Perturbative QCD

Part I : Introduction to QCD
Structure functions for DIS

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QCD

- QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - m)q - \frac{1}{4}G^{\mu\nu,a}G^{a}_{\mu\nu}$$

$$\begin{aligned} D_\mu &= \partial^\mu - igT^a A^a_\mu, \\ G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^a_\mu A^b_\nu \end{aligned}$$

- QCD Beta function : calculated as a negative value

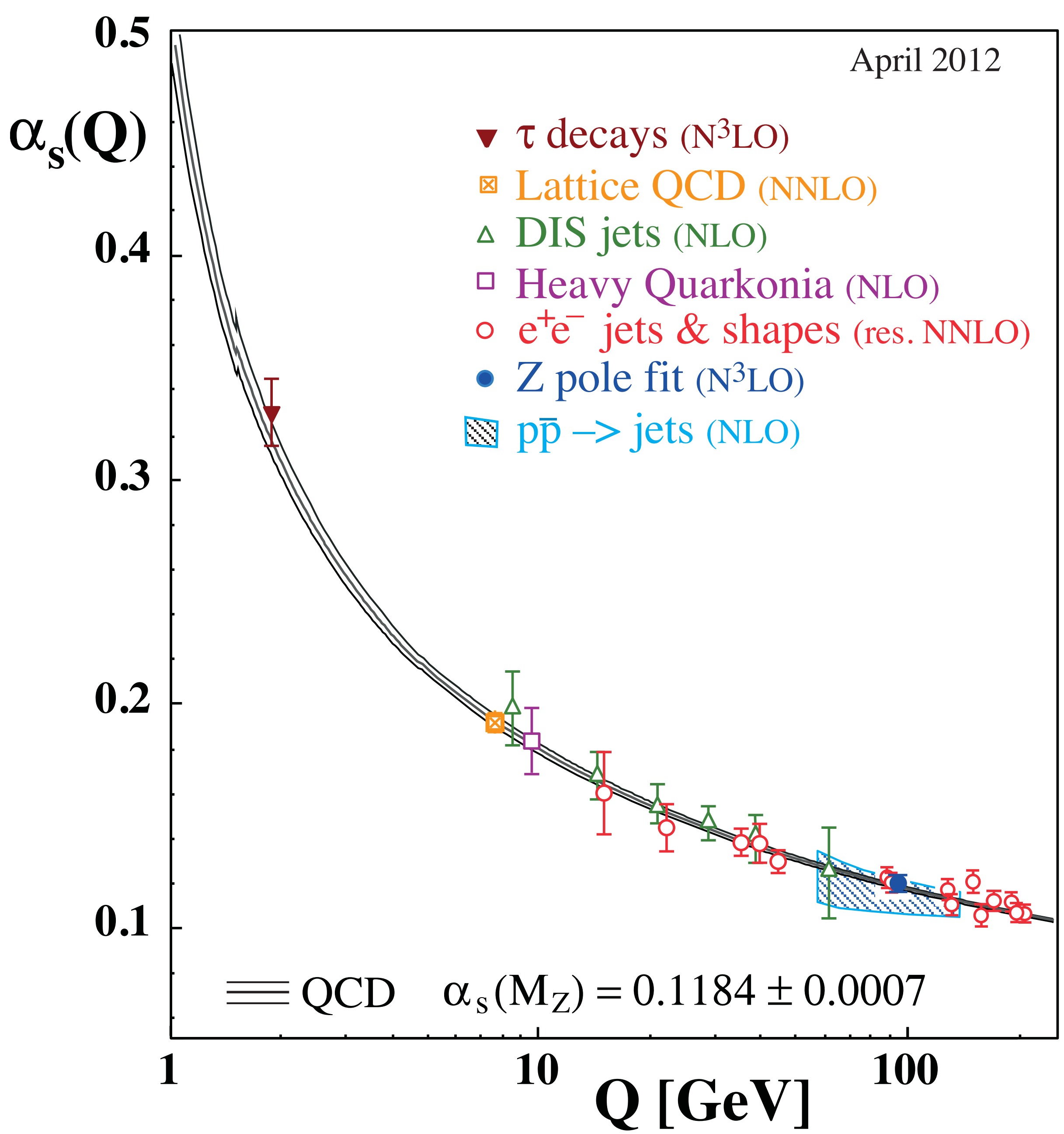
$$\frac{d}{d\ln\mu}g(\mu) = \beta(g(\mu))$$

$$\beta(g) = -g\sum_{k=0}\beta_k\left(\frac{\alpha_s}{4\pi}\right)^{k+1} = -\beta_0\frac{g^3}{16\pi^2} - \beta_1\frac{g^5}{(16\pi^2)^2} + \cdots$$

$$\beta_0 = \frac{11N_c - 2n_f}{3}, \quad \beta_1 = \frac{34}{3}N_c^2 - \frac{10}{3}N_c n_f - 2C_F n_f$$

