SUSY and Localization

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Part I

Equivariant Volume & Localization

Why do we care ?

[1] Nekrasov Instanton Partition Functions and Seiberg-Witten Theories

[2] Volume of Sasaki-Einstein Manifolds and AdS/CFT correspondence

Long-Standing Problem

Question

What measures # of degrees of freedom that decreases monotonically along the renormalization group flow ?

Renormalization 101

- Describe a system by microscopic d.o.f and their interactions
- Zoom out, or coarse grain: average out **“heavy modes”** that are irrelevant to long-distance physics
- At each typical energy scale, the d.o.f describing the same system may look very different to each other
- This procedure of zooming out and ignoring the small irrelevant details is known as Renormalization Group (RG) flow

A Little History

We know the answer in 2 dimensions due to the work of Zamoldchikov [86]

- One can define a real number **c** for any 2d QFTs, even for strongly interacting system, from 2-point function of energy-momentum tensor

- He has shown that this number always decrease monotonically along RG flow,

$$\boxed{C_{\mathrm{UV}} > C_{\mathrm{IR}}}$$

- It counts # of d.o.f, generalizing the notion of counting in free theory

$\Longrightarrow$ **“ Irreversibility of Renormalization Group Flow ! ”**

Two Questions Arise

[1] Similar measure in higher dimensions ?

[2] Counterpart in the bulk geometry via AdS/CFT correspondence

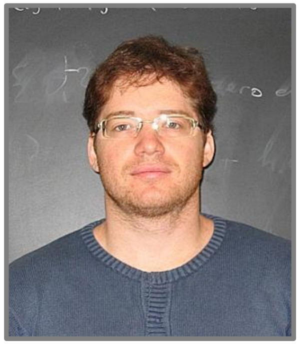
Higher Dimensions

In 4D, it has been conjecture by John Cardy soon after Zamoldchikov's work

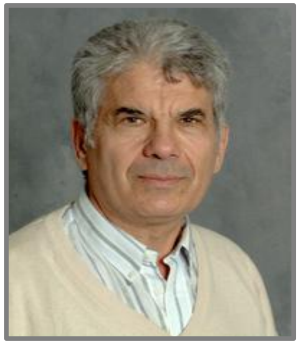
- One can easily generalize the definition of c-function in 2D to 4 dimensions, known as a-function

- Conjecture

$$\boxed{a_{\mathrm{UV}} > a_{\mathrm{IR}}}$$



A proof is recently proposed by Komargodski and Schwimmer [11]



What about three dimensions ?

Not even clue until very recently. This is because the definition of c- and a-function cannot be generalized to any odd dimensions

Higher Dimensions

Answer turns out to be S^3 partition function

More precisely, defining $F_{S^3} = -\log Z_{S^3}$,

[Jafferis]

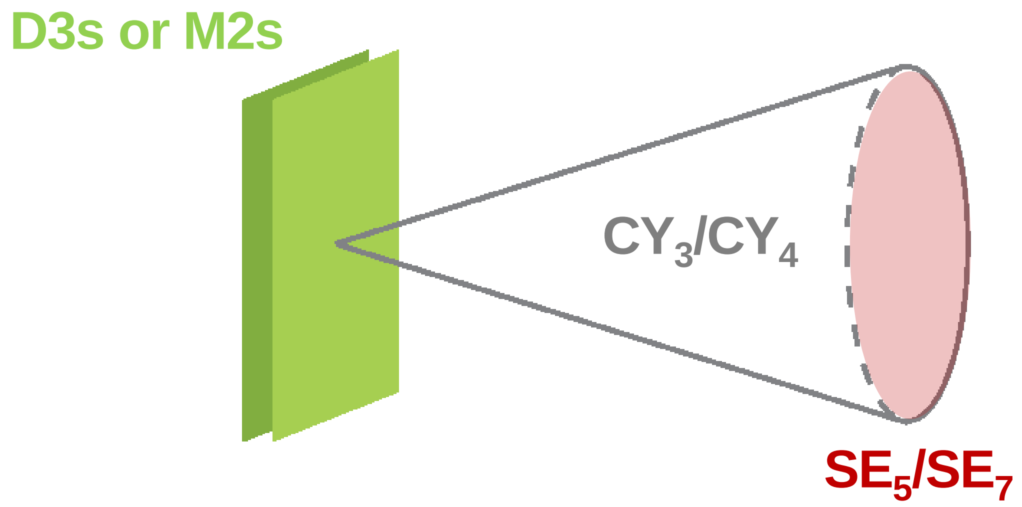
[Jafferis,Klebanov,Pufu,Safdi]

[Closset,Dumitrescu,Festuccia, Komargodski,Seiberg]

$$\boxed{F_{\rm UV} > F_{\rm IR}}$$

I may have a chance to introduce this story a little bit on Saturday…

AdS/CFT & Volume Minimization



NB **$AdS_{5(4)}/CFT_{4(3)}$** : **a(F)**-maximization vs **Vol[$SE_{5(7)}$]**-minimization

Equivariant Volume & Localization Method play a key role

SYMPLECTIC MANIFOLD (X, ω)

- $\dim[X] = 2n$
- $\omega = \frac{1}{2}\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu$, symplectic 2-form satisfying "$d\omega = 0$"

SYMPLECTIC VOLUME

- Volume form is defined as $\frac{1}{n!}\omega^n$

$$\mathrm{Vol}(X) = \int_X \frac{1}{n!}\omega^n = \int_X e^\omega$$

SYMPLECTIC VOLUME

i.e., 'symplectic mfd.'

