Perturbative QCD

Part III : NLO corrections to PDF

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▪ Parton Distribution Function (PDF)

$$
\begin{aligned}
f_{q/N}(y,\mu) &= \frac{1}{\overline{n}\cdot P}\sum_{X_n}\langle N(P)|\bar{q}_n W_n \frac{\overline{\not n}}{2}|X_n\rangle\langle X_n|\delta(y-\frac{\overline{n}\cdot\mathcal{P}}{\overline{n}\cdot P})W_n^\dagger q_n|N(P)\rangle \\
&= \langle N(P)|\bar{q}_n W_n \frac{\overline{\not n}}{2}\delta(y\overline{n}\cdot P-\overline{n}\cdot\mathcal{P})W_n^\dagger q_n|N(P)\rangle \\
&= \int \frac{dn\cdot z}{4\pi}e^{-ix\overline{n}\cdot Pn\cdot z/2}\langle N(P)|\bar{q}_n(\frac{n\cdot z}{2})\left[\frac{n\cdot z}{2},0\right]q_n(0)|N(P)\rangle
\end{aligned}
$$

$$\sum_{X_n}|X_n\rangle\langle X_n| = 1$$

gauge invariant

$$
\begin{aligned}
[\bar{z},0] &= W_n(\bar{z})W_n^\dagger = \mathrm{P}\exp\left[ig\int_{-\infty}^{\bar{z}}d\bar{z}'\overline{n}\cdot A_n(\bar{z}')\right]\bar{\mathrm{P}}\exp\left[-ig\int_{-\infty}^{0}d\bar{z}'\overline{n}\cdot A_n(\bar{z}')\right] \\
&= \mathrm{P}\exp\left[ig\int_0^{\bar{z}}d\bar{z}'\overline{n}\cdot A_n(\bar{z}')\right]
\end{aligned}
$$

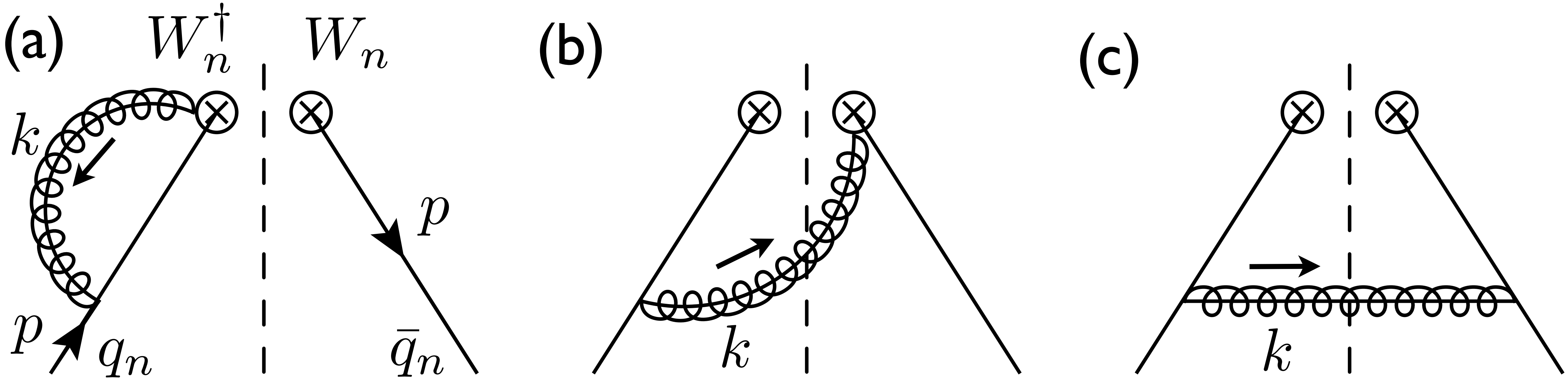
- PDF at parton level

$$f_{q/q}(y) = \langle q(p)|\bar{q}_n W_n\frac{\overline{\not n}}{2}\delta(y\overline{n}\cdot p-\overline{n}\cdot\mathcal{P})W_n^\dagger q_n|q(p)\rangle$$