- Coupling Constant

$$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi} = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda_{QCD}^2}}\left[1 - \frac{\beta_1 \ln\ln \frac{\mu^2}{\Lambda_{QCD}^2}}{\beta_0^2 \ln \frac{\mu^2}{\Lambda_{QCD}^2}}\right]$$



$$\alpha_s(\mu \to \Lambda_{QCD}) \to \infty$$

Confinement : long distance interactions

$$\alpha_s(\mu \to \infty) \to 0$$

Asymptotically Free : short distance interactions

- Operator Product Expansion (OPE)

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = \sum_n C_{12}^n(x,\mu)\langle \mathcal{O}_n(\mu)\rangle$$

$C_{12}^n(x,\mu)$: Complex function including Wilson coefficient
Expanded by the short distance *x*

$\langle \mathcal{O}_n(\mu)\rangle$: Includes all the information on the long distance interactions

EX) Hadronic tensor for DIS

$$W_{\mu\nu} = \frac{1}{2\pi}\int d^4 z e^{iq\cdot z}\langle N|J_\mu^\dagger(z)J_\nu(0)|N\rangle$$

$z \to 0$: Short distance expansion

▪ QCD Factorization Theorem

- Systematically separate the short and long distance interactions

EX) Factorization theorem of DIS structure function

$$F_1(x) = \int_x^1 \frac{dz}{z} H(Q^2, z, \mu) f_{q/p}(\frac{x}{z}, \mu)$$

- Describe the short distance interactions
- Corresponding to Wilson coefficient
- Can be computed by perturbation

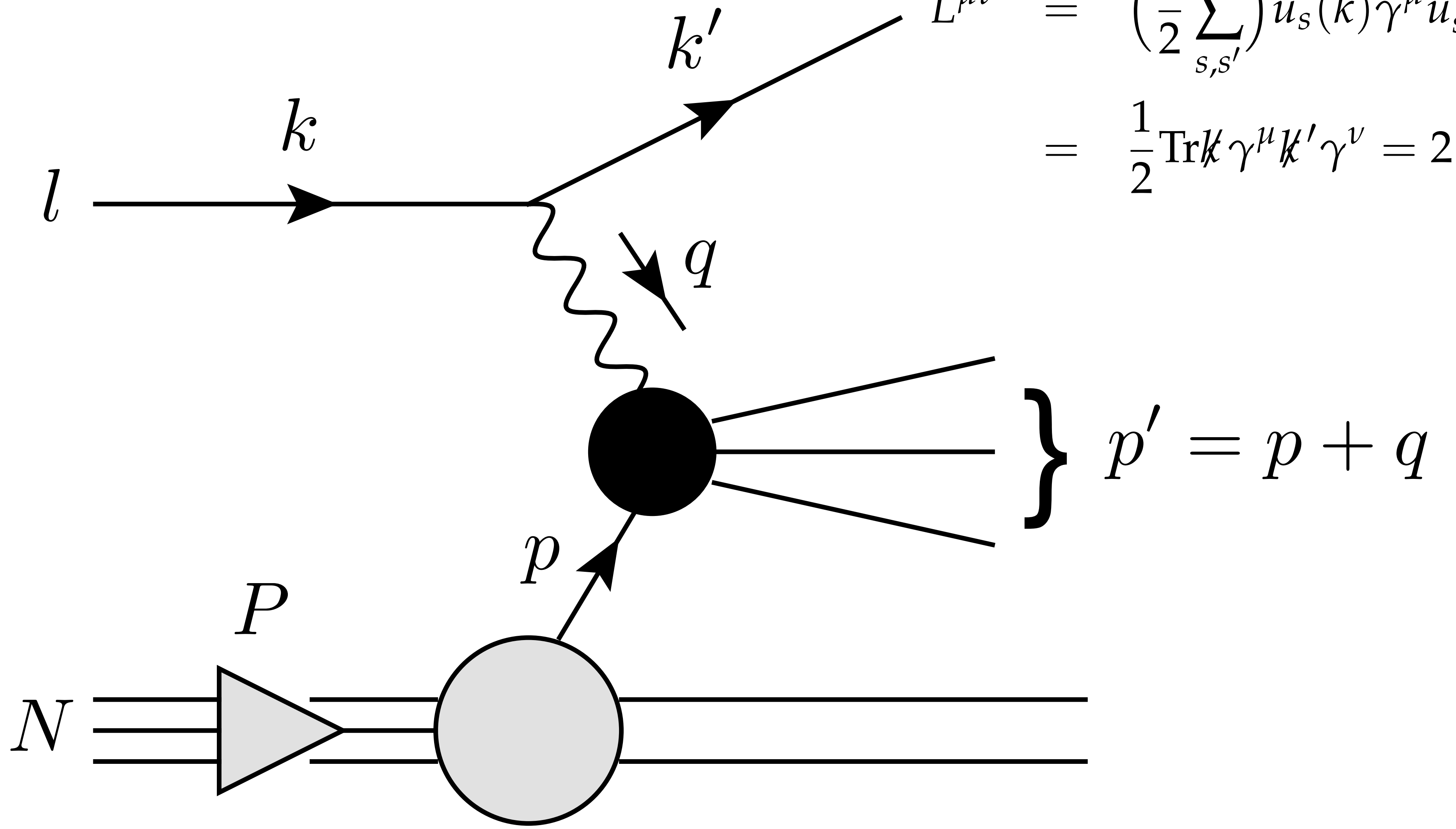
- Describe the long distance interactions
- Corresponding to the matrix element of the nonlocal operator
- Cannot be computed, instead fit to experiments