- **Example**: Kähler manifold (= a complex mfd. w/ a closed 2-form)

  - metric $ds^2 = g_{a\underline{b}}\, dz^a d\bar{z}^{\underline{b}}$

  - Kähler 2-form $\omega = \frac{i}{2} g_{a\underline{b}}\, dz^a \wedge d\bar{z}^{\underline{b}}$ & $d\omega = 0$

  - symplectic volume

$$\mathrm{Vol}(X) = \int_X \frac{1}{n!} \left(\frac{i}{2}\right)^n g_{a_1\underline{b_1}} \cdots g_{a_n\underline{b_n}} \left[\epsilon^{a_1\cdots a_n}\epsilon^{\underline{b_1}\cdots\underline{b_n}}\right] dz^1 d\bar{z}^{\underline{1}} \cdots dz^n d\bar{z}^{\underline{n}}$$

$$= \int_X \sqrt{\det G} \qquad G : \text{metric on } X$$

NB SYMPLECTIC VOLUME = CANONICAL VOLUME for Kähler mfd.

SUPERSPACE & SUPERSYMMETRY

- Superspace $T[1]X = (x^\mu, \psi^\mu)$ where x^μ : coordinates of X
  ψ^μ : Grassmanian Variable

- function $f(x,\psi)$ on $T[1]X$ = space of differential form $\Omega^*(X)$

$$f(x,\psi) = f^{(0)}(x) + f^{(1)}_\mu(x)\psi^\mu + \cdots + \frac{1}{n!} f^{(n)}_{\mu_1\cdots\mu_n}(x)\psi^{\mu_1}\cdots\psi^{\mu_n}$$

- INTEGRATION

'top-form'

$$\int_{T[1]X}[dx\,d\psi]f(x,\psi) = \int_X [dx^\mu]\frac{1}{n!}f^{(n)}_{\mu_1\cdots\mu_n}(x)\epsilon^{\mu_1\cdots\mu_n} = \int_X f^{(n)}(x)$$

- $\omega = \frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu$ is invariant under

$$\delta X^\mu = \psi^\mu \qquad \delta\psi^\mu = 0$$

- $\delta^2 * = 0$ ... nilpotent!

- supercharge $Q \doteq \psi^\mu \partial_\mu \doteq dx^\mu \partial_\mu = d$ (exterior derivative)

- $[dX^\mu][d\psi^\mu]$ is obviously an invariant measure under SUSY transf. rule

$\Rightarrow \int_{T[1]X} [dX^\mu][d\psi^\mu]\,\delta\# = 0$! (nothing but. Stokes thm)

- $\delta\omega = \frac{1}{2}\partial_\rho\omega_{\mu\nu}\,\delta X^\rho\psi^\mu\psi^\nu + \omega_{\mu\nu}\,\delta\psi^\mu\psi^\nu = \frac{1}{2}\partial_\rho\omega_{\mu\nu}\,\psi^\rho\psi^\mu\psi^\nu = 0$! ($\because d\omega=0$)

- Adding a Q-exact term doesn't affect the volume integral

$$I[t] = \int [dx^\mu][d\psi^\mu]\, e^{\omega + t\delta\mathcal{V}} \qquad I[0] = \mathrm{vol}(X)$$

<u>PROOF</u>:

$$\frac{d}{dt} I[t] = \int [dx^\mu][d\psi^\mu]\, \delta\mathcal{V}\, e^{\omega + t\delta\mathcal{V}} = \int [dx^\mu][d\psi^\mu]\, \delta\left[\mathcal{V}\cdot e^{\omega + t\delta\mathcal{V}}\right]$$

$$= 0\,! \quad \blacksquare \qquad \because \ \delta\omega = 0 \ \& \ \delta^2\mathcal{V} = 0$$

- This implies that $I[\infty] = \mathrm{Vol}(X)$

BUT, it is not helpful! $\because$ $\delta\mathcal{V}$ doesn't contain any bosonic terms

EQUIVARANT VOLUME

– Hamiltonian flow : (H, V^μ) s.t. $i_V \omega = dH$

function   vector field on X

• $\mathcal{L}_V \omega = [d\, i_V + i_V d]\omega = d^2 H = 0$ $\blacksquare$

NB In many physical systems (e.g. QM with target space X)

$$V^\mu \cong \text{global symmetry}$$

– DEFORM THE SUSY !

$$\delta_\epsilon X^\mu = \psi^\mu \qquad \delta_\epsilon \psi^\mu = \epsilon V^\mu(\vec{x})$$

- Then, one can show that $\delta_\epsilon^2 X^\mu = \epsilon V^\mu \quad \delta_\epsilon^2 \psi^\mu = \epsilon \partial_\nu V^\mu \psi^\nu$

$$\delta_\epsilon^2 = \mathcal{L}_V \left(= d\, i_V + i_V d\right)$$

- Equivariant Volume

$$\mathrm{Vol}_\epsilon[X] = \int e^{\omega - \epsilon H}, \quad \text{where} \quad dH = i_V \omega$$

Note that $\underline{\delta_\epsilon[\omega - \epsilon H] = 0}$!

$$\left(\begin{array}{c} \delta_\epsilon \omega = \epsilon \omega_{\mu\nu} V^\mu \psi^\nu = \partial_\nu H \cdot \psi^\nu \\ \qquad\qquad\qquad\qquad\qquad\qquad \parallel \\ \qquad\qquad\qquad\qquad\qquad\qquad \delta_\epsilon H \end{array} \right)$$

Example S^2

$$\omega = d\theta \wedge \sin\theta d\phi \qquad V = \partial_\phi \qquad i_V\omega = + d\cos\theta \rightarrow \; H = \cos\theta$$

Then,

$$Vol_\epsilon[S^2] = \int_{\Pi[1]S^2} e^{\omega - \epsilon H} \qquad (\omega = \sin\theta\, \psi^\theta \psi^\phi)$$

$$= \int d\theta d\phi \, \sin\theta \, e^{-\epsilon\cos\theta}$$

$$= \int_{-1}^{1} dx \cdot 2\pi \cdot e^{-\epsilon x} = \frac{2\pi}{\epsilon}\left[e^{\epsilon} - e^{-\epsilon}\right] = 4\pi\frac{\sinh\epsilon}{\epsilon}$$

$$\lim_{\epsilon\to 0} Vol_\epsilon[S^2] = 4\pi \,!$$

Example $\mathbb{R}^2$ $\quad \omega = dx \wedge dy \quad V = -x\partial_y + y\partial_x \quad i_V\omega = +\frac{1}{2}d(x^2+y^2)$

$$\Rightarrow H = +\frac{1}{2}(x^2+y^2)$$

$$\mathrm{Vol}_\epsilon[\mathbb{R}^2] = \int_{\Pi[1]\mathbb{R}^2} e^{\omega - \epsilon H} \quad \text{with} \quad \omega = \psi^x \psi^y$$

$$= \int_{\mathbb{R}^2} e^{-\frac{\epsilon}{2}(x^2+y^2)} = \left(\sqrt{\frac{2\pi}{\epsilon}}\right)^2 = \frac{2\pi}{\epsilon}$$

Equivariant volume of a non-compact space can be finite!

