : 2d N=(2,2) U(1) gauge theory

$$Z_{S^2}^{\text{GLSM}} = \frac{(l\Lambda)^{c/3}}{|W|} \sum_B \int_{\mathfrak{t}} d^r\sigma \;\; e^{-4\pi i \xi_{\text{ren}}\sigma + i\theta B} \times Z_{\text{1-loop}}^{\text{reg}}(\sigma)$$

$$Z_{\text{v.m.}} = \prod_{\alpha\in\Delta^+} \left[\left(\frac{\alpha\cdot B}{2}\right)^2 + (\alpha\cdot l\sigma)^2 \right] \qquad Z_{\text{c.m.}} = \prod_{\rho\in\mathbf{R}} \frac{\Gamma\left(\frac{q}{2} - il(\rho\cdot\sigma+m) - \frac{\rho\cdot B}{2}\right)}{\Gamma\left(1-\frac{q}{2} + il(\rho\cdot\sigma+m) - \frac{\rho\cdot B}{2}\right)}$$

simple poles

If ξ_{ren} does not vanish, the matrix integral over the Coulomb branch can be evaluated by residues. Let us assume that $\xi_{\text{ren}} > 0$

Higgs Branch Representation

Factorization of S^2 partition function :

G=U(1) with N_F chiral multiplets of charge +1 and N_F chiral multiplets of charge -1

$$Z_{U(1)}^{\tilde{N}_F = N_F} = 2\pi \sum_{p=1}^{N_F} \left[z^{-i\mathrm{M}_p} \bar{z}^{-i\mathrm{M}_p} \frac{\prod_{s=1}^{N_F} \gamma(-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_p)}{\prod_{s=1}^{N_F} \gamma(1 + i\mathrm{M}_s - i\mathrm{M}_p)} F_p(z) F_p(\bar{z}) \right]$$

$$q = e^{2\pi i \tau}$$

$$\tau = \frac{\theta}{2\pi} + i\xi$$

$$F_p(z) = {}_{N_F}F_{N_F-1}\left(\begin{matrix} -i\mathrm{M}_p - i\widetilde{\mathrm{M}}_1 \cdots -i\mathrm{M}_p - i\widetilde{\mathrm{M}}_{N_F} \\ 1 + i\mathrm{M}_1 - i\mathrm{M}_p \,\widehat{\cdots}\, 1 + i\mathrm{M}_{N_F} - i\mathrm{M}_p \end{matrix} \middle| z \right)$$

[1] Sum over isolated vacua (=Higgs branch roots) in the Higgs branch

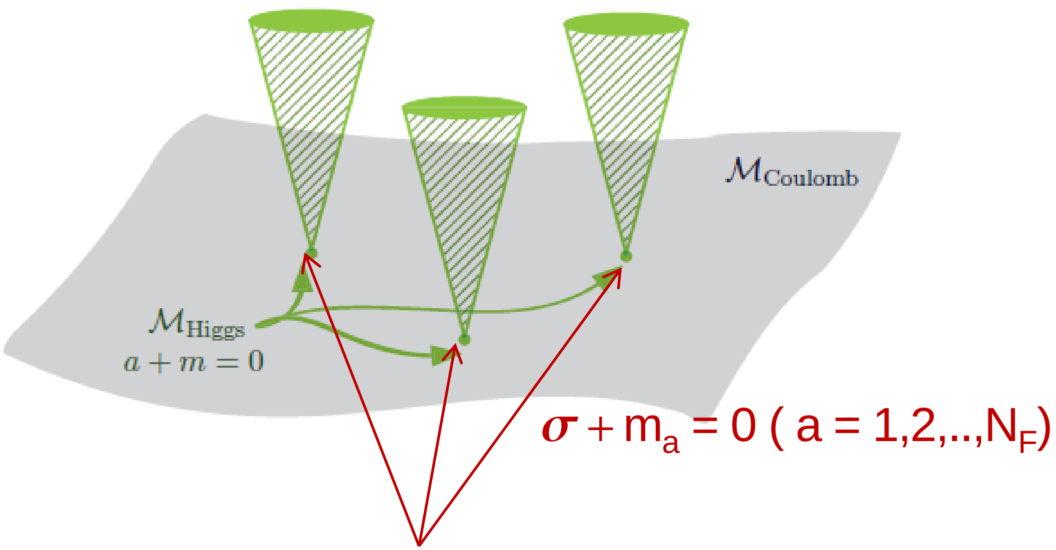
[2] Factorization occurs !

