Introduction to Quantum Anomalies: Lectures at PSI 2013

Kazuo Fujikawa

Mathematical Physics Laboratory,
RIKEN Nishina Center, Wako 351-0198, Japan

Abstract

Some basic aspects of quantum anomalies are discussed.

1 Introduction

The history of quantum anomalies started in 1949, immediately after the formulation of modern quantum field theory by Tomonaga, Schwinger and Feynman. The original indication of chiral anomaly was first recognized by Fukuda and Miyamoto in their calculation of the radiative decay of the neutral pion. This problem was further studied by Tomonaga and his collaborators in Tokyo and Steinberger at Princeton. The anomalous behavior was not eliminated by the Pauli-Villars regulator which was introduced around that time. Schwinger also analyzed the problem some time later.

This problem was re-discovered in 1969 by Bell and Jackiw at CERN and Adler at Princeton. The notion of soft-pion was well-established at that time, and also the spontaneous symmetry breaking of chiral symmetry by Nambu was gradually gathering suport among particle physicists. The notion of quantum anomaly became prominent in connection with the develeopments of the Standard Model initiated by the works of 't Hooft in 1971. The notion of anomaly played further fundamental roles in the developments of superstring theory in 1980s.

From the point of view of mathematics, the Atiyah-Singer index theorem was established around 1969, about the same time as the modern developments of quantum anomalies. The ghost number anomaly which appears in the first quantization of relativistic string theory was later recognized as a manifestation of Riemann-Roch theorem in the theory of Riemann surfaces. In fact, it is said that the Atiyah-Singer index theorem itself is a result of efforts to generalize the Riemann-Roch theorem to dimensions higher than 2.

2 Chiral anomaly and the radiative decay of the neutral pion

We consider the decay of a massive pseudo-scalar particle P into a pair of photons in the Lagrangian given by (in the natural units $c = \hbar = 1$),

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(x)[i\gamma^\mu D_\mu - m]\psi(x) + \frac{1}{2}[\partial_\mu P(x)\partial^\mu P(x) - \mu^2 P(x)^2] \\ &+ 2gmP(x)\bar{\psi}(x)i\gamma_5\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (2.1)\end{aligned}$$

where $D_\mu = \partial_\mu - ieA_\mu$ and a coupling constant g. The field ψ is a charged fermion and may be regarded as a proton in this primitive model. The above lagrangian is equivalent to

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(x)[i\gamma^\mu D_\mu - m]\psi(x) + \frac{1}{2}[\partial_\mu P(x)\partial^\mu P(x) - \mu^2 P(x)^2] \\ &- g\partial_\mu P(x)\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (2.2)\end{aligned}$$

if one uses the free equations of motion $[i\gamma^\mu\partial_\mu - m]\psi(x) = 0$ for the fermion ψ in the spirit of interaction picture.

The decay of the pion $P(x)$ into a photon pair

$$P(p) \rightarrow A_\alpha(k) + A_\beta(l) \qquad (2.3)$$

takes place through fermion triangle diagrams in the lowest order in perturbation theory. Starting with the Lagrangian, we evaluate the two Feynman diagrams such as

$$\begin{aligned}&p_\mu \int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{[\gamma^\mu\gamma_5\frac{1}{\not{q} - m}\gamma^\alpha\frac{1}{\not{q} - \not{k} - m}\gamma^\beta\frac{1}{\not{q} - \not{k} - \not{l} - m}] \qquad (2.4)\\ &+[\gamma^\mu\gamma_5\frac{1}{\not{q} - m}\gamma^\beta\frac{1}{\not{q} - \not{l} - m}\gamma^\alpha\frac{1}{\not{q} - \not{l} - \not{k} - m}]\}A_\alpha(k)A_\beta(l)\end{aligned}$$

One may now use

$$\begin{aligned}\not{p}\gamma_5 &= (\not{k} + \not{l})\gamma_5 \\ &= (\not{k} + \not{l} - \not{q} + m)\gamma_5 + \gamma_5(-\not{q} + m) - 2m\gamma_5 \qquad (2.5)\end{aligned}$$

then the above expression is replaced by

$$\begin{aligned}-2m \int \frac{d^4q}{(2\pi)^4}&\mathrm{Tr}\{[\gamma_5\frac{1}{\not{q} - m}\gamma^\alpha\frac{1}{\not{q} - \not{k} - m}\gamma^\beta\frac{1}{\not{q} - \not{k} - \not{l} - m}] \\ &+[\gamma_5\frac{1}{\not{q} - m}\gamma^\beta\frac{1}{\not{q} - \not{l} - m}\gamma^\alpha\frac{1}{\not{q} - \not{l} - \not{k} - m}]\}A_\alpha(k)A_\beta(l)\end{aligned}$$

$$
\begin{aligned}
&-\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{\gamma_5[\frac{1}{\not q - m}\gamma^\alpha\frac{1}{\not q - \not k - m}\gamma^\beta \\
&\quad +\gamma^\alpha\frac{1}{\not q - \not k - m}\gamma^\beta\frac{1}{\not q - \not k - \not l - m} \\
&\quad +\frac{1}{\not q - m}\gamma^\beta\frac{1}{\not q - \not l - m}\gamma^\alpha \\
&\quad +\gamma^\beta\frac{1}{\not q - \not l - m}\gamma^\alpha\frac{1}{\not q - \not l - \not k - m}]\}A_\alpha(k)A_\beta(l)
\end{aligned}
\tag{2.6}
$$

The second group in (2.6) is rewritten as

$$
\begin{aligned}
&-\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{\gamma_5[\frac{1}{\not q - m}\gamma^\alpha\frac{1}{\not q - \not k - m}\gamma^\beta \\
&\quad +\gamma^\alpha\frac{1}{\not q - \not k - m}\gamma^\beta\frac{1}{\not q - \not k - \not l - m} \\
&\quad -\gamma^\alpha\frac{1}{\not q - m}\gamma^\beta\frac{1}{\not q - \not l - m} \\
&\quad -\frac{1}{\not q - \not l - m}\gamma^\alpha\frac{1}{\not q - \not l - \not k - m}\gamma^\beta]\}A_\alpha(k)A_\beta(l)
\end{aligned}
\tag{2.7}
$$

by using the relation

$$\{\gamma_5, \gamma^\alpha\} = 0. \tag{2.8}$$

If one replaces the integration variable as $q^\mu \to q^\mu - k^\mu$ in the 3rd term and as $q^\mu - l^\mu \to q^\mu$ in the fourth term, respectively, then one can conform that all the terms cancel each other in (2.7). Eq.(2.4) is thus replaced by

$$-2m\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{[\gamma_5\frac{1}{\not q - m}\gamma^\alpha\frac{1}{\not q - \not k - m}\gamma^\beta\frac{1}{\not q - \not k - \not l - m} \tag{2.9}$$

$$+[\gamma_5 \frac{1}{\not{q}-m}\gamma^\beta \frac{1}{\not{q}-\not{l}-m}\gamma^\alpha \frac{1}{\not{q}-\not{l}-\not{k}-m}]\}A_\alpha(k)A_\beta(l)$$

which establishes the equivalence of (2.1) and (2.2).