What are they good for? [DH-formula]

- can use the localization techique

$$I_\epsilon[t] = \int_{T[1]X} e^{\omega - \epsilon H - t\delta\mathcal{V}}$$

IF $\mathcal{V}(x,\psi)$ IS INVARIANT UNDER $\mathcal{L}_V$, $I_\epsilon[t]$ IS INDEP. OF t

- Choose $\mathcal{V} = g_{\mu\nu}\psi^\mu V^\nu$ $\quad \delta\Psi^\mu = V^\mu$:

$$S_{def} = \delta\mathcal{V} = \underbrace{\epsilon g_{\mu\nu} V^\mu V^\nu}_{\to \text{positive def.}} + \partial_\rho g_{\mu\nu}\psi^\rho\psi^\mu V^\nu + g_{\mu\nu}\psi^\mu \partial_\rho V^\nu \psi^\rho$$

- TAKE A LIMIT $t \to \infty$,

$$\mathrm{vol}_\epsilon[X] = \int_{T[1]X} e^{\omega - \epsilon H - t[\underbrace{\epsilon g_{\mu\nu} V^\mu V^\nu + \cdots}_{= S_{\mathrm{def}}}]}$$

[1] THE INTEGRAL LOCALIZES NEAR $\|V\| = 0$! $\quad V^\mu(x_*) = 0$

[2] SADDLE-POINT APPROXITION BECOMES EXACT $\quad \delta\Psi^\mu = V^\mu = 0$

SUSY condition

$$V^\mu \cong V^\mu_\rho(x_*)\, x^\rho + \cdots$$

$$S_{\mathrm{def}} = \epsilon g_{\mu\nu}(x_*) V^\mu_\alpha V^\nu_\beta x^\alpha x^\beta + g_{\mu\nu}(x_*) \psi^\mu \psi^\rho V^\nu_\rho(x_*)$$

$$\therefore \quad \mathrm{vol}_\epsilon[X] = \sum_{x^*} e^{-\epsilon H(x_*)} \left[\frac{2\pi}{\epsilon}\right]^n \frac{\mathrm{Pf}\left[g_{[\mu\nu}(x_*) V^\nu_{\rho]}\right]}{\det^{1/2}\left[g_{\mu\nu}(x_*) V^\mu_\alpha V^\nu_\beta\right]}$$

EXAMPLE S^2 REVISITED! $\quad \mathrm{Vol}(S^2) = \int_{T[1]S^2} e^{\omega - \epsilon H - S_{def}}$

$$S_{def} = \delta_\epsilon [g_{ab}\psi^a V^b] = \delta_\epsilon[\sin^2\theta\,\psi^\phi] = \epsilon\sin^2\theta + 2\sin\theta\cos\theta\,\psi^\theta\psi^\phi$$

In the limit $t\to\infty$, the volume integral is localized onto $\theta=0$ (N) & $\theta=\pi$ (S)

① Near $\theta=0$ Around N-pole, $S^2 \sim R^2$

$x \equiv \theta\cos\phi$, $y \equiv \theta\sin\phi$

$\psi^x = \cos\phi\,\psi^\theta + y\,\psi^\phi$

$\psi^y = \sin\phi\,\psi^\theta - x\,\psi^\phi$

$$S_{def} = -t[\epsilon(x^2+y^2) - 2\psi^x\psi^y]$$

$$\Rightarrow -e^{-\epsilon}\int dx\,dy\,d\psi^x d\psi^y\, e^{-\epsilon(x^2+y^2)+2\psi^x\psi^y}$$

$$= -\frac{2\pi}{\epsilon}e^{-\epsilon}$$

② Near $\theta=\pi$, one can obtain $\frac{2\pi}{\epsilon}e^{+\epsilon}$ $\quad \therefore \mathrm{Vol}(S^2) = \frac{4\pi}{\epsilon}\sinh\epsilon$

HARISH-CHANDRA-ITZYKSON-ZUBER [HCIZ] INTEGRAL

$$I(A,B) = \int dU \, e^{-S[A,B;U]}$$ with $S[A,B;U] = \mathrm{tr}[AUBU^{+}]$

U : unitary

$A = \mathrm{diag}(a_1, \cdots, a_n)$

$B = \mathrm{diag}(b_1, \cdots, b_n)$

Use the localization technique to show

$$I(A,B) \propto \frac{\det(e^{-a_i b_j})}{\prod_{i<j} (a_i - a_j)(b_i - b_j)}$$

This integral arises from Brownian motions

Note that an integration domain $X = U(n)/U(1)^n$ they don't rotate B

Mathematical Facts

I. X is a Kähler manifold

II. Coordinates $(z^1, \cdots, z^N)$ $N = n(n-1)/2$

Kähler 2-form $\omega = \frac{i}{2} g_{a\underline{b}} dz^a \wedge d\bar{z}^{\underline{b}}$

Then, one can rewrite the HCIZ integral as follow

$$I(A,B) = \int_{T[1]X} e^{-S[A,B;z]+\omega}$$

Supersymmetry

- SUSY transf. rules

$$\delta z^a = \psi^a \qquad \delta\psi^a = -2i\, g^{a\underline{b}}\, \partial_{\underline{b}} S$$

$$\delta \bar{z}^{\underline{b}} = \bar{\psi}^{\underline{b}} \qquad \delta\bar{\psi}^{\underline{b}} = +2i\, g^{a\underline{b}}\, \partial_a S$$