[3] Vortex partition function (= 2d counterpart of Nekrasov's partition function)

Higgs Branch Representation

Factorization of S^2 partition function :

G=U(1) with N_F chiral multiplets of charge +1 and N_F chiral multiplets of charge -1

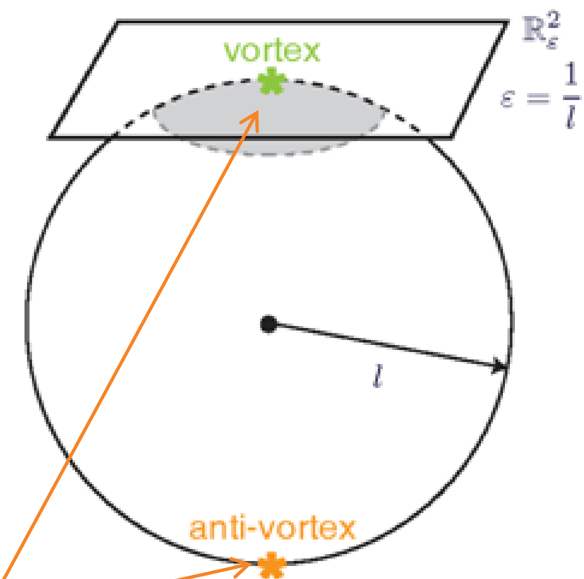


[1] Sum over isolated vacua (=Higgs branch roots) in the Higgs branch

Higgs Branch Representation

Factorization of S^2 partition function :

G=U(1) with N_F chiral multiplets of charge +1 and N_F chiral multiplets of charge -1



- Choosing a different Q-exact def. can localize the path-integral onto isolated Higgs vacua

- New SUSY saddle point configurations :

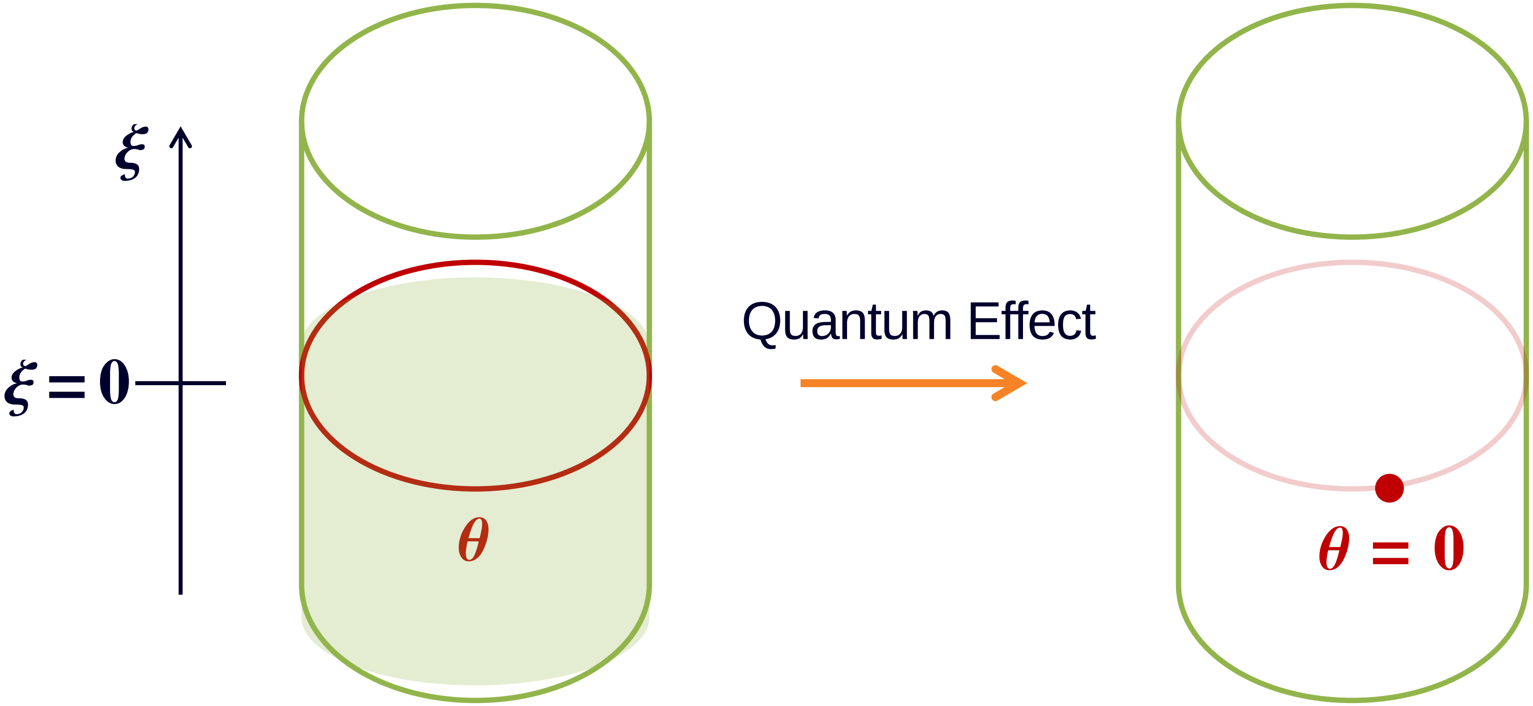
point-like (anti-) vortex at NP (SP) in omega background