However, Fukuda and Miyamoto recognized that the combination (2.7) does not quite vanish, namely, the naive shift of the integration momentum is not justified in this linearly divergent integral. One of the ways to see this subtlety is to use the Pauli-Villars regaularization which amounts to add a bosonic massive fermion to the Lagrangian

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(x)[i\gamma^\mu D_\mu - m]\psi(x) + \frac{1}{2}[\partial_\mu P(x)\partial^\mu P(x) - \mu^2 P(x)^2] \\ &\quad -g\partial_\mu P(x)\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad +\bar{\Psi}(x)[i\gamma^\mu D_\mu - M]\Psi(x) - g\partial_\mu P(x)\bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x) \end{aligned} \tag{2.10}$$

Eq.(2.4) is then replaced by

$$\begin{aligned} &p_\mu \int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{[\gamma^\mu\gamma_5 \frac{1}{\not{q}-m}\gamma^\alpha \frac{1}{\not{q}-\not{k}-m}\gamma^\beta \frac{1}{\not{q}-\not{k}-\not{l}-m}] \\ &+[\gamma^\mu\gamma_5 \frac{1}{\not{q}-m}\gamma^\beta \frac{1}{\not{q}-\not{l}-m}\gamma^\alpha \frac{1}{\not{q}-\not{l}-\not{k}-m}]\}A_\alpha(k)A_\beta(l) \\ &-p_\mu \int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{[\gamma^\mu\gamma_5 \frac{1}{\not{q}-M}\gamma^\alpha \frac{1}{\not{q}-\not{k}-M}\gamma^\beta \frac{1}{\not{q}-\not{k}-\not{l}-M}] \\ &+[\gamma^\mu\gamma_5 \frac{1}{\not{q}-M}\gamma^\beta \frac{1}{\not{q}-\not{l}-M}\gamma^\alpha \frac{1}{\not{q}-\not{l}-\not{k}-M}]\}A_\alpha(k)A_\beta(l) \end{aligned} \tag{2.11}$$

namely, the massive fermion contributions have relatively minus signs due to the bose statistics of the massive bosonic fermion Ψ. The equivalent expressions (2.9) are now replaced by

$$-2m\int\frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{[\gamma_5\frac{1}{\not q-m}\gamma^\alpha\frac{1}{\not q-\not k-m}\gamma^\beta\frac{1}{\not q-\not k-\not l-m}] \tag{2.12}$$
$$+[\gamma_5\frac{1}{\not q-m}\gamma^\beta\frac{1}{\not q-\not l-m}\gamma^\alpha\frac{1}{\not q-\not l-\not k-m}]\}A_\alpha(k)A_\beta(l)$$
$$+2M\int\frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{[\gamma_5\frac{1}{\not q-M}\gamma^\alpha\frac{1}{\not q-\not k-M}\gamma^\beta\frac{1}{\not q-\not k-\not l-M}]$$
$$+[\gamma_5\frac{1}{\not q-M}\gamma^\beta\frac{1}{\not q-\not l-M}\gamma^\alpha\frac{1}{\not q-\not l-\not k-M}]\}A_\alpha(k)A_\beta(l)$$

where the divergent terms in (2.7), which are independent of masses, cancel among the fermion ψ contributions and the regulator Ψ contributions (note that the non-ambiguous parts cancel among themselves in any case). This is the essence of the Pauli-Villars regularization for the fermion fields.

The expression (2.11) defines the *regularized* axial-vector current and the second term in (2.12) gives a possible extra term in the limit $M\to\infty$. The evaluation of the second term in (2.12) proceeds as

$$+2M\int\frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{[\gamma_5\frac{\not q+M}{\not q^2-M^2}\gamma^\alpha\frac{\not q-\not k+M}{(\not q-\not k)^2-M^2}\gamma^\beta\frac{\not q-\not k-\not l+M}{(\not q-\not k-\not l)^2-M^2}]$$
$$+[\gamma_5\frac{\not q+M}{\not q^2-M^2}\gamma^\beta\frac{\not q-\not l+M}{(\not q-\not l)^2-M^2}\gamma^\alpha\frac{\not q-\not l-\not k+M}{(\not q-\not l-\not k)^2-M^2}]\}A_\alpha(k)A_\beta(l)$$
$$=2M^2\int\frac{d^4q}{(2\pi)^4}\mathrm{Tr}\{\gamma_5\frac{(\not q)\gamma^\alpha(-\not k)\gamma^\beta+(\not q)\gamma^\alpha\gamma^\beta(-\not k-\not l)+\gamma^\alpha(\not q-\not k)\gamma^\beta(}{[q^2-M^2][(q-k)^2-M^2][(q-k-l)^2-M^2]}$$

$$+[\gamma_5 \frac{(\not q)\gamma^\beta(-\not l)\gamma^\alpha + (\not q)\gamma^\beta\gamma^\alpha(-\not k - \not l) + \gamma^\beta(\not q - \not l)\gamma^\alpha(-\not k)}{[q^2 - M^2][(q-l)^2 - M^2][(q-l-k)^2 - M^2]}\} A_\alpha(k$$

We evaluate the first term by using

$$\frac{1}{[q^2 - M^2][(q-k)^2 - M^2][(q-k-l)^2 - M^2]}$$
$$= 2! \int \delta(1-x-y-z) dxdydz \frac{1}{(x[q^2 - M^2] + y[(q-k)^2 - M^2] + z[(q-}$$
$$= 2! \int \delta(1-x-y-z) dxdydz \frac{1}{[(q')^2 - D]^3}$$

with

$$\begin{aligned} D &= k^2(y^2 - y) + (k+l)^2(z^2 - z) + M^2 \\ q' &= q - ky - (k+l)z \\ q &= q' + ky + (k+l)z \equiv q' + a \end{aligned} \quad (2.15)$$