- Under the above transf. rules, one can show $S[A,B] - w$ is invariant

$$\delta S[A,B] = \partial_a S\, \delta z^a + \partial_{\underline{b}} S\, \delta\bar{z}^{\underline{b}} = \partial_a S\, \psi^a + \partial_{\underline{b}} S\, \bar{\psi}^{\underline{b}}$$

they vanish $\because$ $X=$ Kähler

$$\delta w = -\frac{i}{2} g_{a\underline{b}}\, \delta\psi^a \bar{\psi}^{\underline{b}} - \frac{i}{2} g_{a\underline{b}}\, \psi^a \delta\bar{\psi}^{\underline{b}} + \frac{i}{2}\partial_c g_{a\underline{b}}\, \delta z^c \psi^a \bar{\psi}^{\underline{b}} + \frac{i}{2}\partial_{\underline{c}} g_{a\underline{b}}\, \delta\bar{z}^{\underline{c}} \psi^a \bar{\psi}^{\underline{b}}$$

$$= \partial_{\underline{b}} S\, \bar{\psi}^{\underline{b}} + \partial_a S\, \psi^a$$

$$\Rightarrow \delta[S - w] = 0\,!$$

- Q-exact terms : We choose $S_{def} = QV(z, \bar{z}, \psi, \bar{\psi})$

$$\begin{cases} V = i\partial_a S\psi^a - i\partial_{\underline{b}} S\bar{\psi}^{\underline{b}} \\ S_{def} = 4g^{a\underline{b}}\partial_a S\partial_{\underline{b}} S - 2i\partial_a\partial_{\underline{b}} S\,\psi^a\bar{\psi}^{\underline{b}} \end{cases}$$

→ 'positive def. bosonic terms'

- Now, it's time to identify the saddle pts.

<u>SOLVE</u> $\partial_a S = \partial_{\underline{b}} S = 0$!

Suppose that $U_0 \in X = U(n)/U(1)^n$ satisfies the saddle pt. eqn., i.e.,

$$\delta S[U_0] = S[e^{ih}U_0] - S[U_0] = \mathrm{tr}[A\,ih\,U_0BU_0^+] - \mathrm{tr}[A\,U_0BU_0^+\,ih]$$

$$= \mathrm{tr}[\,ih\,\underbrace{[U_0BU_0^+, A]}\,] = 0!$$

"linear part"

$U_0BU_0^+$ & A commute to each other!

$\underline{\text{saddle pts}}$: $U_0 B U_0^\dagger = \mathrm{diag}(b_{p(1)}, \cdots, b_{p(N)})$ where $p \in S_N$, permutation group.

– ONE-LOOP DETERMINANT

Fix a saddle pt. $U_0 = \mathrm{id}_N$

quadratic piece of S_{def} around U_0 ;

$$U = e^{ih} \quad \text{where } h = \begin{bmatrix} 0 & & y_{ij} \\ & \ddots & \\ \bar{y} & & 0 \end{bmatrix} \quad i<j$$

[a] $\delta^{(2)} S = \mathrm{tr}[AhBh] - \frac{1}{2}\mathrm{tr}[Ah^2B] - \frac{1}{2}\mathrm{tr}[ABh^2]$

$$= \sum_{i<j} \underbrace{\left[a_i b_j + a_j b_i - a_i b_i - a_j b_j\right]}_{= -(a_i - a_j)(b_i - b_j)} y_{ij} \bar{y}_{ij}$$

[b] $$g^{a\underline{b}}\partial_a S\,\partial_{\underline{b}} S = \sum_{i<j} (a_i-a_j)^2(b_i-b_j)^2\, y_{ij}\,\bar{y}_{ij} \qquad \left[\partial_a \doteq \partial_{y_{ij}}\,,\ \partial_{\underline{b}} \doteq \partial_{\bar{y}_{ij}}\right]$$

$$\partial_a\partial_{\underline{b}} S\cdot \psi^a\bar{\psi}^{\underline{b}} = -\sum_{i<j}(a_i-a_j)(b_i-b_j)\,\psi_{ij}\,\bar{\psi}_{ij}$$

[c] $$S[\mathrm{id}] = \sum_i a_i b_i$$

$$\det{}_f/\det{}_b\Big|_{u_0=\mathrm{id}.} = \prod_{i<j}\frac{(a_i-a_j)(b_i-b_j)}{(a_i-a_j)^2(b_i-b_j)^2} = \prod_{i<j}\frac{1}{(a_i-a_j)(b_i-b_j)}$$

[d] Sum over other saddle pts.

$$\Longrightarrow \quad I(AB) \propto \sum_{p\in S_N}\frac{\prod_i e^{-a_i b_{p(i)}}}{\prod_{i<j}(a_i-a_j)(b_{p(i)}-b_{p(j)})}$$

$$= \frac{\det(e^{-a_i b_j})}{\prod_{i<j}(a_i-a_j)(b_i-b_j)} \qquad \blacksquare$$

In general, it is not easy to compute $Val_\epsilon[X]$

① Suppose that the space X is embedded into a (flat) ambient space M in a 'nice' way

② Can we compute $Val_\epsilon[X]$ in the ambient space M ?