[2] Factorization occurs !

[3] Vortex partition function (= 2d counterpart of Nekrasov's partition function)

Phase of GLSM

Due to the quantum correction, physics with $\xi_{\text{ren}} > 0$ can be analytically continued to physics with $\xi_{\text{ren}} < 0$, if θ is non-zero [Flop Transition]



Higgs Branch Representation

Factorization of S^2 partition function :

G=U(1) with N_F chiral multiplets of charge +1 and N_F chiral multiplets of charge -1

$$Z_{U(1)}^{\tilde{N}_F=N_F} = 2\pi \sum_{p=1}^{N_F} \left[z^{-i\mathrm{M}_p} \bar{z}^{-i\mathrm{M}_p} \frac{\prod_{s=1}^{N_F} \gamma(-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_p)}{\prod_{s=1}^{N_F} \gamma(1 + i\mathrm{M}_s - i\mathrm{M}_p)} F_p(z) F_p(\bar{z}) \right]$$

$$q = e^{2\pi i \tau}$$

$$\tau = \frac{\theta}{2\pi} + i\xi$$

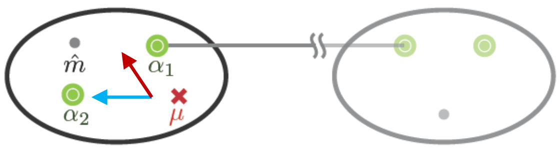
$$F_p(z) = {}_{N_F}F_{N_F-1}\left(\begin{matrix} -i\mathrm{M}_p - i\widetilde{\mathrm{M}}_1 \cdots -i\mathrm{M}_p - i\widetilde{\mathrm{M}}_{N_F} \\ 1+i\mathrm{M}_1 - i\mathrm{M}_p \,\widehat{\cdots}\, 1+i\mathrm{M}_{N_F} - i\mathrm{M}_p \end{matrix} \middle| z \right)$$

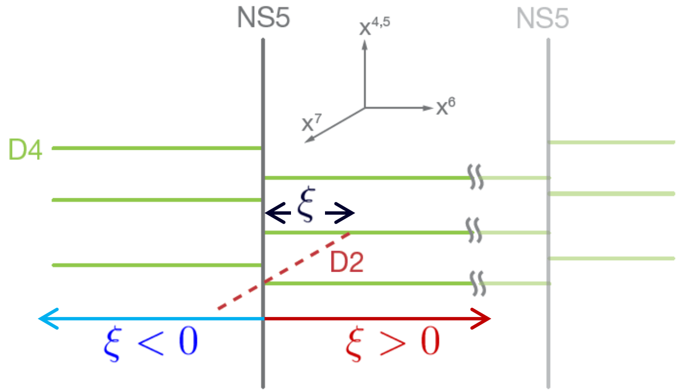
[1] Vortex partition function $F_p(z)$ is analytic across $\xi = 0$, if θ is non-zero !