The numerator is evaluated by keeping the antisymmetrization by $Tr\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 4i\epsilon^{\mu\nu\alpha\beta}$ in mind.

$$\begin{aligned} &\mathrm{Tr}\gamma_5[(\not q)\gamma^\alpha(-\not k)\gamma^\beta + (\not q)\gamma^\alpha\gamma^\beta(-\not k - \not l) + \gamma^\alpha(\not q - \not k)\gamma^\beta(-\not l)] \\ &= \mathrm{Tr}\gamma_5[(\not a)\gamma^\alpha(-\not k)\gamma^\beta + (\not a)\gamma^\alpha\gamma^\beta(-\not k - \not l) + \gamma^\alpha(\not a - \not k)\gamma^\beta(-\not l)] \\ &= \mathrm{Tr}\gamma_5[-z \not l\gamma^\alpha \not k\gamma^\beta - y \not k\gamma^\alpha\gamma^\beta \not l + (1-y-z)\gamma^\alpha \not k\gamma^\beta \not l] \\ &= -4i\epsilon^{\alpha\beta\lambda\rho}k_\lambda l_\rho \end{aligned} \quad (2.16)$$

where we used $q = q' + a$ and the terms linear in q' vanish after the integration over d^4q'. We also use

$$\int \frac{d^4q'}{(2\pi)^4} \frac{1}{[(q')^2 - D]^3}$$

$$= i(-1) \int \frac{d^4 q'}{(2\pi)^4} \frac{1}{[-(q')^2 + D]^3}$$

$$= i(-1) \frac{1}{(2\pi)^4} \frac{\pi^2}{2} \frac{1}{D} = -i \frac{1}{32\pi^2} \frac{1}{D} \tag{2.17}$$

where we Wick-rotated the integration variable $q'_0 \rightarrow i q'_0$ to make $-(q')^2 \geq 0$.

The first term in (2.13) becomes

$$-8M^2 \frac{1}{32\pi^2} \frac{1}{D} \epsilon^{\alpha\beta\lambda\rho} k_\lambda l_\rho A_\alpha(k) A_\beta(l)$$

$$= -\frac{1}{4\pi^2} \epsilon^{\alpha\beta\lambda\rho} k_\lambda l_\rho A_\alpha(k) A_\beta(l) \tag{2.18}$$

where we used

$$M^2 \frac{1}{D} = M^2 \frac{1}{D = k^2(y^2 - y) + (k+l)^2(z^2 - z) + M^2} \rightarrow 1 \tag{2.19}$$

for $M^2 \rightarrow \infty$. The total contribution is thus

$$-2 \times \frac{1}{4\pi^2} \epsilon^{\alpha\beta\lambda\rho} k_\lambda l_\rho A_\alpha(k) A_\beta(l) \rightarrow \frac{1}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{2.20}$$

in the coordinate representation. The extra factor of 2 comes from the contraction of external electromagnetic fields with the variables in the anomaly.

In the operator notation, we have the anomalous relation

$$\partial_\mu [\bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x)]_{reg} = 2mi \bar{\psi}(x) \gamma_5 \psi(x) + \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{2.21}$$

2.1 PCAC and soft-pion

We write the model (2.1) in a slightly more realistic way as

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(x)[i\gamma^\mu D_\mu - m]\psi(x) + \frac{1}{2}[\partial_\mu \pi^a(x)\partial^\mu \pi^a(x) - \mu^2(\pi^a(x))^2] \\ &+ 2gm\sum_{a=1}^{3}\pi^a(x)\bar{\psi}(x)i\gamma_5\frac{1}{2}\tau^a\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{2.22}$$

with Pauli matrices τ^a and

$$\psi(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}. \tag{2.23}$$

and

$$D_\mu = \partial_\mu - ie\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A_\mu(x) \tag{2.24}$$

Then the pion decay amplitude is given by

$$\langle\gamma\gamma|\pi^0\rangle = g\langle\gamma\gamma|\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}|0\rangle/\sqrt{2E} \tag{2.25}$$

where the extra 1/2 arises from the isospin convention. If one uses the Goldberger-Treiman relation

$$\frac{1}{f_\pi} = \frac{g_{NN\pi}}{2m} = \frac{2mg}{2m} = g \tag{2.26}$$

we have

$$\langle\gamma\gamma|\pi^0\rangle = \frac{1}{f_\pi}\langle\gamma\gamma|\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}|0\rangle/\sqrt{2E} \tag{2.27}$$

It is known that this relation is good from an experimental point of view.

To understand this relation from the point of view of PCAC, we define the $SU(2)$ current

$$J_5^{a,\mu}(x) = \bar{\psi}(x)\frac{1}{2}\tau^a\gamma^\mu\gamma_5\psi(x) \tag{2.28}$$

then (by ignoring the anomaly, for a while)

$$\partial_\mu J_5^{a,\mu}(x) = 2m\bar{\psi}(x)\frac{1}{2}\tau^a i\gamma_5\psi(x) \tag{2.29}$$

For the on-shell pion, we define the pion decay constant

$$\langle 0|J_5^{a,\mu}(0)|\pi^a\rangle = f_\pi p^\mu/\sqrt{2E}. \tag{2.30}$$

Then PCAC means that the right-hand side of (2.29) is an interpolating field of the soft pion

$$\partial_\mu J_5^{a,\mu}(x) = 2m\bar{\psi}(x)\frac{1}{2}\tau^a i\gamma_5\psi(x) \simeq m_\pi^2 f_\pi\phi_\pi(x) \tag{2.31}$$

for $p^\mu \simeq 0$.