SYMPLECTIC QUOTIENT $X = M /\!/ U(1)$

On M, G generate $U(1)$ rotation $\Rightarrow$ Hamiltonian flow = moment map

$$i_G \omega = d\mu$$

[1] Define a level surface $\mu^{-1}(\xi) = \{ x \in M \mid \mu(x) = \xi \}$

[2] $\mu^{-1}(\xi)$ is invariant under $U(1)$

$$\mu(\vec{x}+\vec{G}) - G^\mu \partial_\mu \mu = G^\mu G^\nu \omega_{\mu\nu} = 0 \,!$$

[3] $X = \mu^{-1}(\xi) / U(1)$ (well-defined)

IN GLSM, $U(1)$ = gauge symmetry & μ = D-term

FORMULA $X = \mu^{-1}(\xi)/U(1)$

$$\mathrm{Vol}_\epsilon[X] = \frac{1}{2\pi\,\mathrm{Vol}[U(1)]}\int_{T[1]M} [dx^M][d\psi^M][d\phi]\; e^{\omega + i\phi[\mu(x^M)-\xi]}$$

- introduce an auxiliary variable ϕ (Lagrangian multiplier)

- SUSY transf. rules

generate $U(1)$!

$$\delta x^M = \psi^M \qquad \delta\psi^M = -i\phi V^M \qquad \delta\phi = 0$$

Then, one can show

$$\delta[\omega + i\phi\mu] = -i\phi\,\omega_{MN}V^M\psi^N + i\phi\,\partial_N\mu\,\psi^N = 0$$

$\Rightarrow$ $\mathrm{Vol}_\epsilon[X]$ is invariant under SUSY !

Proof of formula

- Introduce a coordinate system $\{x^i, x^v, x^n\} \equiv \{x^M\}$

$V = \frac{\partial}{\partial x^v}$

normal to $\mu^{-1}(\xi)$

$X = M/\!/U(1)$

$d\mu = i_v \omega$ implies that $\omega_{vi} = \frac{\partial}{\partial x^i}\mu$ & $\omega_{vn} = \frac{\partial}{\partial x^n}\mu$

- $$\mathrm{Vol}(X) = \int_{T[1]X} [dx^i][d\psi^i]\, e^{\omega_X} \quad \text{with } \omega_X = \frac{1}{2}\omega_{ij}\psi^i\psi^j$$

$$= \frac{1}{\mathrm{Vol}[U(1)]}\int [dx^v][dx^i][d\psi^i]\, e^{\omega_X} \quad (\because \mu^{-1}(\xi) \text{ is inv. under } U(1)_v)$$

$$= \frac{1}{\mathrm{Vol}[U(1)]}\int [dx^n][dx^v][dx^i][d\psi^i]\, e^{\omega_X}\, \delta(\mu-\xi)\cdot\frac{\partial\mu}{\partial x^n}$$

Note that $\delta(\mu-\xi) = \frac{1}{2\pi}\int d\phi\, e^{i\phi(\mu-\xi)}$ & $\frac{\partial\mu}{\partial x^n} = \int d\psi^v d\psi^n\, e^{\frac{\partial\mu}{\partial x^n}\psi^v\psi^n}$

$$= \frac{1}{2\pi\,\mathrm{Vol}[U(1)]}\int_{T[1]M} e^{\omega_M} \quad \text{with } \omega_M = \frac{1}{2}\omega_{ij}\psi^i\psi^j + \omega_{vn}\psi^v\psi^n + \omega_{vi}\psi^v\psi^i$$

don't affect the result!

- It is convenient to introduce more auxiliary variables

$$\bar{\phi} : B \qquad \eta : F$$

$$h : B \qquad \chi : F$$

SUSY transf. $\delta\bar{\phi} = \eta \qquad \delta\eta = 0 \qquad \delta\chi = h \qquad \delta h = 0$

- Q-exact term $S_{def} = t\,\delta\mathcal{V}$ s.t. $\mathcal{V}$ is invariant under $U(1)$

$$\mathcal{V} = \chi\left(\mu(x^M) - \xi - \tfrac{1}{2}h\right) + \bar{\phi} f_M(x) \psi^M$$

$\rightarrow$ gauge charge "0"

$$S_{def} = t\left[h(\mu(x) - \xi) - \tfrac{1}{2}h^2 + -i\phi\bar{\phi} f_M(x) V^M - \chi \partial_M \mu \psi^M + \bar{\eta} \psi^M f_M(x)\right]$$

$$\mathrm{Vol}_\epsilon[X] = \int [dx^M][d\psi^M][d\phi][d\bar{\phi}][dh][d\chi][d\eta]\, e^{\omega + i\phi\mu(x) + t\delta\mathcal{V}}$$

Localization symmetry of $X \Rightarrow$ symmetry of M, generated by V

$$\delta_\epsilon X^M = \psi^M \qquad \delta_\epsilon \psi^M = -i\phi G^M(x) + \varepsilon V^M(x) \qquad \delta_\epsilon \phi = 0$$

NB $[V, G] = 0$!

Follow the same steps

$$\mathrm{Vol}_\epsilon[X] = \frac{1}{2\pi \, \mathrm{Vol}[U(1)]} \int [dX^M][d\psi^M][d\phi] \, e^{\omega + i\phi\mu(x) - \varepsilon H(x) + \delta_\epsilon V}$$

Example $\mathbb{P}^1 \equiv S^2$

$\mathbb{C}^2 /\!\!/ U(1)$ $\quad (z_1, z_2) \xrightarrow{U(1)} e^{i\alpha}(z_1, z_2)$ $\quad \mathbf{V} = z_i \partial_i - \bar{z}_i \partial_{\underline{i}}$

[1] isometry $\mathbf{R} = z_1\partial_1 - z_2\partial_2 - \bar{z}_1\bar{\partial}_1 + \bar{z}_2\bar{\partial}_2$: $(z_1, z_2) \longrightarrow (e^{i\lambda} z_1, e^{-i\lambda} z_2)$