- Structure function has no renormalization scale variance

- Perturbative QCD has a predictive power

Structure Function for DIS

Deep inelastic scattering



$$
\begin{aligned}
L^{\mu\nu} &= \left(\frac{1}{2}\sum_{s,s'}\right)\bar{u}_s(k)\gamma^\mu u_{s'}(k')\,\bar{u}_{s'}(k')\gamma^\nu u_s(k) \\
&= \frac{1}{2}\mathrm{Tr}\not{k}\gamma^\mu\not{k}'\gamma^\nu = 2(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}k\cdot k')
\end{aligned}
$$

$$
\begin{aligned}
\sigma(lN \to lX) &= \frac{1}{2s}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2k'_0}\sum_X(2\pi)^4\delta(k+P-k'-p_X)\langle N|J_\mu^\dagger|X\rangle\langle X|J_\nu|N\rangle\frac{e^2Q_f^2}{Q^4}L^{\mu\nu} \\
&= \frac{\pi}{s}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2k'_0}\frac{e^2Q_f^2}{Q^4}L_{\mu\nu}(k,k')W^{\mu\nu}(q,P)
\end{aligned}
$$

▪ Hadronic tensor and the structure functions

$$\begin{aligned} W_{\mu\nu}(q,P) &= \frac{1}{2\pi}\sum_X (2\pi)^4 \delta(q+P-p_X)\langle N|J_\mu^\dagger|X\rangle\langle X|J_\nu|N\rangle, \\ &= \frac{1}{2\pi}\int d^4z e^{iq\cdot z}\langle N|J_\mu^\dagger(z)J_\nu(0)|N\rangle \\ &= (-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2})F_1 + (P_\mu - q_\mu\frac{P\cdot q}{q^2})(P_\nu - q_\nu\frac{P\cdot q}{q^2})F_2 \end{aligned}$$

$$q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$$

- **Breit frame :** $q^\mu = Q(-n^\mu + \overline{n}^\mu)/2 \quad n^2 = \overline{n}^2 = 0,\ n\cdot\overline{n} = 2$

$$n^\mu = (1,0,0,1),\ \overline{n}^\mu = (1,0,0,-1)$$

- Incoming hadron: $P^\mu = \overline{n}\cdot P\frac{n^\mu}{2} + \frac{m^2}{\overline{n}\cdot P}\frac{\overline{n}^\mu}{2}$

- n-collinear momentum

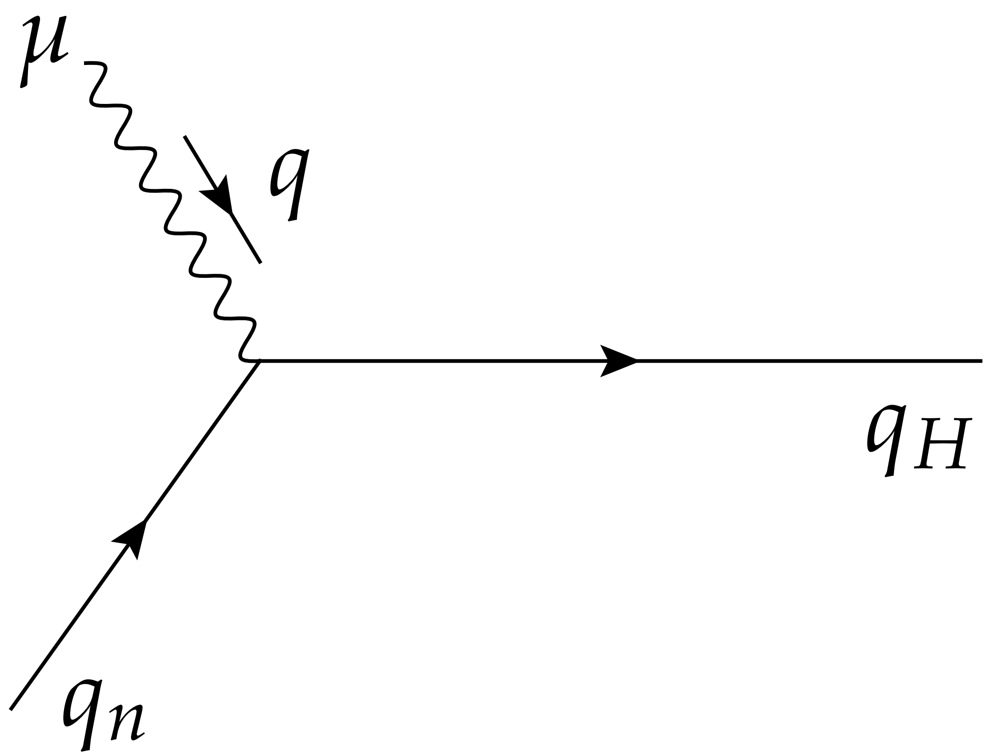
$$p^\mu = \overline{n}\cdot p\frac{n^\mu}{2} + p_\perp^\mu + n\cdot p\frac{\overline{n}^\mu}{2} = \mathcal{O}(Q) + \mathcal{O}(Q\lambda) + \mathcal{O}(Q\lambda^2)$$