In the presence of the anomaly, we have

$$\partial_\mu J_5^{3,\mu}(x) = 2m\bar{\psi}(x)\frac{1}{2}\tau^3 i\gamma_5\psi(x) + \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{2.32}$$

Then we have

$$\begin{aligned}\partial_\mu\langle\gamma\gamma|J_5^{3,\mu}(x)|0\rangle &= \langle\gamma\gamma|2m\bar{\psi}(x)\frac{1}{2}\tau^3 i\gamma_5\psi(x)|0\rangle \\ &\quad +\langle\gamma\gamma|\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}|0\rangle\end{aligned} \tag{2.33}$$

In the soft-pion limit $p_\mu \rightarrow 0$, the left-hand side goes to zero. We thus have (by noting the reduction formula $(p^2 - m_\pi^2)\phi_\pi(p)|0 >= |\pi(p^2)\rangle\sqrt{2E}$)

$$f_\pi\langle\gamma\gamma|\pi^0(p^2=0)\rangle\sqrt{2E} = \langle\gamma\gamma|\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}|0\rangle \quad (2.34)$$

and assuming

$$\langle\gamma\gamma|\pi^0(m_\pi^2)\rangle \simeq \langle\gamma\gamma|\pi^0(p^2=0)\rangle \quad (2.35)$$

we get the correct decay amplitude

$$\langle\gamma\gamma|\pi^0(m_\pi^2)\rangle \simeq \frac{1}{f_\pi}\langle\gamma\gamma|\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}|0\rangle/\sqrt{2E}. \quad (2.36)$$

In the picture of Nambu, the mass of the fundamental fermions (quarks) $m = 0$, and we have

$$J_5^{3,\mu}(x) = \frac{1}{2}[\bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d] \quad (2.37)$$

and

$$\partial_\mu J_5^{3,\mu}(x) = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad (2.38)$$

but the pion is massless $m_\pi^2 = 0$. Then we have

$$\partial_\mu\langle\gamma\gamma|J_5^{3,\mu}(x)|0\rangle \;=\; \langle\gamma\gamma|\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}|0\rangle \quad (2.39)$$

In the soft-momentum limit $p^2 \to 0$, we have symbolically

$$
\begin{aligned}
&\int d^4x e^{-ipx}\langle\gamma\gamma|\partial_\mu J_5^{3,\mu}(x)|0\rangle \\
&= \langle\gamma\gamma|\pi(p^2=0)\rangle\sqrt{2E}\langle 0|T(\phi_\pi(-p)\phi_\pi(p))|0> p^2 f_\pi \\
&= \langle\gamma\gamma|\pi(p^2=0)\rangle\sqrt{2E}f_\pi, \qquad (2.40)
\end{aligned}
$$

or one may simply assume $J_5^{3,\mu}(x) = f_\pi\partial^\mu\phi_\pi(x)$ in the soft-momentum limit, and we get the same result as above.

3 Gauge invariant regularization of currents

We now discuss the above evaluation of Feynman diagrams form a slightly different point of view. We regularize the current in a gauge invariant manner and evaluate the anomaly without recourse to Feynman diagrams. This scheme later leads to the evaluation of anomaly as the Jacobian in path integral.

To make this analysis better defined it is important to consider the Euclidean theory by Wick rotating the Minkowski quantities.

$$
\begin{aligned}
&x^0 \to -ix^4 \\
&\gamma^0 \to -i\gamma^4 \\
&x_0 \to ix_4 \\
&p_0 \to ip_4 \\
&A_0(x) \to iA_4(x) \qquad (3.1)
\end{aligned}
$$

and regard x^4 and p_4 as real quantities. The basic idea is to keep the inner product invariant under this rotation such as

$$\not{p} = \sum_{\mu=0}^{3} \gamma^\mu p_\mu = \sum_{\mu=1}^{4} \gamma^\mu p_\mu,$$
$$\not{D} = \sum_{\mu=0}^{3} \gamma^\mu (\partial_\mu - ieA_\mu(x)) = \sum_{\mu=1}^{4} \gamma^\mu (\partial_\mu - ieA_\mu(x)) \quad (3.2)$$

but the metric is changed as

$$g_{\mu,\nu} = (1, -1, -1, -1) \to g_{\mu,\nu} = (-1, -1, -1, -1) = g^{\mu,\nu} \quad (3.3)$$

and

$$(\gamma^\mu)^\dagger = -\gamma^\mu, \quad \{\gamma^\mu \gamma^\mu\} = 2g^{\mu,\nu} \quad (3.4)$$

In this notation, the Dirac operator $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu(x))$ becomes hermitian for the $SO(4)$ invariant inner product

$$\begin{aligned} (\Psi, \not{D}\Psi) &= \int d^4x \Psi(x)^\dagger \not{D}\Psi(x) \\ &= \int d^4x (\not{D}\Psi(x))^\dagger \Psi(x) \\ &= (\not{D}\Psi, \Psi) \end{aligned} \quad (3.5)$$

The path integral, which is explained later in more detail, becomes

$$\begin{aligned} &\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp\{i \int d^4x [\bar{\psi}(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}]\} \\ \to &\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp\{\int d^4x [\bar{\psi}(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}]\} \end{aligned} \quad (3.6)$$

Note that $F_{\mu\nu}F^{\mu\nu}$ is positive definite in Euclidean metric and thus the path intregral is better-defined.

We now define the axial-current

$$
\begin{aligned}
\langle j_5^\mu(x)\rangle &= \lim_{y\to x}\langle T^\star\bar\psi(y)\gamma^\mu\gamma_5\psi(x)\rangle \\
&= -\lim_{y\to x}\langle T^\star(\gamma^\mu\gamma_5\psi(x))_\alpha\bar\psi(y)_\alpha\rangle \\
&= \lim_{y\to x}\mathrm{tr}\{\gamma^\mu\gamma_5\frac{1}{i\,\not{D}-m}\delta^4(x-y)\}
\end{aligned}
\tag{3.7}
$$

where the trace stands for the sum over the spinor indices. Here we used

$$
\langle T^\star\psi(x)_\beta\bar\psi(y)_\alpha\rangle = (\frac{-1}{i\,\not{D}-m})_{\beta\alpha}\delta^4(x-y) \tag{3.8}
$$