[2] Q_ϵ - exact terms $\quad S_{def} = t[\delta_\epsilon \mathcal{V}]$

$$\epsilon \mathcal{V} = -g_{a\underline{b}}\, \delta_\epsilon \psi^a \bar{\psi}^{\underline{b}} - g_{a\underline{b}}\, \psi^a \delta_\epsilon \bar{\psi}^{\underline{b}} + \chi\left(\mu(z,\bar{z}) - 1 - \frac{h}{2}\right) + \bar{\Phi}\left[\partial_a \mu \psi^a - \partial_{\underline{b}} \mu \bar{\psi}^{\underline{b}}\right]$$

$$S^b_{def} = +2\left[\phi V^a + i\epsilon R^a\right]\left[\phi V^{\underline{a}} + i\epsilon R^{\underline{a}}\right] g_{a\underline{a}} + h\left(\mu - 1 - \frac{h}{2}\right)$$
$$+ \bar{\Phi}\left[-i\phi V^a + \epsilon R^a\right]\partial_a \mu - \bar{\Phi}\left[-i\phi V^{\underline{a}} + \epsilon R^{\underline{a}}\right]\partial_{\underline{a}} \mu$$

$$S^f_{def} = \left[i\phi V^a{}_b - \epsilon R^a{}_b\right]\psi^b \bar{\psi}^{\underline{b}} g_{a\underline{b}} - \left[i\phi V^{\underline{a}}{}_{\underline{b}} - \epsilon R^{\underline{a}}{}_{\underline{b}}\right]\psi^a \bar{\psi}^{\underline{b}} g_{a\underline{a}}$$
$$+ (\eta - \chi)\psi^a \partial_a \mu + (\eta + \chi)\bar{\psi}^{\underline{b}} \partial_{\underline{b}} \mu \big]$$

[3] saddle points $\quad \delta_\epsilon \psi^a = \delta_\epsilon \bar{\psi}^{\bar{b}} = \delta_\epsilon \chi = 0$ & $\delta S_{def}[\Phi_*] = 0$

① $\phi V^a + i\epsilon R^a = \phi(z^1, z^2) + i\epsilon(z^1, -z^2) = 0$!

$\Rightarrow$ two solutions : $(\phi = -i\epsilon,\ z^2 = 0), (\phi = +i\epsilon,\ z^1 = 0)$

② $h = 0$

③ $\frac{\delta S_{def}}{\delta h} = 0 \longrightarrow h = \mu(z, \bar{z}) - 1$

$\Big\}$ $h = 0$ & $\mu = 1$

④ $\frac{\delta S_{def}}{\delta \phi} = 0 \longrightarrow \bar{\phi}\,[V^a \partial_a \mu - V^{\bar{b}} \partial_{\bar{b}} \mu] \propto \bar{\phi}\mu = 0 \quad \therefore \bar{\phi} = 0$!

$\therefore\ \exists$ two saddle pts $(z^1 = 1, z^2 = 0, \phi = -i\epsilon, \bar{\phi} = 0, h = 0)$ N

$(z^1 = 0, z^2 = 1, \phi = +i\epsilon, \bar{\phi} = 0, h = 0)$ S

HW: work out the one-loop determinant around these two fixed points and see if one can get the same answer

NB Quotient with Non-abelian group

$$\delta\varphi^M = \psi^M \qquad \delta\psi^M = [\phi, \varphi^M] \qquad \delta\bar{\Phi} = \eta \qquad \delta\eta = [\phi, \bar{\Phi}]$$

$$\delta\chi = h \qquad \delta h = [\phi, h] \qquad \delta^2 = \mathrm{Gauge}(\phi)$$

Adjoint Representation

NEKRASOV INSTANTON PARTION FUCTION

N D4s

= EQUIVARIANT VOLUME OF INSTANTON MODULI SPACE

	0	1	2	3	4	5	6	7	8	9
D4	x	x	x	x	x					
D0	x									

Global symmetry

$$G = SO(4) \times Sp(4) \times SU(N)_F$$

Low-energy dynamics on D0 branes can be described as $\overset{U(k)}{\text{"GAUGED QM}}$ with 8 supercharges", involving

(i) Vector $(A_0, \varphi^I, \bar{\lambda}^{i}_{\dot{\alpha}})$

(ii) adjoint hyper (a_m, λ^i_α)

(iii) fund. hyper $(q_{\dot{\alpha}}{}^A, \psi^{iA})$

$I = 1, 2, \cdots, 5$ $SO(5)$ indices

$i = 1, 2, \cdots, 4$ $Sp(4)$ indices

SUSY Action

$$
\begin{aligned}
L_{SYM} \;=\; & \mathrm{tr}_k \Big(\frac{1}{2} D_t \varphi_I D_t \varphi_I + \frac{1}{2} D_t a_m D_t a_m + \frac{1}{4}[\varphi_I, \varphi_J]^2 + \frac{1}{2}[a_m, \varphi_I]^2 + \frac{1}{4}[a_m, a_n]^2 \\
& + \frac{i}{2}(\bar{\lambda}^{i\dot{\alpha}})^\dagger D_t \bar{\lambda}^{i\dot{\alpha}} + \frac{1}{2}(\bar{\lambda}^{i\dot{\alpha}})^\dagger (\gamma^I)^i{}_j [\varphi_I, \bar{\lambda}^{j\dot{\alpha}}] + \frac{i}{2}(\lambda^i_\alpha)^\dagger D_t \lambda^i_\alpha - \frac{1}{2}(\lambda^i_\alpha)^\dagger (\gamma^I)^i{}_j [\varphi_I, \lambda^j_\alpha] \\
& - \frac{i}{2}(\lambda^i_\alpha)^\dagger (\sigma^m)_{\alpha\dot{\beta}} [a_m, \bar{\lambda}^{i\dot{\beta}}] + \frac{i}{2}(\bar{\lambda}^{i\dot{\alpha}})^\dagger (\bar{\sigma}^m)^{\dot{\alpha}\beta} [a_m, \lambda^i_\beta] \Big) \; .
\end{aligned}
$$

$$
\begin{aligned}
L_f \;=\; & D_t q_{\dot{\alpha}} D_t \bar{q}^{\dot{\alpha}} - (\varphi_I \bar{q}^{\dot{\alpha}} - \bar{q}^{\dot{\alpha}} v_I)(q_{\dot{\alpha}} \varphi_I - v_I q_{\dot{\alpha}}) + i(\psi^i)^\dagger D_t \psi^i + (\psi^i)^\dagger (\gamma^I)^i{}_j \;\; \psi^j \varphi_I \\
& + \sqrt{2} i \left((\bar{\lambda}^{i\dot{\alpha}})^\dagger \bar{q}^{\dot{\alpha}} \psi^i - (\psi^i)^\dagger q_{\dot{\alpha}} \bar{\lambda}^{i\dot{\alpha}} \right) \; ,
\end{aligned}
$$