- LO result

$$
\begin{aligned}
f_{q/q}^{(0)}(y) &= \frac{1}{2}\sum_s \bar{u}_s(p)\frac{\overline{\not n}}{2}\delta(y\overline{n}\cdot p-\overline{n}\cdot p)u_s(p) = \frac{1}{2}\frac{1}{\overline{n}\cdot p}\delta(1-y)\mathrm{Tr}\not p\frac{\overline{\not n}}{2} \\
&= \delta(1-y)
\end{aligned}
$$

One loop computation of PDF



- Rapidity divergence

$\overline{n} \cdot k \to \infty$:When Wilson line become divergent

- In order to regularize the rapidity divergences

$$W_n \to W_n(\Delta) = 1 - \frac{g}{\overline{n} \cdot \mathcal{P} + \Delta + i\epsilon} \overline{n} \cdot A_n + \cdots ,$$

$$W_n^\dagger \to W_n^\dagger(\Delta) = 1 - g\overline{n} \cdot A_n \frac{1}{\overline{n} \cdot \mathcal{P}^\dagger - \Delta - i\epsilon} = 1 + \frac{g}{\overline{n} \cdot \mathcal{P} + \Delta + i\epsilon} \overline{n} \cdot A_n + \cdots .$$

$$W_n(\Delta) W_n^\dagger(\Delta) = W_n^\dagger(\Delta) W_n(\Delta) = 1$$

- Virtual Corrections

$$\tilde{M}_a = \mu_{\overline{MS}}^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} \langle \bar{q}_n \frac{\not{\overline{n}}}{2} \delta(x\overline{n}\cdot p - \overline{n}\cdot p)(-g\frac{\overline{n}^\mu T^a}{\overline{n}\cdot k - \Delta}) \frac{i(\not{k} + \not{p})}{(k+p)^2} (+igT^a\gamma_\mu) q_n \rangle \frac{-i}{k^2}$$

$$= -2ig^2 C_F \mu_{\overline{MS}}^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} \frac{\overline{n}\cdot(k+p)}{k^2(k+p)^2(\overline{n}\cdot k - \Delta)} \delta(1-x)$$

- Feynman Integral **Homework**

$$\frac{1}{k^2(k+p)^2(\overline{n}\cdot k-\Delta)} = 2\Gamma(3)\int_0^1 dz \int_0^\infty dt \frac{1}{(k^2+2zk\cdot p+2t\overline{n}\cdot k - 2t\Delta)^3}$$

$$= 4\int_0^1 dz \int_0^\infty dt \, \frac{1}{(l^2-\Lambda^2)^3}.$$

$$l = k + zp + t\overline{n}, \;\; \Lambda = 2(z\overline{n}\cdot p + \Delta)t$$

$$\tilde{M}_a = -8ig^2 C_F \mu_{\overline{MS}}^{2\varepsilon} \delta(1-x) \int_0^1 dz \int_0^\infty dt \int \frac{d^D l}{(2\pi)^D} \frac{(1-z)\overline{n}\cdot p}{(l^2-\Lambda^2)^3}$$

$$= -\frac{\alpha_s C_F}{\pi}(\mu^2 e^\gamma)^\varepsilon \overline{n}\cdot p \Gamma(1+\varepsilon)\delta(1-x)\int_0^1 dz(1-z)(2z\overline{n}\cdot p + 2\Delta)^{-1-\varepsilon}\int_0^\infty dt\, t^{-1-\varepsilon}$$

- Dimensionless Integral

$$\mu^{\varepsilon}\int_0^{\infty} dt\, t^{-1-\varepsilon} = \int_0^{\infty} dt'\, t'^{-1-\varepsilon} = \frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}$$

$$\int_0^1 dz \frac{1-z}{2z\overline{n}\cdot p + 2\Delta} = -\frac{1}{2\overline{n}\cdot p}\left(1+\ln\frac{\Delta}{\overline{n}\cdot p}\right)$$