We now use the relations

$$
\begin{aligned}
\frac{1}{i\,\not{D}-m} &= \frac{1}{i\,\not{\partial}+e\,\not{A}(x)-m} \\
&= \frac{1}{i\,\not{\partial}-m}-\frac{1}{i\,\not{\partial}-m}e\,\not{A}(x)\frac{1}{i\,\not{\partial}-m} \\
&\quad +\frac{1}{i\,\not{\partial}-m}e\,\not{A}(x)\frac{1}{i\,\not{\partial}-m}e\,\not{A}(x)\frac{1}{i\,\not{\partial}-m}+\ldots
\end{aligned}
\tag{3.9}
$$

and

$$
\delta^4(x-y) = \int\frac{d^4q}{(2\pi)^4}e^{-iq(x-y)} \tag{3.10}
$$

we have

$$
\begin{aligned}
\langle j_5^\mu(x)\rangle &= \int \frac{d^4q}{(2\pi)^4} e^{iqx} \mathrm{tr}\gamma^\mu\gamma_5 [\frac{1}{i\not\partial - m} e\gamma^\alpha e^{ikx} \frac{1}{i\not\partial - m} e\gamma^\beta e^{ilx} \frac{1}{i\not\partial - m} \\
&+ \frac{1}{i\not\partial - m} e\gamma^\beta e^{ilx} \frac{1}{i\not\partial - m} e\gamma^\alpha e^{ikx} \frac{1}{i\not\partial - m}] e^{-iqx} A_\alpha(k) A_\beta(l) \\
&= e^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\gamma^\mu\gamma_5 [\frac{1}{\not q - \not k - \not l - m}\gamma^\alpha \frac{1}{\not q - \not l - m}\gamma^\beta \frac{1}{\not q - m} \\
&+ \frac{1}{\not q - \not k - \not l - m}\gamma^\beta \frac{1}{\not q - \not k - m}\gamma^\alpha \frac{1}{\not q - m}] A_\alpha(k) e^{ikx} A_\beta(l) e^{ilx}
\end{aligned}
\tag{3.11}
$$

and similarly

$$
\begin{aligned}
2m\langle j_5^\mu(x)\rangle &= 2me^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\gamma_5 [\frac{1}{\not q - \not k - \not l - m}\gamma^\alpha \frac{1}{\not q - \not l - m}\gamma^\beta \frac{1}{\not q - m} \\
&+ \frac{1}{\not q - \not k - \not l - m}\gamma^\beta \frac{1}{\not q - \not k - m}\gamma^\alpha \frac{1}{\not q - m}] A_\alpha(k) e^{ikx} A_\beta(l) e^{ilx}
\end{aligned}
\tag{3.12}
$$

and this mass term is explicitly evaluated as

$$
\begin{aligned}
2m\langle j_5^\mu(x)\rangle &= -2\frac{e^2}{4\pi^2}\epsilon^{\alpha\beta\lambda\rho} k_\lambda l_\rho A_\alpha(k) e^{ikx} A_\beta(l) e^{ilx} \\
&= -2\frac{e^2}{4\pi^2}\epsilon^{\alpha\beta\lambda\rho} \partial_\lambda(A_\alpha(k) e^{ikx}) \partial_\rho(A_\beta(l) e^{ilx}) \\
&= -\frac{e^2}{4\pi^2}\epsilon^{\alpha\beta\lambda\rho} \partial_\lambda(A_\alpha(k) e^{ikx} + A_\alpha(l) e^{ilx}) \partial_\rho(A_\beta(l) e^{ilx} + A_\beta(k) e^{ikx}) \\
&= -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\lambda\rho} \partial_\lambda A_\alpha(x) \partial_\rho A_\beta(x)
\end{aligned}
$$

$$= -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\lambda\rho}F_{\lambda\alpha}F_{\rho\beta} \quad (3.13)$$

for the large m, and we defined

$$A_\mu(x) = A_\mu(k)e^{ikx} + A_\mu(l)e^{ilx} \quad (3.14)$$

by assuming two frequency modes which are sufficient for our calculation. This decay amplitude through the mass term, which is different from the anomaly and given here just for an illustration, is (-1) times the anomaly given by the Pauli-Villars regularization.

The above definition of currents thus reproduces the results of Feynman diagram calculations.

Our definition of the current can also be written as

$$\begin{aligned}\langle j_5^\mu(x)\rangle &= \lim_{y\rightarrow x}\mathrm{tr}\{\gamma^\mu\gamma_5\frac{1}{i\not{D}-m}\delta^4(x-y)\}\\ &= \sum_n \lim_{y\rightarrow x}\{\left(\gamma^\mu\gamma_5\frac{1}{i\not{D}-m}\phi_n(x)\right)_\alpha \phi_n(y)^\dagger_\alpha\}\\ &= \sum_n \phi_n(x)^\dagger\gamma^\mu\gamma_5\frac{1}{i\not{D}-m}\phi_n(x)\\ &= \sum_n \phi_n(x)^\dagger\gamma^\mu\gamma_5\frac{1}{i\lambda_n-m}\phi_n(x) \quad (3.15)\end{aligned}$$

by using the complete orthonormal set

$$\begin{aligned}&\not{D}\phi_n(x) = \lambda_n\phi_n(x),\\ &\int d^4x\phi_n^\dagger(x)\phi_m(x) = \delta_{mn},\\ &\sum_n \phi_n(x)_\beta\phi_n(y)^\dagger_\alpha = \delta_{\alpha\beta}\delta^4(x-y) \quad (3.16)\end{aligned}$$

of eigenfunctions of the hermitian operator $\not{D}$. Since the eigenvalue λ_n of $\not{D}$ is gauge invariant, as is seen under the gauge transformation generated by $U(\omega)$,

$$U(\omega)\not{D}U(\omega)^\dagger = \not{D}' \tag{3.17}$$

then

$$\not{D}'(U(\omega)\phi_n(x)) = U(\omega)\not{D}\phi_n(x) = U(\omega)\lambda_n\phi_n(x) = \lambda_n(U(\omega)\phi_n(x)). \tag{3.18}$$