SUSY Transformation Rules

$$\bar{Q}^{i\dot{\alpha}} A_t = i\bar{\lambda}^{i\dot{\alpha}} , \quad \bar{Q}^{i\dot{\alpha}}\varphi^I = -i(\gamma^I)^i{}_j \bar{\lambda}^{j\dot{\alpha}}$$

$$\bar{Q}^{i\dot{\alpha}}\bar{\lambda}^{j\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}(\gamma^I\omega)^{ij} D_0\varphi^I - \frac{i}{2}\epsilon^{\dot{\alpha}\dot{\beta}}(\gamma^{IJ}\omega)^{ij}[\varphi^I, \varphi^J] - 2i\omega^{ij} D^{\dot{\alpha}}{}_{\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}}$$

$$\bar{Q}^{i\dot{\alpha}} a^m = (\bar{\sigma}^m)^{\dot{\alpha}\beta}\lambda^i_\beta$$

$$\bar{Q}^{i\dot{\alpha}}\lambda^j_\beta = (\bar{\sigma}^m)^{\dot{\alpha}\gamma}\epsilon_{\gamma\beta}\left(i\omega^{ij} D_t a_m + (\gamma^I\omega)^{ij}[\varphi_I, a_m]\right)$$

$$\bar{Q}^{i\dot{\alpha}} q_{\dot{\beta}} = \sqrt{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\psi^i$$

$$\bar{Q}^{i\dot{\alpha}}\psi^j = \sqrt{2}\left[i\omega^{ij} D_t q_{\dot{\beta}} - (\gamma^I\omega)^{ij} q_{\dot{\beta}}\varphi_I\right]\epsilon^{\dot{\beta}\dot{\alpha}}$$

Why do we study D0-D4 QM (or dim'l reduction to matrix model)?

<u>SW theory</u> can determine the low-energy effective theory in the Coulomb branch of 4d $\mathcal{N}=2$ SUSY gauge theories

Question derivation from first principles?

$\Rightarrow$ equivariant Volume of $\mathcal{M}^{k,N}_{inst}$ moduli space of YM instantons!

partition function of D0-D4 QM can be identified as the volume of $\mathcal{M}^{k,N}_{inst}$

$SU(2)_\ell \times SU(2)_\lambda \times SU(2)_L \times SU(2)_R \subset$ global sym. G

BRST charge

supercharge $\bar{Q}^i_{\dot\alpha}$: $(1,2,2,1) \oplus (1,2,1,2)$

Vector multiplet $\quad A_0 + i\varphi_5 \equiv \phi$: $(1,1,1,1)$

$A_0 - i\varphi_5 \equiv \bar{\phi}$: $(1,1,1,1)$

φ_m : $(1,1,2,2)$

λ^i_α : $(2,1,2,1) \oplus (2,1,1,2)$

adj. hyper $\quad a_m$: $(2,2,1,1)$

$\bar{\lambda}^i_{\dot\alpha}$: $(1,2,2,1) \oplus (1,2,1,2)$

fund. hyper $\quad q_{\dot\alpha}$: $(1,2,1,1)$

ψ^i : $(1,1,2,1) \oplus (1,1,1,2)$

$$SU(2)_\ell \times \widehat{SU(2)}_r \times SU(2)_L \subset SU(2)_\ell \times SU(2)_r \times SU(2)_L \times SU(2)_R$$

supercharge $\bar{Q}^{\dot{i}}_{\dot{\alpha}}$: $(1,2,2,1)\oplus(1,2,1,2) \to \cdots \oplus (1,1,1) \oplus (1,3,1)$

Vector multiplet $A_0 + i\varphi_5 \equiv \phi$: $(1,1,1,1) \to (1,1,1)$

$A_0 - i\varphi_5 \equiv \bar{\phi}$: $(1,1,1,1) \to (1,1,1)$

φ_m : $(1,1,2,2) \to (1,2,2)$

$\lambda^{\dot{i}}_{\alpha}$: $(2,1,2,1)\oplus(2,1,1,2) \to \underset{\vec{\chi}}{(2,1,2)} \oplus \underset{\Psi_m}{(2,2,1)}$

adj. hyper

a_m : $(2,2,1,1) \to (2,2,1)$

$\bar{\lambda}^{\dot{i}}_{\dot{\alpha}}$: $(1,2,2,1)\oplus(1,2,1,2) \to \underset{\Psi_{m+4}}{(1,2,2)} \oplus \underset{\eta}{(1,1,1)} \oplus \underset{\vec{\chi}}{(1,3,1)}$

fund. hyper

$q_{\dot{\alpha}}$: $(1,2,1,1) \to (1,2,1)$

$\psi^{\dot{i}}$: $(1,1,2,1)\oplus(1,1,1,2) \to \underset{\xi_a}{(1,1,2)} \oplus \underset{\psi_{\dot{\alpha}}}{(1,2,1)}$

SUSY transf. = SUSY transf. in the case of symplectic quotient.

$$\delta\phi = 0\ ,\quad \delta\bar{\phi} = \eta\ ,\quad \delta\eta = [\phi, \bar{\phi}]\ ,\quad \delta a_m = \Psi_m\ ,\quad \delta\varphi_m = \Psi_{m+4}$$

$$\delta\Psi_m = [\phi, a_m]\ ,\quad \delta\Psi_{m+4} = [\phi, \varphi_m]\ ,\qquad \delta q_{\dot\alpha} = \psi_{\dot\alpha}\ ,\quad \delta\psi_{\dot\alpha} = \phi\, q_{\dot\alpha}$$

$$\delta\vec{\chi} = \overset{7}{\vec{H}}\ ,\quad \delta\vec{H} = [\phi, \vec{\chi}]\ ,\quad \delta\xi_a = \overset{2}{h_a}\ ,\quad \delta h_a = \phi\,\xi_a$$

auxiliary variables!