- Naive collinear computation at one loop

$$\tilde{M}_a = \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\left(1+\ln\frac{\Delta}{\overline{n}\cdot p}\right)$$

- **The zero-bin subtraction**
  - In computing collinear one loop diagram, we have the soft limit

$$\overline{n}\cdot k \to \mathcal{O}(Q\lambda) \ll Q$$

  - Consistent computation for collinear interactions to avoid double counting

$$\int d^D k\; M(k\sim col) - \int d^D k\; M(k\sim soft)$$

The zero-bin subtraction

- The zero-bin subtraction for virtual contribution

$$\tilde{M}_a = -2ig^2 C_F \mu^{2\varepsilon}_{\overline{MS}} \int \frac{d^D k}{(2\pi)^D} \frac{\overline{n}\cdot(k+p)}{k^2(k+p)^2(\overline{n}\cdot k - \Delta)} \delta(1-x) \qquad (\overline{n}\cdot k, k_\perp, n\cdot k) \sim (\lambda^0, \lambda, \lambda^2)$$

$$\to M_a^0 = -2ig^2 C_F \mu^{2\varepsilon}_{\overline{MS}} \int \frac{d^D k}{(2\pi)^D} \frac{\overline{n}\cdot p}{k^2(\overline{n}\cdot p n\cdot k)(\overline{n}\cdot k - \Delta)} \delta(1-x) \quad (\overline{n}\cdot k, k_\perp, n\cdot k) \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$\begin{aligned} M_a^0 &= -2ig^2 C_F \mu^{2\varepsilon}_{\overline{MS}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 n\cdot k(\overline{n}\cdot k - \Delta)} \delta(1-x) \\ &= -4ig^2 C_F \mu^{2\varepsilon}_{\overline{MS}} \delta(1-x) \underline{\int_0^\infty du \int_0^\infty dt \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2-\Lambda^2)^3}} \\ &= -\frac{\alpha_s C_F}{2\pi} (\mu^2 e^\gamma)^\varepsilon \Gamma(1+\varepsilon) \delta(1-x) \int_0^\infty du (u+\Delta)^{-1-\varepsilon} \int_0^\infty dt\, t^{-1-\varepsilon} \end{aligned}$$

- Feynman Integral **Homework**

$$\begin{aligned} \frac{1}{k^2 n\cdot k(\overline{n}\cdot k - \Delta)} &= 4\Gamma(3) \int_0^\infty du' \int_0^\infty dt' \frac{1}{(k^2 + 2u' n\cdot k + 2t' \overline{n}\cdot k - 2t'\Delta)^3} \\ &= 2 \int_0^\infty du \int_0^\infty dt \frac{1}{(l^2-\Lambda^2)^3}, \quad 2u' \equiv u, \quad 2t' \equiv t \end{aligned}$$

$$l = k + (un + t\overline{n})/2, \quad \Lambda = (u+\Delta)t$$

- Dimensionless Integral

$$\mu^{\varepsilon}\int_0^{\infty} du(u+\Delta)^{-1-\varepsilon} = \int_0^{\infty} d\bar{u}(\bar{u}+\frac{\Delta}{\mu})^{-1-\varepsilon} = \frac{1}{\varepsilon_{\text{UV}}} - \ln\frac{\Delta}{\mu}$$

- Zero-bin contribution

$$M_a^0 = -\frac{\alpha_s C_F}{2\pi}\Big(\frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}}\Big)\Big(\frac{1}{\varepsilon_{\text{UV}}} - \ln\frac{\Delta}{\mu}\Big)\delta(1-x)$$