- Bjorken variable: $x = \frac{Q^2}{2P\cdot q} \sim \frac{Q}{\overline{n}\cdot P}$

- Final state momentum: $p_X^2 = (P+q)^2 = m^2 + 2P\cdot q - Q^2 \sim Q^2\frac{1-x}{x}$

$$0 \le x \le 1$$

- Electromagnetic current



$$J^\mu = \bar{q}\gamma^\mu q \to \bar{q}_H \gamma_\perp^\mu q_n$$

$$\not{n}\, q_n = \mathcal{O}(\lambda)$$

- Hadronic tensor in the Breit frame

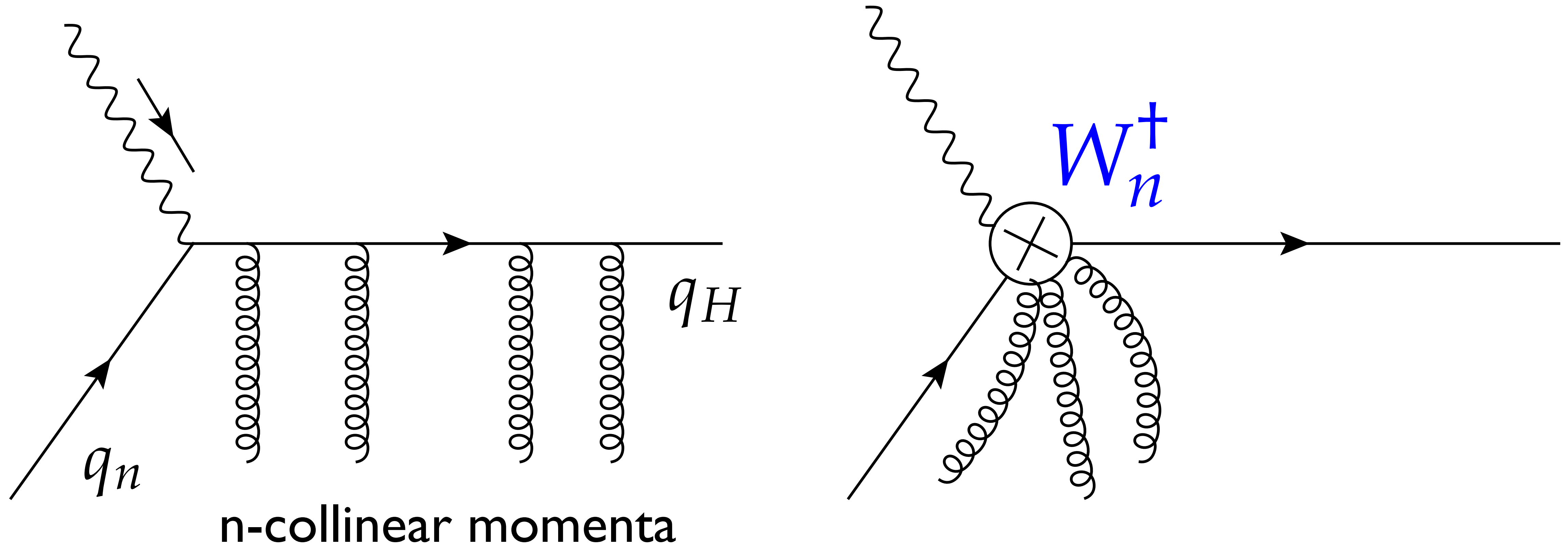
$$W^{\mu\nu} = -g_\perp^{\mu\nu} F_1 + \left(\frac{n^\mu}{2} + \frac{\overline{n}^\mu}{2}\right)\left(\frac{n^\nu}{2} + \frac{\overline{n}^\nu}{2}\right) F_L$$

Suppressed part

- Callan-Gross relation

$$F_L = F_2 \frac{Q^2}{4x} - F_1 \to 0$$ (At leading twist)