Gauge/Toda Duality

Decoupling Limit of AGT Relation with surface operator

- Toda correlator with degenerate operator vs $SQED_2$ coupled to N_f flavors





$$\frac{|1-z|^{-2b\langle \hat{\mathrm{m}}, h_1\rangle}}{|z|^{2\delta}} \frac{\langle V_{\alpha_2}(\infty) V_{\hat{\mathrm{m}}}(1) V_{\mu}(z,\bar{z}) V_{\alpha_1}(0)\rangle}{\langle V_{\alpha_2}(\infty) V_{\hat{\mathrm{m}}-bh_{N_F}}(1) V_{\alpha_1}(0)\rangle} \overset{z=e^{2\pi i\tau}}{=} Z_{S^2}^{\mathrm{SQED}}(M_{\Box}, M_{\bar{\Box}}, \tau, \bar{\tau})$$

Crossing Symmetry = Flop Transition

[Fateev,Litvinov] [Witten]

Exact Partition Function

Landau-Ginzburg (LG) Model, which involves

Twisted chiral multiplets Y coupled by twisted superpotential W(Y)

- SUSY saddle points : Y = const. over S^2 and all other fields vanish

- One-loop determinant : trivial in a sense that it is independent of Y

- **Result**

$$\boxed{Z_{S^2}^{\mathrm{LG}} = \int dY d\bar{Y} \;\; e^{-4\pi i l W(Y) - 4\pi i l \bar{W}(\bar{Y})}}$$

N=(2,2) SCA at IR Fixed Points

Global part of N=(2,2) SCA

$$
\begin{aligned}
\{S_\alpha, Q_\beta\} &= \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \\
\{\bar{S}_\alpha, \bar{Q}_\beta\} &= -\gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \\
\{Q_\alpha, \bar{Q}_\beta\} &= \gamma^m_{\alpha\beta} K_m + \frac{1}{2} C_{\alpha\beta} \mathcal{A} \\
\{S_\alpha, \bar{S}_\beta\} &= \gamma^m_{\alpha\beta} K_m - \frac{1}{2} C_{\alpha\beta} \mathcal{A} \\
[J_m, J_n] &= i\epsilon_{mnp} J^p \\
[K_m, K_n] &= -i\epsilon_{mnp} J^p \\
[J_m, K_n] &= i\epsilon_{mnp} K^p
\end{aligned}
$$

$$
\begin{aligned}
[J_m, S^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} S_\beta & [R, S_\alpha] &= +S_\alpha \\
[J_m, Q^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} Q_\beta & [R, Q_\alpha] &= -Q_\alpha \\
[J_m, \bar{Q}^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{Q}_\beta & [R, \bar{Q}_\alpha] &= +\bar{Q}_\alpha \\
[J_m, \bar{S}^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{S}_\beta & [R, \bar{S}_\alpha] &= -\bar{S}_\alpha \\
[K_m, S^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{Q}_\beta & [\mathcal{A}, S_\alpha] &= \bar{Q}_\alpha \\
[K_m, Q^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{S}_\beta & [\mathcal{A}, Q_\alpha] &= -\bar{S}_\alpha \\
[K_m, \bar{Q}^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} S_\beta & [\mathcal{A}, \bar{Q}_\alpha] &= -S_\alpha \\
[K_m, \bar{S}^\alpha] &= -\frac{1}{2} \gamma_m{}^{\alpha\beta} Q_\beta & [\mathcal{A}, \bar{S}_\alpha] &= Q_\alpha \,.
\end{aligned}
$$

Appendix

Restoration of Axial $U(1)_R$ Symmetry at IR fixed point

There is a one-parameter family of SUSY theories on S^2, $T[\theta]$, related by axial $U(1)_R$ rotation at UV

However, S^2 partition functions of $T[\theta]$ are all the same !

This result confirms in the S^2 partition function framework that the axial $U(1)_R$ symmetry is **restored at IR**!

Part III

Application to String Theory

What are we probing ?

The most useful method in physics is to throw something at an object we want to understand

- Today we are probing “**geometrical space**”
- With “**quantum string**”

Why are we probing ?

Physics: Attempts to get a sensible 4-dimensional model, relevant to nature, from 10-dimensional string theory on small 6-dimensional space

Many physical observables (e.g. Yukawa coupling) in 4d theories are determined by how strings see the given 6-dimensional space.

Mathematics: the spectrum of quantum particle with spin solves many important mathematical problems of classical geometry

e.g. (Supersymmetric) Ground States = de Rham Cohomology

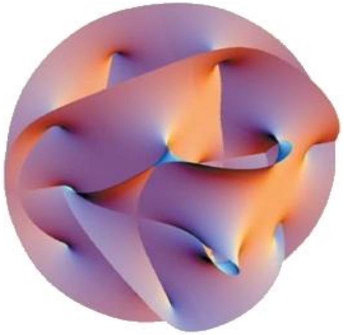
What about quantum string then ? ***New Maths ?***

What did we learn ?