- Final one loop result of virtual contribution

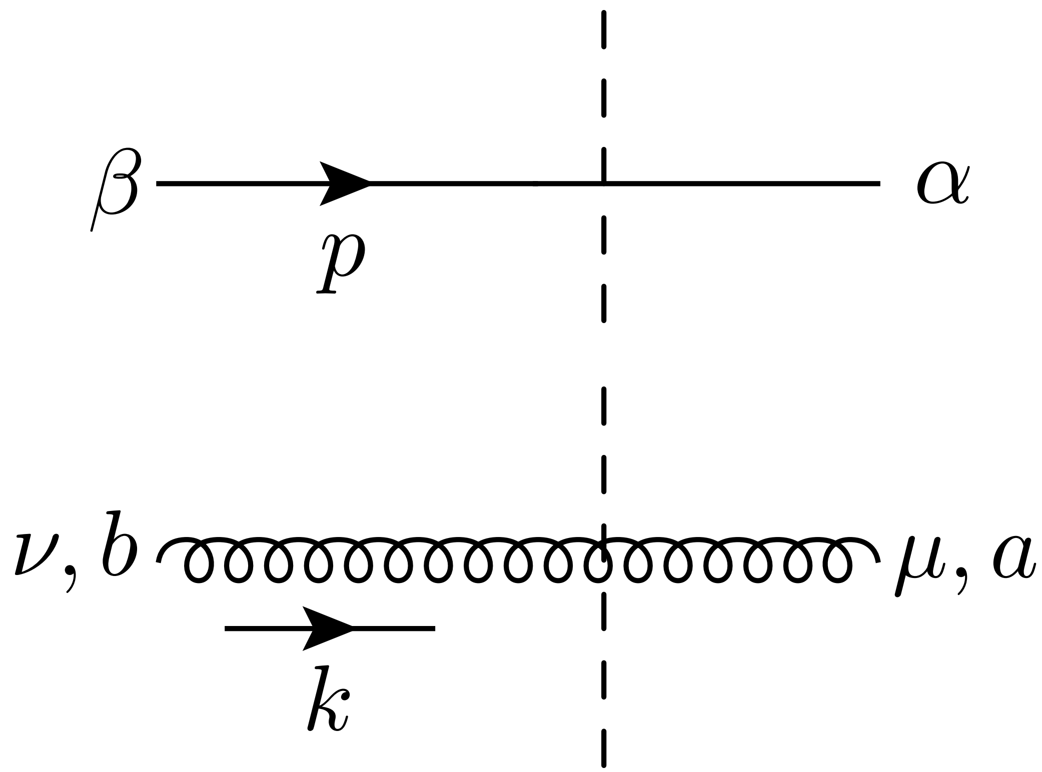
$$M_a = \tilde{M}_a - M_a^0 = \frac{\alpha_s C_F}{2\pi}\Big(\frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}}\Big)\Big(\frac{1}{\varepsilon_{\text{UV}}} + \ln\frac{\mu}{\overline{n}\cdot p} + 1\Big)\delta(1-x)$$

  - Rapidity divergence is gone
  - Very uneasy UV & IR mixed terms : cancelled when combined with the real gluon emission

Real gluon emission

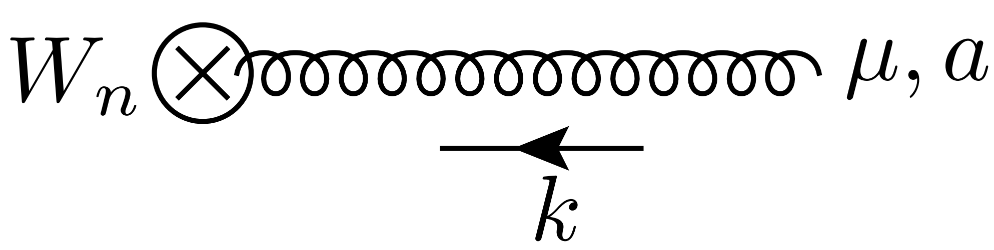
- Some Feynman rules

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_{pol} \epsilon^{*a}_{\mu}(k)\epsilon^{b}_{\nu}(k) = \int \frac{d^{D}k}{(2\pi)^{D}}(2\pi\delta(k^2))(-g_{\mu\nu}\delta^{ab})$$