$$\Rightarrow\ \delta^2 = \text{gauge}_{U(k)}[\phi]$$

Then, the SUSY Lagrangian in '0'-dimensions [dim'l red. of QM] is **Q-exact!!**

SUSY ACTION

$$\vec{\mathcal{E}} = (\mathcal{E}_R^A, \mathcal{E}_L^A, \mathcal{E})$$
$$\equiv \left(\frac{i}{2}\bar{\eta}^A_{mn}\left([\varphi_m,\varphi_n] - [a_m,a_n]\right) - \bar{x}^{\dot\alpha} x_{\dot\beta}(\tau^A)^{\dot\beta}{}_{\dot\alpha} + \zeta^A, \frac{i}{2}\bar{\eta}^A_{mn}\left([\varphi_m,a_n] + [a_m,\varphi_n]\right), -i[\varphi_m,a_m] \right)$$

$$i\mathcal{F}_a \equiv -(\sigma^m)_{a\dot\alpha}\,\epsilon^{\dot\alpha\dot\beta} x_{\dot\beta}\varphi_m$$

$$\underset{\delta\mathcal{V}_0}{\overset{S}{\shortparallel}} = \delta\Big[\frac{1}{8}\eta[\phi,\bar\phi] - \frac{1}{2}\Psi_m[\bar\phi,a_m] - \frac{1}{2}\Psi_{m+4}[\bar\phi,\varphi_m] + \vec{\chi}\cdot\left(\frac{1}{2}\vec{H} - i\vec{\mathcal{E}}\right)$$
$$+ \xi_a\left(\frac{1}{2}h_a^{+} - i\mathcal{F}_a^{+}\right) + \xi_a^{+}\left(\frac{1}{2}h_a - i\mathcal{F}_a\right) + \bar\psi^{\dot\alpha}\bar\phi q_{\dot\alpha} + \bar q^{\dot\alpha}\bar\phi\psi_{\dot\alpha}\Big]$$

$$\Rightarrow\ Z_{D-1} = \int [da_m\, dq\, d\bar q\, d\Psi_m\, d\psi\, d\bar\psi][d\varphi_m\, d\Psi_{m+4}][d\bar\phi\, d\eta\, d\vec\chi\, d\vec H\, d\xi\, d\bar\xi\, d\mathcal{F}\, d\bar{\mathcal{F}}]$$
$$[d\phi]\ \mathrm{Exp}[\omega + i\phi\mu + \delta\mathcal{V}_0]$$

Note also that, in flat ambient space. $\omega + i\phi\mu = \delta(\#)$

Equivariant Volume of $\mathcal{M}^{k,N}_{\text{inst}}$ = "Path"-integral of D-1 matrix theory

$$Z_{D\text{-}1} = \int [da_m\, dq\, d\bar{q}\, d\Psi_m\, d\psi\, d\bar{\psi}][d\varphi_m\, d\Psi_{m+4}][d\bar{\Phi}\, d\eta\, d\vec{\chi}\, d\vec{H}\, d\xi\, d\bar{\xi}\, dF\, d\bar{F}]$$
$$[d\phi]\ \mathrm{Exp}[\omega + i\phi\mu + \delta V_0]$$

- Volume of the instanton moduli space [$U(k)$-quotient], $\mathcal{M}^{k,N}_{\text{inst}}$

- $\mathcal{M}^{k,N}_{\text{inst}}$ is non-compact

- Equivariant parameters $\quad \underbrace{SU(2)_\ell \times \widehat{SU(2)}_r}_{\text{Nekrasov's } \Omega\text{-parameters } [\epsilon_1, \epsilon_2]} \times \overbrace{SU(N)}^{\text{Coulomb branch parameter } a}_F$

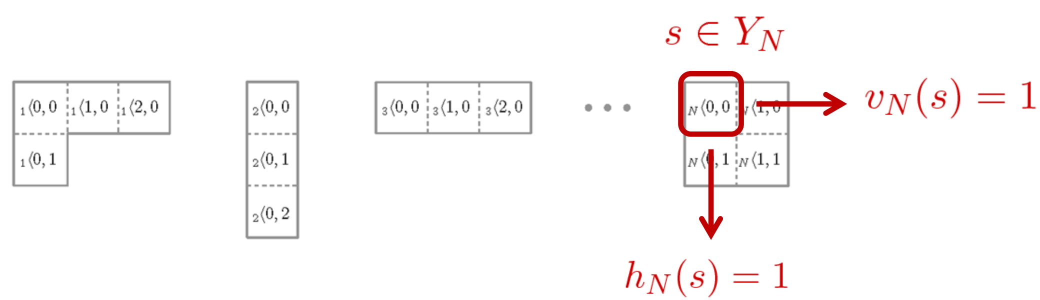
- Turning on FI parameter, $\quad Z_{D\text{-}1} = Z^{k}_{\text{Nek}}[\vec{a}, \epsilon_1, \epsilon_2]$

Nekrasov Partition Function

Result: [Nekrasov]

. Volume can be computed by Gaussian path-integrals over a set of saddle-points (solutions of deformed ADHM), characterized by N-colored Young diagrams

$$\vec{Y} = \{Y_1, Y_2, ..Y_N\} \;:$$



$$\Rightarrow \qquad I(\vec{\mu}, \gamma, q) = \sum_{\vec{Y}=\{Y_1,Y_2,..Y_N\}} q^{|\vec{Y}|} I_{\{Y_1,Y_2,..Y_N\}}$$

where $$I_{\{Y_1,Y_2,\cdots,Y_N\}} = \prod_{i,j=1}^{N} \prod_{s\in Y_i} \frac{\sinh \frac{E_{ij}-i(\gamma_2+\gamma_R)}{2} \sinh \frac{E_{ij}+i(\gamma_2-\gamma_R)}{2}}{\sinh \frac{E_{ij}}{2} \sinh \frac{E_{ij}-2i\gamma_R}{2}}$$

$$E_{ij} = \mu_i - \mu_j + i(\gamma_1 - \gamma_R) h_j(s) + i(\gamma_1 + \gamma_R)(v_i(s) + 1)$$