We can thus regularize the current in a gauge invariant manner by (*gauge invariant mode cut-off*)

$$\langle j_5^\mu(x)\rangle_{cov} = \sum_n \phi_n(x)^\dagger\gamma^\mu\gamma_5\frac{1}{i\lambda_n - m}f(\lambda_n^2/M^2)\phi_n(x) \tag{3.19}$$

with any smooth function $f(x)$ with a large cut-off mass M, and

$$\begin{aligned} &f(0) = 1,\\ &xf'(x)|_{x=0} = 0,\\ &f(\infty) = 0,\\ &xf'(x)|_{x=\infty} = 0 \end{aligned} \tag{3.20}$$

such as

$$f(x) = e^{-x}. \tag{3.21}$$

We then have

$$
\begin{aligned}
\partial_\mu \langle j_5^\mu(x)\rangle_{cov} &= \sum_n \partial_\mu \phi_n(x)^\dagger \gamma^\mu \gamma_5 \frac{1}{i\lambda_n - m} f(\lambda_n^2/M^2)\phi_n(x) \\
&\quad + \phi_n(x)^\dagger \gamma^\mu \gamma_5 \frac{1}{i\lambda_n - m} f(\lambda_n^2/M^2)\partial_\mu \phi_n(x) \\
&= \sum_n [-(\not{D}\phi_n(x))^\dagger \gamma_5 \frac{1}{i\lambda_n - m} f(\lambda_n^2/M^2)\phi_n(x) \\
&\quad - \phi_n(x)^\dagger \gamma_5 \frac{1}{i\lambda_n - m} f(\lambda_n^2/M^2)\, \not{D}\phi_n(x)] \\
&= \sum_n [-2\phi_n(x)^\dagger \gamma^\mu \gamma_5 \frac{\lambda_n}{i\lambda_n - m} f(\lambda_n^2/M^2)\phi_n(x)] \\
&= 2mi \sum_n \phi_n(x)^\dagger \gamma_5 \frac{1}{i\lambda_n - m} f(\lambda_n^2/M^2)\phi_n(x) \\
&\quad + 2i \sum_n \phi_n(x)^\dagger \gamma_5 f(\lambda_n^2/M^2)\phi_n(x) \\
&= 2mi\langle j_5(x)\rangle_{cov} + 2i \sum_n \phi_n(x)^\dagger \gamma_5 f(\lambda_n^2/M^2)\phi_n(x)
\end{aligned} \tag{3.22}
$$

The gauge invariant regularization of the current thus automatically produces an extra term

$$
2i \sum_n \phi_n(x)^\dagger \gamma_5 f(\lambda_n^2/M^2)\phi_n(x) \tag{3.23}
$$

We extract the gauge field dependence from this expression as follows:

$$
\begin{aligned}
&\sum_n \phi_n(x)^\dagger \gamma_5 f(\lambda_n^2/M^2)\phi_n(x) \\
&= \sum_n \phi_n(x)^\dagger \gamma_5 f(\not{D}^2/M^2)\phi_n(x)
\end{aligned}
$$

$$= \mathrm{tr}\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\gamma_5 f(\not{D}^2/M^2) e^{ikx} \tag{3.24}$$

where trace stands for the Dirac indices. We assumed that the operator $\gamma_5 f(\not{D}^2/M^2)$ is well-regularized and transformed the basis set to the complete set of plane waves. Namely

$$\begin{aligned} \phi_n(x) &\to e^{ikx} \\ \sum_n &\to \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}. \end{aligned} \tag{3.25}$$

We also note

$$\begin{aligned} \not{D}^2 &= \gamma^\mu\gamma^\nu D_\mu D_\nu \\ &= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}D_\mu D_\nu + \frac{1}{2}[\gamma^\mu,\gamma^\nu]D_\mu D_\nu \\ &= g^{\mu\nu}D_\mu D_\nu + \frac{1}{4}[\gamma^\mu,\gamma^\nu][D_\mu,D_\nu] \\ &= g^{\mu\nu}D_\mu D_\nu - ie\frac{1}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} \end{aligned} \tag{3.26}$$

where we used

$$[D_\mu, D_\nu] = -ieF_{\mu\nu} \tag{3.27}$$

We thus have

$$\begin{aligned} &\mathrm{tr}\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\gamma_5 f(\not{D}^2/M^2) e^{ikx} \\ &= \mathrm{tr}\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\gamma_5 f((g^{\mu\nu}D_\mu D_\nu - ie\frac{1}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu})/M^2) e^{ikx} \end{aligned}$$

$$= \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\gamma_5 f(\frac{g^{\mu\nu}(ik_\mu + D_\mu)(ik_\nu + D_\nu)}{M^2} - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu})$$
$$= M^4 \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\gamma_5 f(g^{\mu\nu}(ik_\mu + \frac{D_\mu}{M})(ik_\nu + \frac{D_\nu}{M}) - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}) \tag{3.28}$$

where we re-scaled the integration variable

$$k_\mu \to M k_\mu. \tag{3.29}$$

We then expand

$$f(g^{\mu\nu}(ik_\mu + \frac{D_\mu}{M})(ik_\nu + \frac{D_\nu}{M}) - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu})$$
$$= f(-g^{\mu\nu}k_\mu k_\nu)$$
$$+ f'(-g^{\mu\nu}k_\mu k_\nu)\{g^{\mu\nu}(2ik_\mu D_\nu/M + D_\mu D_\nu/M^2) - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}$$
$$+ \frac{1}{2!}f''(-g^{\mu\nu}k_\mu k_\nu)\{g^{\mu\nu}(2ik_\mu D_\nu/M + D_\mu D_\nu/M^2) - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}^2$$
$$+ \frac{1}{3!}f'''(-g^{\mu\nu}k_\mu k_\nu)\{g^{\mu\nu}(2ik_\mu D_\nu/M + D_\mu D_\nu/M^2) - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}^3$$
$$+ \frac{1}{4!}f''''(-g^{\mu\nu}k_\mu k_\nu)\{g^{\mu\nu}(2ik_\mu D_\nu/M + D_\mu D_\nu/M^2) - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}^4$$

We now observe that the trace with $\gamma_5 = \gamma^4\gamma^1\gamma^2\gamma^3$ requires at least 4 Dirac γ matrices with the powers in $1/M^4$ in the limit $M \to \infty$. We thus have the unique term

$$M^4 \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\gamma_5 f(g^{\mu\nu}(ik_\mu + \frac{D_\mu}{M})(ik_\nu + \frac{D_\nu}{M}) - ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu})$$

$$= M^4 \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 \frac{1}{2!} f''(-g^{\mu\nu}k_\mu k_\nu)\{-ie\frac{1}{4M^2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}^2$$

$$= -\frac{e^2}{16}\mathrm{tr}\{\gamma_5[\gamma^\mu, \gamma^\nu]F_{\mu\nu}[\gamma^\alpha, \gamma^\beta]F_{\alpha\beta}\} \int \frac{d^4k}{(2\pi)^4}\frac{1}{2!}f''(-g^{\mu\nu}k_\mu k_\nu) \quad (3.31)$$