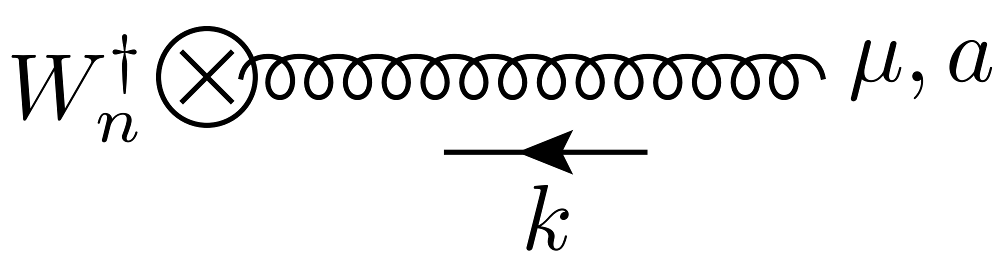


$$= 2\pi\delta(p^2)\not{p}\,\delta_{\alpha\beta}$$

$$= -2\pi\delta(k^2)g^{\mu\nu}\delta^{ab}$$



$$= -\frac{g\overline{n}^{\mu}T^{a}}{\overline{n}\cdot k(+\Delta)+i\epsilon}$$



$$= +\frac{g\overline{n}^{\mu}T^{a}}{\overline{n}\cdot k(+\Delta)+i\epsilon}$$

- Diagram (b)

$$\tilde{M}_b = -4\pi g^2 C_F \mu_{\overline{MS}}^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} \frac{\delta(k^2)\delta(x\overline{n}\cdot p - \overline{n}\cdot(p-k))\overline{n}\cdot(p-k)\overline{n}\cdot p}{(-2p\cdot k)(\overline{n}\cdot k+\Delta)}$$

- Decomposition of the loop integral

$$\int \frac{d^D k}{(2\pi)^D} = \frac{(4\pi)^\varepsilon}{32\pi^3}\frac{1}{\Gamma(1-\varepsilon)} \int d\overline{n}\cdot k \, dn\cdot k \, d\mathbf{k}_\perp^2 \, (\mathbf{k}_\perp^2)^{-\varepsilon}$$

- Integration of component by component

$$\begin{aligned}\tilde{M}_b &= \frac{\alpha_s C_F}{2\pi}\frac{(\mu^2 e^\gamma)^\varepsilon}{\Gamma(1-\varepsilon)} \frac{x((1-x)\overline{n}\cdot p)^{-\varepsilon}}{1-x+\frac{\Delta}{\overline{n}\cdot p}} \int dn\cdot k (n\cdot k)^{-1-\varepsilon} \\ &= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{x}{1-x+\frac{\Delta}{\overline{n}\cdot p}}\end{aligned}$$

$$\frac{x}{1-x+\frac{\Delta}{\overline{n}\cdot p}} = -\ln\frac{\Delta}{\overline{n}\cdot p}\delta(1-x) + \frac{x}{(1-x)_+}$$

- Plus function

Plus function can be defined as

$$\Big(f(x)\Big)_+ = \lim_{\beta\to 0}\Big[f(x)\theta(1-x-\beta) - \delta(1-x-\beta)\int_0^{1-\beta} dy f(y)\Big], \quad (1)$$

and it has the property

$$\int_0^1 dx \Big(f(x)\Big)_+ = 0. \quad (2)$$

So it can be used as

$$\int_0^1 dx \Big(g(x)\Big)_+ f(x) = \int_0^1 dx\, g(x)\Big[f(x) - f(1)\Big], \quad (3)$$

$$\int_y^1 dx \Big(g(x)\Big)_+ f(x) = \int_y^1 dx\, g(x)\Big[f(x) - f(1)\Big] - f(1)\int_0^y dx g(x). \quad (4)$$