Wilson lines



- Effective theory description

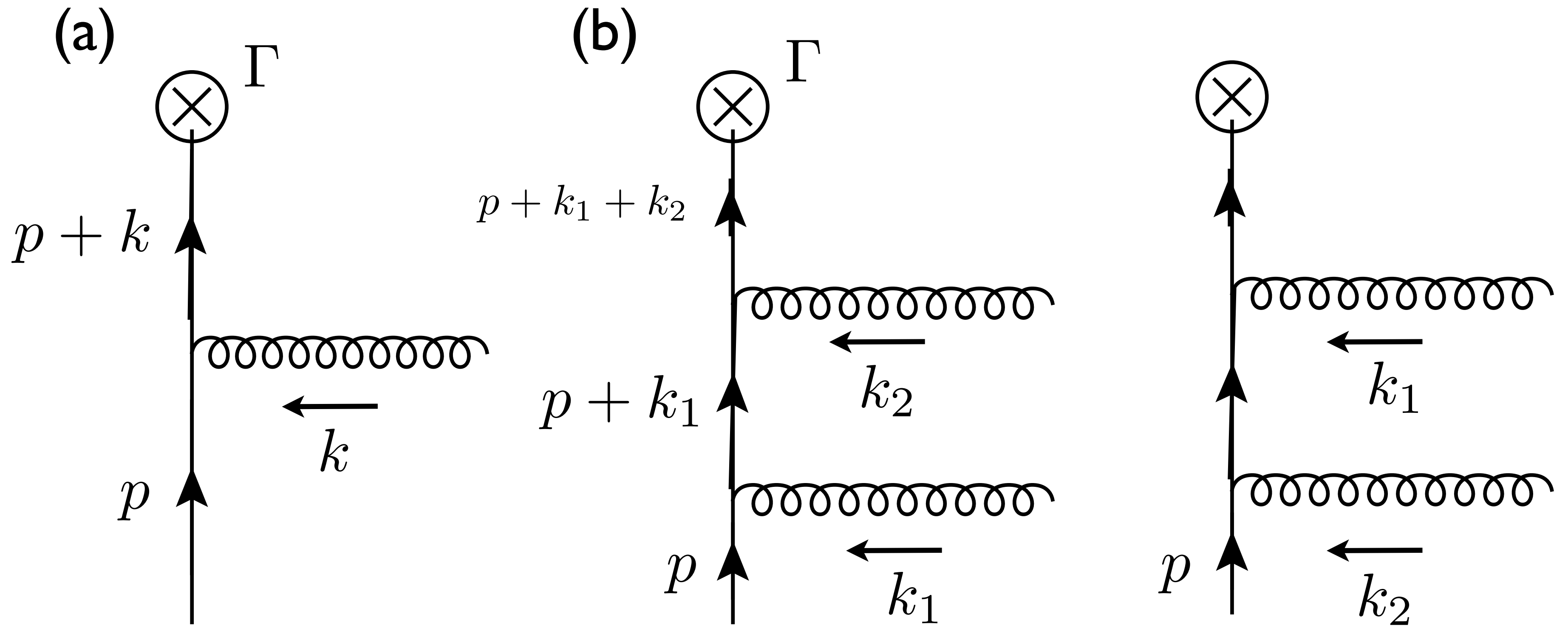
$$J^\mu = \bar{q}\gamma^\mu q \to \bar{q}_H \gamma_\perp^\mu q_n \to \bar{q}_H \gamma_\perp^\mu W_n^\dagger q_n$$

Gauge-invariant combination

$$W_n(x) = \mathrm{P}\exp\left[ig\int_{-\infty}^{x} ds\,\bar{n}\cdot A_n(s\bar{n}^\mu)\right]$$

▪ Derivation of the collinear Wilson lines

When n-collinear gluons radiate from quark other than n-collinear



$$M_a = -g\Gamma\frac{(\not p + \not k)\not A}{(p+k)^2}q = -g\Gamma\frac{2p\cdot A}{2p\cdot k}q = -g\Gamma\frac{\overline{n}\cdot A n\cdot p}{\overline{n}\cdot k n\cdot p}q = \Gamma(-g\frac{\overline{n}\cdot A}{\overline{n}\cdot k})q$$

$$M_{b1} \quad = \quad \Gamma\Big(\frac{g^2\overline{n}\cdot A(k_2)\overline{n}\cdot A(k_1)}{\overline{n}\cdot k_1\overline{n}\cdot(k_1+k_2)}\Big)q$$

$$M_{b2} \quad = \quad \Gamma\Big(\frac{g^2\overline{n}\cdot A(k_1)\overline{n}\cdot A(k_2)}{\overline{n}\cdot k_2\overline{n}\cdot(k_1+k_2)}\Big)q$$

$$A^\mu = \overline{n}\cdot A\frac{n^\mu}{2} + A_\perp^\mu + n\cdot A\frac{\overline{n}^\mu}{2} = \mathcal{O}(Q) + \mathcal{O}(Q\lambda) + \mathcal{O}(Q\lambda^2)$$

$$W_n = 1 - g\frac{1}{\overline{n}\cdot\mathcal{P}+i\epsilon}\overline{n}\cdot A_n + g^2\frac{1}{\overline{n}\cdot\mathcal{P}+i\epsilon}\overline{n}\cdot A_n\frac{1}{\overline{n}\cdot\mathcal{P}+i\epsilon}\overline{n}\cdot A_n + \cdots$$

- Expression in the coordinate space $\overline{n}\cdot A_n(\overline{n}\cdot q)=\int_{-\infty}^{\infty}d\bar{z}e^{i\overline{n}\cdot q\bar{z}}\overline{n}\cdot A_n(\bar{z})$

$$\begin{aligned}-g\frac{1}{\overline{n}\cdot\mathcal{P}+i\epsilon}\overline{n}\cdot A_n(\bar{x}) &= \frac{-g}{2\pi}\int_{-\infty}^{\infty}d\overline{n}\cdot q\frac{e^{-i\overline{n}\cdot q\bar{x}}}{\overline{n}\cdot q+i\epsilon}\overline{n}\cdot A_n(\overline{n}\cdot q)\\ &= \frac{-g}{2\pi}\int_{-\infty}^{\infty}d\bar{z}\int_{-\infty}^{\infty}d\overline{n}\cdot q\frac{e^{-i\overline{n}\cdot q(\bar{x}-\bar{z})}}{\overline{n}\cdot q+i\epsilon}\overline{n}\cdot A_n(\bar{z})\\ &= ig\int_{-\infty}^{\bar{x}}d\bar{z}\overline{n}\cdot A_n(\bar{z})\end{aligned}$$

$$-2\pi i\,\Theta(\bar{x}-\bar{z})$$

Homework: show it.