1. Strings want to change the given geometry until they become happy,

$$\frac{d}{d\mu} g_{ij} = -R_{ij}$$

2. They are happy in a space with $R_{ij} = 0$ known as "**Calabi-Yau manifold**"



Can deform it in two different ways

- change of size: Kahler moduli
- change of shape: Complex structure moduli

3. There are quantum corrections to classical geometry, *suppressed by string size* l_s **,** arising from point-particle approximation of string

2-Dimensional Theories

String which probes the six-dimensional Calabi-Yau space can be studied by 2-dimensional theories. (world-sheet theory)

1. $\frac{d}{d\mu} g_{ij} = -R_{ij}$: nothing but the renormalization group (RG) flow

2. Calabi-Yau space ($R_{ij} = 0$) is a fixed point of RG flow.
   Kahler/Complex Structure moduli can define a "space of RG-fixed points"

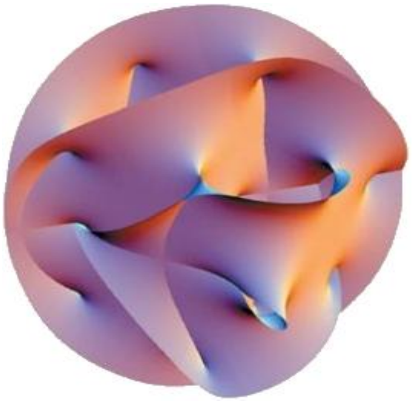
   It is important to know the **metric** on the space of RG-fixed points

3. **Most Challenging Problem**
   How to compute quantum corrections to the metric on the space of RG-fixed points ?

   A solution is the celebrated **Mirror Symmetry**

Mirror Symmetry

Calabi-Yau manifolds come **in pairs**. Strings see the two manifolds as the same, although mathematician saw them differently



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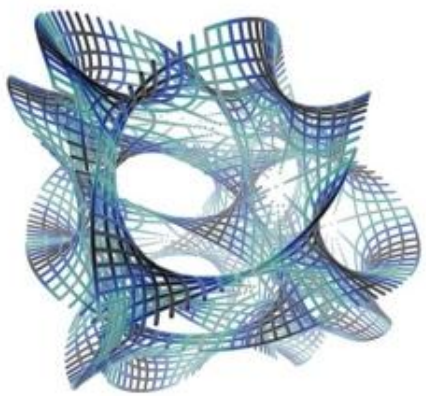


Kahler Moduli:
change of size

Quantum (Hard)

Complex structure Moduli:
change of shape

Classical (Easy)

However, there are not many known examples (well ∞, but still not big enough)

Recent Progress: A New Method

The exact S^2 partition function provides a direct and powerful method to compute both perturbative and nonperturbative quantum corrections **without use of mirror symmetry**

$$Z_{S^2}(\mathrm{GLSM}) = e^{-K(\tau,\bar{\tau})}$$

- Exact Kahler potential **K** encodes all quantum corrections to the metric on the space of RG-fixed points of 2D theories
- **K** also defines a new math: Gromov-Witten Invariant (Quantum Cohomology)
- Z_{S^2} is independent of complex structure moduli of CY_3 (= parameters in superpotential)

Recent Progress: A New Method

The exact S^2 partition function provides a direct and powerful method to compute both perturbative and nonperturbative quantum corrections **without use of mirror symmetry**

$$Z_{S^2}(\mathrm{GLSM}) = e^{-K(\tau,\bar{\tau})}$$

• **K** also defines a new math: Gromov-Witten Invariant (Quantum Cohomology)

$$e^{-K(t,\bar{t})} = -\frac{i}{6}\sum_{a,b,c} C_{abc}(t^a-\bar{t}^a)(t^b-\bar{t}^b)(t^c-\bar{t}^c) + \frac{\zeta(3)}{4\pi^3}\chi(X) + \cdots + \frac{2i}{(2\pi i)^3}\sum_{\beta\in H_2(X)} N_\beta\left(\mathrm{Li}_3(e^{2\pi i\int_\beta t}) + \mathrm{c.c}\right)$$