$$\frac{1}{(1-z)^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(1-z) + \frac{1}{(1-z)_+} - \epsilon\Big(\frac{\ln(1-z)}{1-z}\Big)_+ + \mathcal{O}(\epsilon^2), \quad (5)$$

$$\frac{1}{(1-z)^{2+\epsilon}} = \frac{1}{\epsilon}\delta(1-z)\frac{\partial}{\partial z} - (1-\epsilon)\delta(1-z) + \frac{1}{[(1-z)^2]_\Delta} - \epsilon\Big(\frac{\ln(1-z)}{(1-z)^2}\Big)_\Delta + \mathcal{O}(\epsilon^2), \quad (6)$$

where Δ distribution has been defined as

$$\int_0^1 dz \Big[h(z)\Big]_\Delta f(z) = \int_0^1 dz h(z)\Big[f(z) - f(1) + (1-z)f'(1)\Big]. \quad (7)$$

- Naive one loop result

$$\tilde{M}_b = \frac{\alpha_s C_F}{2\pi}\Big(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\Big)\Big(-\ln\frac{\Delta}{\bar{n}\cdot p}\delta(1-x) + \frac{x}{(1-x)_+}\Big)$$

- Zero-bin subtraction

$$M_b^0 = -4\pi g^2 C_F \mu_{\overline{MS}}^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} \frac{\delta(k^2)\delta(x\overline{n}\cdot p - \overline{n}\cdot p)(\overline{n}\cdot p)^2}{(-\overline{n}\cdot p n\cdot k)(\overline{n}\cdot k + \Delta)}$$

$$\begin{aligned} M_b^0 &= \frac{\alpha_s C_F}{2\pi} \frac{e^{\gamma\varepsilon}}{\Gamma(1-\varepsilon)} \delta(1-x) \left[\mu^\varepsilon \int_0^\infty d\overline{n}\cdot k \frac{(\overline{n}\cdot k)^{-\varepsilon}}{\overline{n}\cdot k + \Delta}\right] \left[\mu^\varepsilon \int_0^\infty dn\cdot k (n\cdot k)^{-1-\varepsilon}\right] \\ &= \frac{\alpha_s C_F}{2\pi} \frac{e^{\gamma\varepsilon}}{\Gamma(1-\varepsilon)} \delta(1-x) \left(\frac{\Delta}{\mu}\right)^{-\varepsilon} \Gamma(\varepsilon)\Gamma(1-\varepsilon) \left(\frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}}\right) \\ &= \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\varepsilon_{\mathrm{UV}}} - \ln\frac{\Delta}{\mu}\right) \left(\frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}}\right) \delta(1-x) \end{aligned}$$

- Final result of diagram (b)

$$M_b = \tilde{M}_b - M_b^0 = \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}}\right) \left[\left(-\frac{1}{\varepsilon_{\mathrm{UV}}} - \ln\frac{\mu}{\overline{n}\cdot p}\right)\delta(1-x) + \frac{x}{(1-x)_+}\right]$$

  - Once again, rapidity divergence is gone
  - UV & IR mixed term can cancel if combined with diagram (a)

- Sum of the zero-bin contribution

$$M^0 = 2(M_a^0 + M_b^0) = 0$$

- For PDF computation, naive collinear computation can give correct result
- However, in generic cases, the zero-bin subtraction has a crucial role
- It simply represents that the decouple soft interactions have been cancelled