$$g^2\frac{1}{\overline{n}\cdot\mathcal{P}+i\epsilon}\overline{n}\cdot A_n(\bar{x})\frac{1}{\overline{n}\cdot\mathcal{P}+i\epsilon}\overline{n}\cdot A_n(\bar{x})=\frac{(-ig)^2}{2!}\mathrm{P}\int_{-\infty}^{\bar{x}}d\bar{z}\overline{n}\cdot A_n(\bar{z})\int_{-\infty}^{\bar{x}}d\bar{y}\overline{n}\cdot A_n(\bar{y})$$

$$W_n(\bar{x})=\mathrm{P}\exp\left[ig\int_{-\infty}^{\bar{x}}d\bar{z}\overline{n}\cdot A_n(\bar{z})\right]$$

$$W_n^\dagger(\bar{x})=\bar{\mathrm{P}}\exp\left[-ig\int_{-\infty}^{\bar{x}}d\bar{z}\overline{n}\cdot A_n(\bar{z})\right]$$

$$W_nW_n^\dagger=W_n^\dagger W_n=1$$

Factorization of the structure Fn.

$$F_1(x,,Q^2) = -\frac{g_\perp^{\mu\nu}}{2}W_{\mu\nu} = -4\pi^3 \sum_X \delta(q+P-p_X)\langle N|J^\dagger_{\perp\mu}|X\rangle\langle X|J^\mu_\perp|N\rangle$$

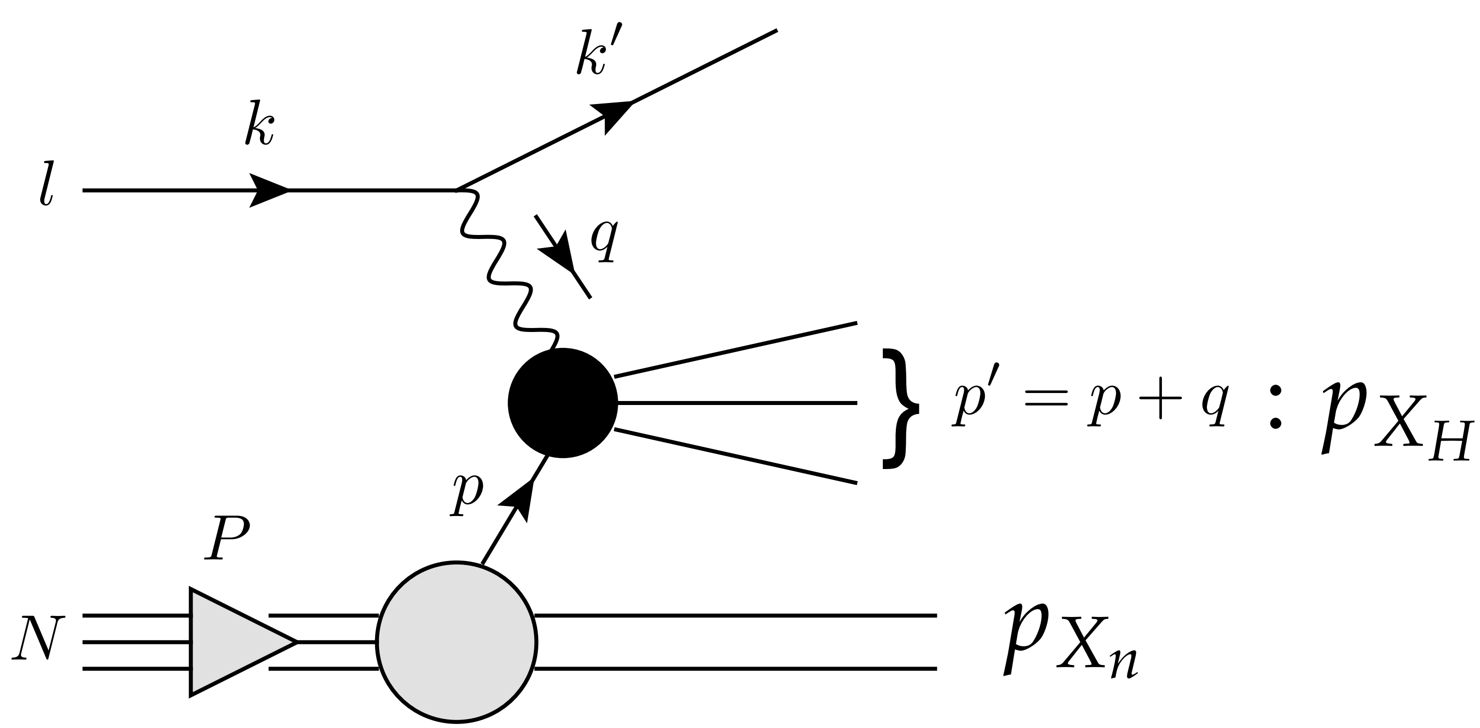
$$= -4\pi^3 \int_x^1 dy \sum_X \delta(q+P-p_X)\langle N|\bar{q}_n W_n \gamma_\perp^\mu q_H|X\rangle\langle X|\bar{q}_H \gamma^\perp_\mu \delta(y - \frac{\overline{n}\cdot\mathcal{P}}{\overline{n}\cdot P}) W_n^\dagger q_n|N\rangle$$