$$t = \sum_{a=1}^{h^{1,1}(X)} t^a\omega_a \quad \omega_a \in H^2(X)$$

$$C_{abc} = \int_X \omega_a \wedge \omega_b \wedge \omega_c$$

t^a: flat coordinate

N_β: GW-invariant

Works for many known examples, for instance, a famous Quintic Threefold

2875, 609250, 317206375, 242467530000, …

Examples

$\tilde{N}_{m_0,m_1}$	$m_0=0$	$1/2$	1	$3/2$	2	$5/2$	3
$m_1=0$	–		56		0		0
$1/2$		192		896		192	
1	56		2 544		23 016		41 056
$3/2$		896		52 928		813 568	
2	0		23 016		1 680 576		35 857 016
$5/2$		192		813 568		66 781 440	
3	0		41 056		35 857 016		3 074 369 392
$7/2$		0		3 814 144		1 784 024 064	
4	0		23 016		284 749 056		96 591 652 016
$9/2$		0		6 292 096		20 090 433 088	
5	0		2 544		933 789 504		1 403 214 088 320
$11/2$		0		3 814 144		105 588 804 096	
6	0		56		1 371 704 192		10 388 138 826 968
$13/2$		0		813 568		277 465 693 248	
7	0		0		933 789 504		41 598 991 761 344
$15/2$		0		52 928		380 930 182 784	
8	0		0		284 749 056		93 976 769 192 864
$17/2$		0		896		277 465 693 248	
9	0		0		35 857 016		122 940 973 764 384
$19/2$		0		0		105 588 804 096	
10	0		0		1 680 576		93 976 769 192 864
$21/2$		0		0		20 090 433 088	
11	0		0		23 016		41 598 991 761 344
$23/2$		0		0		1 784 024 064	
12	0		0		0		10 388 138 826 968
$25/2$		0		0		66 781 440	
13	0		0		0		1 403 214 088 320
$27/2$		0		0		813 568	
14	0		0		0		96 591 652 016
$29/2$		0		0		192	
15	0		0		0		3 074 369 392
$31/2$		0		0		0	

Table 4: Predictions for genus zero Gromov–Witten invariants of the determinantal GN Calabi–Yau threefold Y.

Predicts new results of GW invariants for CY_3's whose mirrors are unknown !

Why does this formula work ?

New Solution

Proof I (Warm-up) [Gomis,**S.L**]

LG theories with twisted superpotential W(Y), which describe N=(2,2) SCFTs

$$Z_{S^2}[\mathrm{LG}] = \int dY\, d\bar{Y}\ e^{-4\pi i l W(Y) - 4\pi i l \bar{W}(\bar{Y})}$$

$\|$ [Cecotti]

$$e^{-K(\tau,\bar{\tau})}$$

New Solution

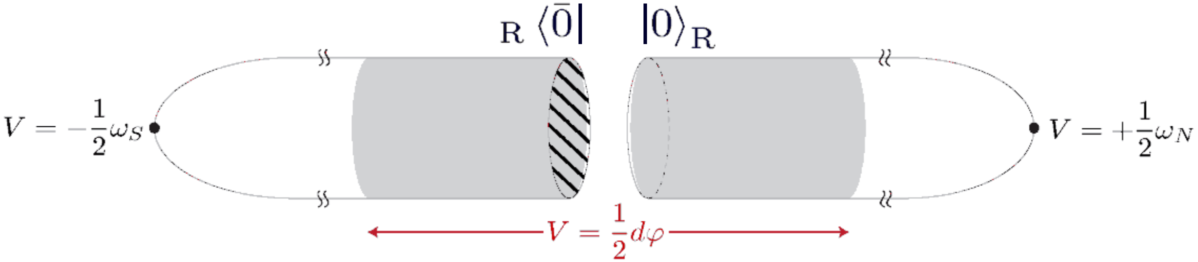
Sketch of Proof II [Gomis,**S.L**]

$$\boxed{Z_{S^2}} \underset{[1]}{=} \boxed{Z_{S^2_b}} \overset{\substack{b=l/\tilde{l} \\ b\to 0}}{\underset{[2]}{=}} \boxed{{}_{\mathrm{R}}\langle\bar{0}|0\rangle_{\mathrm{R}}} \underset{\text{tt* eqn}}{=} \boxed{e^{-K(\vec{\tau},\vec{\bar{\tau}})}}$$

[1] SUSY theory on squashed two-sphere $S^2_b\ :\ \dfrac{x_1^2+x_2^2}{l^2}+\dfrac{x_3^2}{\tilde{l}^2}=1$