We now evaluate

$$\mathrm{tr}\{\gamma_5[\gamma^\mu, \gamma^\nu]F_{\mu\nu}[\gamma^\alpha, \gamma^\beta]F_{\alpha\beta}\} = -16\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad (3.32)$$

for

$$\gamma_5 = \gamma_4\gamma_1\gamma_2\gamma_3 \quad (3.33)$$

which is the same as the Minkowski convention and

$$\begin{aligned} &\{\gamma_5, \gamma_\mu\} = 0, \\ &\epsilon^{1234} = 1. \end{aligned} \quad (3.34)$$

We also have

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4}\frac{1}{2!}f''(-g^{\mu\nu}k_\mu k_\nu) &= \frac{1}{32\pi^2}\int_0^\infty dx x f''(x) \\ &= \frac{1}{32\pi^2}\int_0^\infty dx(-f'(x)) \\ &= \frac{1}{32\pi^2} \end{aligned} \quad (3.35)$$

with $x = --g^{\mu\nu}k_\mu k_\nu \geq 0$, and

$$\sum_n \phi_n(x)^\dagger \gamma_5 f(\lambda_n^2/M^2)\phi_n(x) = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad (3.36)$$

independently of the regularization function $f(x)$ for large M.

We thus have the anomaly relation

$$\partial_\mu \langle j_5^\mu(x) \rangle_{cov} \;=\; 2mi\langle j_5(x)\rangle_{cov} + 2i\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{3.37}$$

The result of Minkowski space is obtained by removing the imaginary factor i from the anomaly term and

$$\epsilon^{1230} = \epsilon_{0123} = 1. \tag{3.38}$$

These two relations are valid even for the non-Abelian gauge field if one defines

$$\begin{aligned}
&A_\mu(x) = A^a_\mu(x)T^a, \\
&D_\mu = \partial_\mu - igA_\mu(x), \\
&[D_\mu, D_\mu] = -igF_{\mu\nu} \\
&= -igT^aF^a_{\mu\nu} = -igT^a(\partial_\mu A_\nu - \partial_{nu}A_\mu + gf^{abc}A^b_\mu A^c_\nu)
\end{aligned} \tag{3.39}$$

with the structure constant

$$\begin{aligned}
&[T^a, T^b] = if^{abc}T^c, \\
&\mathrm{tr}T^aT^b = \frac{1}{2}\delta^{ab}.
\end{aligned} \tag{3.40}$$

Then

$$\sum_n \phi_n(x)^\dagger\gamma_5 f(\lambda_n^2/M^2)\phi_n(x) = \frac{g^2}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{3.41}$$

where tr stands for the trace over Yang-Mills freedom, and

$$\partial_\mu \langle j_5^\mu(x)\rangle_{cov} \;=\; 2mi\langle j_5(x)\rangle_{cov} + 2i\frac{g^2}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{3.42}$$

In the presence of the instanton solution of the Yang-Mills field we have

$$\nu = \int d^4x \frac{g^2}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \text{integer} \tag{3.43}$$

and the Atiyah-Singer index relation

$$\begin{aligned} n_+ - n_- \;&=\; \int d^4x \sum_n \phi_n(x)^\dagger \gamma_5 f(\lambda_n^2/M^2)\phi_n(x) \\ &=\; \int d^4x \frac{g^2}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \\ &=\; \nu. \end{aligned} \tag{3.44}$$

Here we used

$$\begin{aligned} &\not{D}\phi_n(x) = \lambda_n\phi_n(x), \\ &\not{D}\gamma_5\phi_n(x) = -\lambda_n\gamma_5\phi_n(x) \end{aligned} \tag{3.45}$$

using $\gamma_5 \not{D} = -\not{D}\gamma_5$. Namely, $\phi_n(x)$ and $\gamma_5\phi_n(x)$ are orthogonal for $\lambda_n \neq 0$

$$\int d^4x \phi_n^\dagger(x)\gamma_5\phi_n(x) = 0, \text{for } \lambda_n \neq 0. \tag{3.46}$$

On the other hand, for $\lambda_n = 0$

$$\begin{aligned} &\not{D}\phi_n(x) = 0, \\ &\not{D}\gamma_5\phi_n(x) = 0 \end{aligned} \tag{3.47}$$

namely

$$\not{D}\frac{(1\pm\gamma_5)}{2}\phi_n(x)=0 \tag{3.48}$$

Thus we can choose the eigenstates of chirality γ_5,

$$\begin{aligned}
&\phi_{\lambda_n=0}(x)_\pm=\frac{(1\pm\gamma_5)}{2}\phi_{\lambda_n=0}(x),\\
&\gamma_5\phi_{\lambda_n=0}(x)_\pm=\pm\phi_{\lambda_n=0}(x)_\pm.
\end{aligned} \tag{3.49}$$

We thus evaluate

$$\begin{aligned}
&\int d^4x\sum_n\phi_n(x)^\dagger\gamma_5 f(\lambda_n^2/M^2)\phi_n(x)\\
=&\int d^4x\sum_n\phi_{\lambda_n=0}(x)_\pm^\dagger\gamma_5 f(0)\phi_{\lambda_n=0}(x)_\pm\\
=&\ n_+-n_-
\end{aligned} \tag{3.50}$$

where $n_\pm$ stand for the number of normaizable eigenstates with vanishing $\lambda_n=0$ and chirality ± 1 in the background of topologically non-trivial Yang-Mills field. This number n_+-n_- is called "index" associated with the operator $\not{D}$.

Intuitively, the square matrix $M=\not{D}$, which has a vanishing index, is deformed to a rectangular matrix with a non-trivial index in the presence of the topologically non-trivial background gauge field;

$$\begin{aligned}
\text{index}&=\text{dim ker}M-\text{dim ker}M^\dagger\\
&=\text{dim ker}M^\dagger M-\text{dim ker}MM^\dagger
\end{aligned} \tag{3.51}$$

where "dim ker" counts the number of normalizable eigenstates with the 0 eigenvalue.

It is important to recognize that the chiral anomaly, which may be called "a local form of index", is valid even for topologically trivial gauge fields, such as $d = 4$ Abelian gauge field.