- Diagram (c)

$$\begin{aligned} M_c = \tilde{M}_c &= -2\pi g^2 C_F \mu_{\overline{MS}}^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} \frac{\delta(k^2)\delta(x\overline{n}\cdot p - \overline{n}\cdot(p-k))}{(-2p\cdot k)^2} \langle \bar{q}_n \gamma^\mu (\not{p} - \not{k}) \frac{\not{\overline{n}}}{2} (\not{p} - \not{k}) \gamma_\mu q_n \rangle \\ &= \alpha_s C_F \mu_{\overline{MS}}^{2\varepsilon} (D-2)(1-x) \int \frac{d^{D-2}\mathbf{k}_\perp}{(2\pi)^{D-2}} \frac{1}{\mathbf{k}_\perp^2} \\ &= \frac{\alpha_s C_F}{2\pi} \Big(\frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}} \Big)(1-x) \end{aligned}$$

$$\int \frac{d^{D-2}\mathbf{k}_\perp}{(2\pi)^{D-2}} \frac{1}{\mathbf{k}_\perp^2} = \frac{(4\pi)^\varepsilon}{4\pi} \Big(\frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}} \Big)$$

- No double pole appears

- Final result at one loop

$$
\begin{aligned}
f_{q/q}^{(1)}(x,\mu) &= 2\mathrm{Re}(M_a + M_b) + M_c + (Z_q^{(1)} + R_q^{(1)})\delta(1-x) \\
&= \frac{\alpha_s C_F}{2\pi}\Big(\frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}}\Big)\Big[\frac{3}{2}\delta(1-x) + \frac{1+x^2}{(1-x)_+}\Big]
\end{aligned}
$$

- Quark field strength renormalization (Minimal Subtraction)

$$
Z_q^{(1)} = -\frac{\alpha_s C_F}{4\pi}\frac{1}{\varepsilon_{\mathrm{UV}}}
$$

- Residue : depends on renormalization scheme

$$
R_q^{(1)} = \frac{\alpha_s C_F}{4\pi}\frac{1}{\varepsilon_{\mathrm{IR}}}
$$

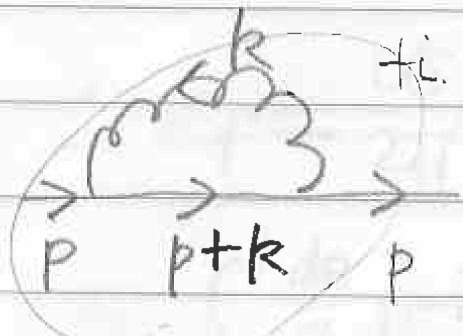
Homework

Compute the residue of the heavy quark

• Computation of residue

$$Z_q = -\frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon}. \qquad ※ \ \langle S \rangle = (Z^{\frac{1}{2}})^n (R^{\frac{1}{2}})^n \cdot \mathrm{AmpG}.$$

※ Residue : $R_q = +\frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon_{IR}}$



$$+ i = \Sigma(\not p)$$

$$= i \mu_{\overline{MS}}^{2\varepsilon} \int \frac{d^D k}{(2\pi)^D} (i g T^a \gamma^\mu) \frac{i(\not p + \not k)}{(p+k)^2} (+ i g \gamma_\mu T^a) \frac{-i}{k^2}$$

$$= -i g^2 C_F \mu_{\overline{MS}}^{2\varepsilon} (2-D) \int \frac{d^D k}{(2\pi)^D} \frac{\not k + \not p}{k^2 (k^2 + 2k\cdot p + p^2)}$$

$$\frac{1}{k^2(k^2+2k\cdot p+p^2)} = \int_0^1 dx \frac{1}{(l^2 - \Delta)^2}, \quad l = k + xp, \ k = l - xp, \ \Delta = -x(1-x)p^2.$$

$$= -g^2 C_F \mu_{\overline{MS}}^{2\varepsilon} (D-2) \otimes \int_0^1 dx \int \frac{d^D l_E}{(2\pi)^D} \frac{1}{(l_E^2 + \Delta)^2} (1-x) \not p.$$

$$\int \frac{d^D l_E}{(2\pi)^D} \frac{1}{(l_E^2 + \Delta)^2} = \frac{(4\pi)^\varepsilon}{(4\pi)^2} \Gamma(\varepsilon) \left(\frac{1}{\Delta}\right)^{+\varepsilon}.$$