- Need a background gauge field V for $U(1)_R$ Symmetry
- Partition function on S^2_b : independent of squashing parameter **b**

[2] Infinite squashing limit



Mirror Symmetry

Hori-Vafa Method: Abelian GLSM vs Landau-Ginzburg model

- T-duality (scalar-scalar duality)

$$\partial_i \varphi = \epsilon_{ij} \partial^j y$$

NB: Chiral multiplet $\mathbf{\Phi}$ $\longleftrightarrow$ Twisted chiral multiplets **Y**

- Nonperturbative effect: dynamically generation of (ADS-type) twisted superpotential W

$$W = e^{-Y}$$

Examples

- Non-compact toric Calabi-Yau:

GLSM: G=U(1), n chiral multiplets of charge Q_a (a=1,2,..,n)

LG: n twisted chiral multiplets Y=Y+2πi with $W = -\frac{1}{4\pi}\left[\Sigma\Big(\sum_{a=1}^{n} Q_a Y^a + 2\pi i \tau\Big) + i\sum_{a=1}^{n} e^{-Y^a}\right]$

Mirror Symmetry Revisited

Non-compact toric CY_3

GLSM: G=U(1), n chiral multiplets of charge Q_a (a=1,2,..,n) with $\sum_a Q_a = 0$

LG: n twisted chiral multiplets Y=Y+2πi with $W = -\frac{1}{4\pi}\left[\Sigma\Big(\sum_{a=1}^{n} Q_a Y^a + 2\pi i \tau\Big) + i\sum_{a=1}^{n} e^{-Y^a}\right]$

$$Z_{S^2}^{\mathrm{GLSM}} = \sum_{B\in\mathbb{Z}} e^{+iB\vartheta} \int d\sigma\ e^{-4\pi i r\sigma\xi} \prod_{a=1}^{n} (-1)^{\frac{|BQ_a|+BQ_a}{2}} \frac{\Gamma\left(\frac{1}{2}|BQ_a| - irQ_a\sigma\right)}{\Gamma\left(1+\frac{1}{2}|BQ_a| + irQ_a\sigma\right)}$$

$$\| \qquad \int_{-\infty}^{+\infty} dx\ e^{-qx} J_\alpha(2e^{-x}) = (-1)^{\frac{|\alpha|-\alpha}{2}} \frac{1}{2} \frac{\Gamma\left(\frac{q}{2}+\frac{1}{2}|\alpha|\right)}{\Gamma\left(1-\frac{q}{2}+\frac{1}{2}|\alpha|\right)}$$

$$\begin{aligned} Z_{S^2}^{\mathrm{LG}} &= \sum_{B\in\mathbb{Z}} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} dx^a \int_{-\pi}^{+\pi} dy^a\ e^{+2ir\sigma(Q_a x^a - 2\pi\xi) + iB(Q_a y^a + \vartheta)} \cdot e^{2ie^{-x^a}\sin y^a} \\ &= \int d\Sigma\, d\bar{\Sigma} dY_a d\overline{Y}_a\ e^{-4\pi i r W - 4\pi i r \overline{W}} \end{aligned}$$

Mirror Symmetry Revisited

Compact CY_3 : Complete intersection in toric variety

GLSM: G=U(1), n chiral multiplets of charge Q_a and $U(1)_R$ charge q_a with

superpotential *W*

LG: n twisted chiral multiplets $Y=Y+2\pi i$ with the **same** twisted superpotential W

$$W = -\frac{1}{4\pi}\left[\Sigma\Big(\sum_{a=1}^{n} Q_a Y^a + 2\pi i\tau\Big) + i\sum_{a=1}^{n} e^{-Y^a}\right]$$

Subtlety in choosing fundamental variables of mirror LG models

However, S^2 partition function can resolve the subtlety automatically !