- Separation of the final state

$$\sum_X = \sum_{X_H}\sum_{X_n}, \quad p_X = p_{X_H} + p_{X_n}, \quad |X\rangle = |X_H\rangle|X_n\rangle$$

$$P = p + p_{X_n}, \quad q + P - p_X = q + p - p_{X_H}$$

incoming parton's momentum



▪ Parton Distribution Function (PDF)

$$f_{q/N}(y,\mu) = \frac{1}{\overline{n}\cdot P}\sum_{X_n}\langle N(P)|\bar{q}_n W_n \frac{\overline{\not{n}}}{2}|X_n\rangle\langle X_n|\delta(y-\frac{\overline{n}\cdot\mathcal{P}}{\overline{n}\cdot P})W_n^\dagger q_n|N(P)\rangle$$

$$= \langle N(P)|\bar{q}_n W_n \frac{\overline{\not{n}}}{2}\delta(y\overline{n}\cdot P-\overline{n}\cdot\mathcal{P})W_n^\dagger q_n|N(P)\rangle$$

$$\sum_{X_n}|X_n\rangle\langle X_n| = 1$$

$$= \int\frac{dn\cdot z}{4\pi}e^{-ix\overline{n}\cdot Pn\cdot z/2}\langle N(P)|\bar{q}_n(\frac{n\cdot z}{2})\left[\frac{n\cdot z}{2},0\right]q_n(0)|N(P)\rangle$$

gauge invariant

$$[\bar{z},0] = W_n(\bar{z})W_n^\dagger = \mathrm{P}\exp\left[ig\int_{-\infty}^{\bar{z}}d\bar{z}'\overline{n}\cdot A_n(\bar{z}')\right]\bar{\mathrm{P}}\exp\left[-ig\int_{-\infty}^{0}d\bar{z}'\overline{n}\cdot A_n(\bar{z}')\right]$$

$$= \mathrm{P}\exp\left[ig\int_0^{\bar{z}}d\bar{z}'\overline{n}\cdot A_n(\bar{z}')\right]$$

- PDF at parton level

$$f_{q/q}(y) = \langle q(p)|\bar{q}_n W_n\frac{\overline{\not{n}}}{2}\delta(y\overline{n}\cdot p-\overline{n}\cdot\mathcal{P})W_n^\dagger q_n|q(p)\rangle$$

- LO result

$$f_{q/q}^{(0)}(y) = \frac{1}{2}\sum_s \bar{u}_s(p)\frac{\overline{\not{n}}}{2}\delta(y\overline{n}\cdot p-\overline{n}\cdot p)u_s(p) = \frac{1}{2}\frac{1}{\overline{n}\cdot p}\delta(1-y)\mathrm{Tr}\not{p}\frac{\overline{\not{n}}}{2}$$

$$= \delta(1-y)$$

- PDF projection

$$\langle N|\Big[\delta(y-\frac{\overline{n}\cdot\mathcal{P}}{\overline{n}\cdot P})\Psi_n\Big]^\alpha_a\Big(\bar{\Psi}_n\Big)^\beta_b|N\rangle=\frac{\overline{n}\cdot P}{2N_c}\delta^{\alpha\beta}\Big(\frac{\not{n}}{2}\Big)_{ab}f_{q/N}(x),\ \Psi_n=W_n^\dagger q_n$$