$$\int_0^1 dx (1-x)^{1-\varepsilon} x^{-\varepsilon} = \frac{\Gamma(2-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(3-2\varepsilon)}$$

$$= -\frac{\alpha_s}{4\pi} C_F \left(\frac{\mu^2 e^\gamma}{-p^2}\right)^\varepsilon \Gamma(\varepsilon) \underbrace{}_{D-2} \int_0^1 dx (1-x)^{1-\varepsilon} x^{-\varepsilon} \not p$$

$$(1 + z_q^{(1)}) \qquad R^{(1)} \not p$$

$$\Sigma = \not{p} R^{(1)} \quad \therefore \frac{1}{\not{p}-\Sigma} = \frac{1}{\not{p}(1-R^{(1)})} = \frac{1+R^{(1)}}{\not{p}} \to \frac{R}{\not{p}}, \quad \not{p}-\Sigma \Rightarrow Z_q \not{p} - \Sigma$$

$$= \not{p} - (\Sigma - Z_q^{(1)} \not{p})$$

$$\therefore R^{(1)} = -\frac{\alpha_s}{4\pi} C_F \left(-\frac{M^2}{p^2} e^{\gamma}\right)^{\varepsilon} \Gamma(\varepsilon) \frac{\Gamma(2-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(3-2\varepsilon)} \cdot 2(1-\varepsilon) + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon_{UV}}$$

$$= -\frac{\alpha_s}{2\pi} C_F \left(\frac{M^2}{-p^2} e^{\gamma}\right)^{\varepsilon} \Gamma(\varepsilon)(1-\varepsilon)^2 \frac{\Gamma^2(1-\varepsilon)}{\Gamma(3-2\varepsilon)} + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon_{UV}}$$

$$= \frac{1}{2}\left(\frac{1}{\varepsilon_{UV}} + \ln\frac{M^2}{-p^2} + 1\right) \text{ or } \frac{1}{2}\left(\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}\right)$$

$$= -\frac{\alpha_s}{4\pi} C_F \left(\ln\frac{M^2}{-p^2} + 1\right) \text{ or } + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon_{IR}}.$$

▪ RG behavior of PDF

$$f^B_{q/q}(x) = \int_x^1 \frac{dz}{z} Z_f(z,\mu) f^R_{q/q}(\frac{x}{z},\mu)$$

$$Z_f(x,\mu) = \delta(1-x) + \frac{\alpha_s C_F}{2\pi}\frac{1}{\varepsilon_{\rm UV}}\left[\frac{3}{2}\delta(1-x) + \frac{1+x^2}{(1-x)_+}\right]$$

$$\frac{d}{d\ln\mu} f^B_{q/q}(x) = \int_x^1 \frac{dz}{z}\left[\frac{d}{d\ln\mu} Z_f(z,\mu)\right] f^R_{q/q}(\frac{x}{z},\mu) + \int_x^1 \frac{dz}{z} Z_f(z,\mu)\left[\frac{d}{d\ln\mu} f^R_{q/q}(\frac{x}{z},\mu)\right] = 0$$

$$\frac{d}{d\ln\mu} f^R_{q/q}(x,\mu) = \int_x^1 \frac{dz}{z}\gamma_f(z,\mu) f^R_{q/q}(\frac{x}{z})$$

$$\frac{d}{d\ln\mu} Z_f(x,\mu) = -\int_x^1 \frac{dz}{z}\gamma_f(z,\mu) Z_f(\frac{x}{z},\mu)$$

$$\frac{d}{d\ln\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}$$

$$\beta(g) = -\varepsilon g - \beta_0 g^3/(16\pi^2) - \cdots$$

$$\gamma_f(x,\mu) = -\varepsilon g\frac{\partial}{\partial g} Z_f = \frac{\alpha_s C_F}{\pi}\left[\frac{3}{2}\delta(1-x) + \frac{1+x^2}{(1-x)_+}\right]$$

DGLAP evolution