Mirror Symmetry Revisited

GLSM: G=U(1), n chiral multiplets of charge Q_a and $U(1)_R$ charge $\mathbf{q_a}$

LG: n twisted chiral multiplets with the **same** twisted superpotential W

$$Z_{S^2}^{\mathrm{GLSM}} = \sum_{B\in\mathbb{Z}} e^{+iB\vartheta} \int d\sigma\ e^{-4\pi i r\sigma\xi} \prod_{a=1}^{n} (-1)^{\frac{|BQ_a|+BQ_a}{2}} \frac{\Gamma\left(\frac{q_a}{2} + \frac{1}{2}|BQ_a| - irQ_a\sigma\right)}{\Gamma\left(1-\frac{q_a}{2} + \frac{1}{2}|BQ_a| + irQ_a\sigma\right)}$$

$$\parallel \quad \int_{-\infty}^{+\infty} dx\ e^{-qx} J_\alpha(2e^{-x}) = (-1)^{\frac{|\alpha|-\alpha}{2}} \frac{1}{2} \frac{\Gamma\left(\frac{q}{2} + \frac{1}{2}|\alpha|\right)}{\Gamma\left(1 - \frac{q}{2} + \frac{1}{2}|\alpha|\right)}$$

$$Z_{S^2}^{\mathrm{GLSM}} = \int d\Sigma\, d\bar{\Sigma} \int \underbrace{\left[\prod_{a=1}^{n} dY_a d\overline{Y}_a e^{-\frac{q_a}{2}(Y^a+\overline{Y}^a)}\right]} \ e^{-4\pi i W(Y) - 4\pi i \overline{W}(Y)}$$

Fundamental LG variables $\boxed{X_a = e^{-\frac{q_a}{2} Y^a}}$

New Idea in Mirror Symmetry

Mirror beyond toric ? [Hori,Vafa] method (due to T-duality) cannot extend to 2D non-abelian GLSM describing CY_3 beyond toric variety

e.g. G=U(N) with chiral multiplets in rep. **R**

Complete-intersection in Grassmannian mfd.

S^2 partition function of 2d non-abelian GLSM can be computed exactly

Same as the S^2 partition function of a following Landau-Ginzburg model

$$Z^{\mathrm{GLSM}}_{S^2} = \frac{1}{|\mathcal{W}(G)|}\int \Big[\prod_{j=1}^{\mathrm{rk}(G)} d\Sigma_j d\bar{\Sigma}_j\Big]\Big[\prod_{\rho\in\mathbf{R}} dY^\rho d\overline{Y}^\rho\Big] \prod_{j<k}|\Sigma_j-\Sigma_k|^2 \, e^{-4\pi i W - 4\pi i \overline{W}}$$

$$W = -\frac{1}{4\pi}\left[\sum_{i=1}^{\mathrm{rk}(G)} \Sigma_i \left(\sum_{\rho\in\mathbf{R}} \rho^i Y^\rho + 2\pi i \tau\right) + i\sum_{\rho} e^{-Y^\rho}\right] \quad \Leftarrow$$

Nontrivial evidence of Hori-Vafa conjecture !

N=(2,2) SUSY on S²

SUSY on Two-Sphere: **SU(2|1)**

- Subalgebra of N=(2,2) SCA:

$$\{S_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \qquad [J_m, S^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} S_\beta \qquad [R, S_\alpha] = +S_\alpha$$

$$\{\bar{S}_\alpha, \bar{Q}_\beta\} = -\gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R \qquad [J_m, Q^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} Q_\beta \qquad [R, Q_\alpha] = -Q_\alpha$$

$$\{Q_\alpha, \bar{Q}_\beta\} = \gamma^m_{\alpha\beta} K_m + \frac{1}{2} C_{\alpha\beta} \mathcal{A} \qquad [J_m, \bar{Q}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{Q}_\beta \qquad [R, \bar{Q}_\alpha] = +\bar{Q}_\alpha$$

$$\{S_\alpha, \bar{S}_\beta\} = \gamma^m_{\alpha\beta} K_m - \frac{1}{2} C_{\alpha\beta} \mathcal{A} \qquad [J_m, \bar{S}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{S}_\beta \qquad [R, \bar{S}_\alpha] = -\bar{S}_\alpha$$

$$[J_m, J_n] = i\epsilon_{mnp} J^p \qquad [K_m, S^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{Q}_\beta \qquad [\mathcal{A}, S_\alpha] = \bar{Q}_\alpha$$

$$[K_m, K_n] = -i\epsilon_{mnp} J^p \qquad [K_m, Q^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{S}_\beta \qquad [\mathcal{A}, Q_\alpha] = -\bar{S}_\alpha$$

$$[J_m, K_n] = i\epsilon_{mnp} K^p \qquad [K_m, \bar{Q}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} S_\beta \qquad [\mathcal{A}, \bar{Q}_\alpha] = -S_\alpha$$

$$[K_m, \bar{S}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} Q_\beta \qquad [\mathcal{A}, \bar{S}_\alpha] = Q_\alpha \,.$$