Factorization theorem

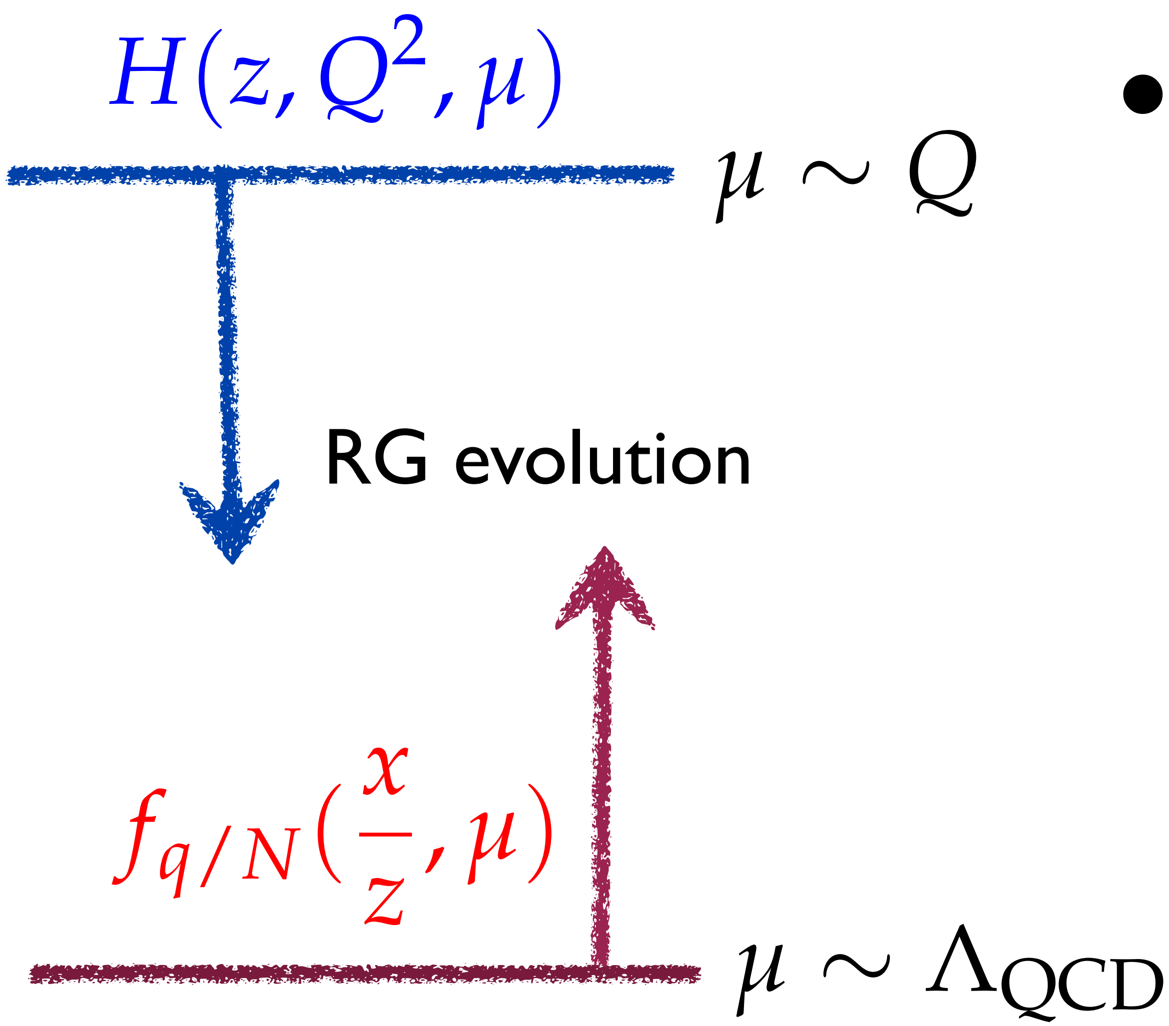
$$\begin{aligned}F_1(x,Q^2) &= -4\pi^3\frac{\overline{n}\cdot P}{2}\int_x^1 dy f_{q/N}(y)\sum_{X_H}\delta(q+p-p_{X_H})\\ &\quad\times\frac{1}{N_c}\langle 0|\Big(\frac{\not{n}}{2}\gamma_\perp^\mu q_H\Big)^\alpha_a|X_H\rangle\langle X_H|\Big(\bar{q}_H\gamma_\mu^\perp\Big)^\alpha_a|0\rangle\\ &= \int_x^1\frac{dy}{y}f_{q/N}(y,\mu)H(\frac{x}{y},Q^2,\mu)=\int_x^1\frac{dz}{z}H(z,Q^2,\mu)f_{q/N}(\frac{x}{z},\mu)\end{aligned}$$

$$H(\frac{x}{y},\mu)=-2\pi^3\overline{n}\cdot p\sum_{X_H}\delta(q+p-p_{X_H})\frac{1}{N_c}\langle 0|\Big(\frac{\not{n}}{2}\gamma_\perp^\mu q_H\Big)^\alpha_a|X_H\rangle\langle X_H|\Big(\bar{q}_H\gamma_\mu^\perp\Big)^\alpha_a|0\rangle$$

- LO hard function

$$\begin{aligned}H^{(0)}(\frac{x}{y},\mu) &= -2\pi^3\overline{n}\cdot p\int\frac{d^3k}{(2\pi)^3}\frac{1}{2k_0}\delta(q+p-k)\mathrm{Tr}\frac{\not{n}}{2}\gamma_\perp^\mu\not{k}\gamma_\mu^\perp\\ &= \overline{n}\cdot p\, n\cdot k\,\delta(k^2)\Big|_{k=q+p}=\overline{n}\cdot pQ\delta(\overline{n}\cdot pQ-Q^2)=y\overline{n}\cdot P\delta(y\overline{n}\cdot P-Q)\\ &= y\delta(y-x)=\delta(1-\frac{x}{y})\end{aligned}$$

$$F_1(x, Q^2) = \int_x^1 \frac{dz}{z} H(z, Q^2, \mu) f_{q/N}\left(\frac{x}{z}, \mu\right)$$



- At the higher order
  - The hard function should be IR-finite
  - All the IR divergence should reside in PDF
  - All the IR divergence should reside in PDF

- Structure function should scale invariant

$$\frac{d}{d\ln\mu} F_1 = 0 \to \left(\frac{d}{d\ln\mu} H\right) \otimes f_{q/N} + H \otimes \left(\frac{d}{d\ln\mu} f_{q/N}\right)$$