4 Path integral and Ward-Takahashi identities

The (Euclidean) path integral for QED is defined by the vaccum-to-vacuum transition amplitude in the presence of localized source terms

$$\begin{aligned} &\langle 0|0\rangle_\eta \\ &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu]\exp\{\int d^4x[\bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\} \end{aligned} \tag{4.1}$$

where $[\mathcal{D}A_\mu]$ includes the gauge fixing and compensating terms, which are not essential for the analysis of chiral symmetries. The last two terms with the classical (Grassmann) source η stand for the source terms essential in the Schwinger's action principle. The fermionic variables including source terms are all Grassmann numbers. We treat ψ and $\bar{\psi}$ as independent variables in path integral. The Grassmann numbers are defined as totally anti-commuting "classical numbers". For example

$$\begin{aligned} &\psi(x)\psi(y) + \psi(y)\psi(x) = 0, \\ &\psi(x)\bar{\psi}(y) + \bar{\psi}(y)\psi(x) = 0, \\ &\bar{\psi}(x)\bar{\psi}(y) + \bar{\psi}(y)\bar{\psi}(x) = 0, \\ &\psi(x)\eta(y) + \eta(y)\psi(x) = 0, \end{aligned} \tag{4.2}$$

in particular,

$$\psi(x)\psi(x) + \psi(x)\psi(x) = 2\psi(x)\psi(x) = 0 \tag{4.3}$$

namely, the Grassmann numbers have no magnitude.

The path integral measure is defined by *left-derivative*

$$\mathcal{D}\psi = \prod_x \frac{\delta}{\delta\psi(x)} \tag{4.4}$$

where the product runs over all the points of Minkowski space or Euclidean space R^4. The left-derivative means, for example,

$$\begin{aligned} \frac{\delta}{\delta\psi(x)} \int d^4y \bar{\eta}(y)\psi(y) &= \frac{\delta}{\delta\psi(x)} \int d^4y(-\psi(y)\bar{\eta}(y)) \\ &= -\bar{\eta}(x) \end{aligned} \tag{4.5}$$

where we used

$$\frac{\delta}{\delta\psi_\alpha(x)}\psi_\beta(y) \equiv \delta^4(x-y)\delta_{\alpha\beta} \tag{4.6}$$

if one writes the spinor index explicitly. Namely, we first move the variable which is integrated (in fact, differentiated) to the left of all the rest of the variables, and then perform the differentiation.

The left derivative satisfies the condition of linear projection from Grassmann numbers to complex numbers, which is defined as "integral".

The Schwinger's action priciple states that

$$\frac{\delta}{\delta\bar{\eta}(x)}\langle 0|0\rangle_\eta = \langle 0|\hat{\psi}(x)|0\rangle_\eta \tag{4.7}$$

and the quantized equation of motion is given by (by treating the gauge field as a background for a moment)

$$\langle 0|(i \not{D} - m)\hat{\psi}(x) + \eta(x)|0\rangle_\eta = 0 \tag{4.8}$$

and thus the vacuum-to-vacuum amplitude $\langle 0|0\rangle_\eta$ should satisfy

$$[(i \not{D} - m)\frac{\delta}{\delta\bar{\eta}(x)} + \eta(x)]\langle 0|0\rangle_\eta = 0 \tag{4.9}$$

In fact, the path integral representation satisfies this condition

$$\begin{aligned}&\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu]\{(i \not{D} - m)\psi(x) + \eta(x)\}\\ &\times \exp\{\int d^4x[\bar{\psi}(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\} = 0\end{aligned} \tag{4.10}$$

Note that all the variables are classical variables in this expression. We rewrite this path integral as

$$\begin{aligned}&\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu]\\ &\times \frac{\delta}{\delta\bar{\psi}(x)}\exp\{\int d^4x[\bar{\psi}(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\} = 0\end{aligned} \tag{4.11}$$

To prove this relation, we start with the identity

$$\begin{aligned}&\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu]\\ &\times \exp\{\int d^4x[\bar{\psi}(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\}\\ &= \int \mathcal{D}\bar{\psi}'\mathcal{D}\psi[\mathcal{D}A_\mu]\\ &\times \exp\{\int d^4x[\bar{\psi}'(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}'\eta]\}\end{aligned} \tag{4.12}$$

which is analogous to $\int dx f(x) = \int dy f(y)$. We choose

$$\bar{\psi}'(x) = \bar{\psi}(x) + \bar{\epsilon}(x) \tag{4.13}$$

for any fixed function $\bar{\epsilon}(x)$ independent of $\bar{\psi}(x)$. We can confirm

$$\begin{aligned} \mathcal{D}\bar{\psi}' &= \prod_x \frac{\delta}{\delta\bar{\psi}'(x)} \\ &= \prod_x \frac{\delta}{\delta\bar{\psi}(x)} \\ &= \mathcal{D}\bar{\psi} \end{aligned} \tag{4.14}$$

Namely, the path integral measure is *translational invariant in the functional space*

$$\mathcal{D}(\bar{\psi} + \bar{\epsilon}) = \mathcal{D}\bar{\psi} \tag{4.15}$$

which ensures the equation of motion of quantized theory, as is seen from (4.12) as

$$\begin{aligned} &\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu] \\ &\times \exp\{\int d^4x[\bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\} \\ &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu] \\ &\times \exp\{\int d^4x[\bar{\psi}'(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}'\eta]\} \end{aligned} \tag{4.16}$$

by expanding the exponential factor in the second expression in powers of $\bar{\epsilon}$.

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu]\{\int d^4y[\bar{\epsilon}(y)(i\not{D} - m)\psi(y) + \bar{\epsilon}(y)\eta(y)]\}$$

$$\times \exp\{\int d^4x[\bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\} = 0,$$

which implies

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu]\{(i\not{D} - m)\psi(x) + \eta(x)\} \tag{4.17}$$
$$\times \exp\{\int d^4x[\bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\} = 0.$$

by differentiating by $\bar{\epsilon}(x)$. This shows tha equation of motion.

We next analyze the Ward-Takahashi identity by taking the chiral symmetry as an example. The chiral symmetry is defined by the transformation of variables by

$$\begin{aligned}\psi(x) &\to \psi'(x) = e^{i\alpha\gamma_5}\psi(x),\\ \bar{\psi}(x) &\to \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha\gamma_5},\end{aligned} \tag{4.18}$$

with an infinitesimal α. Under this transformation, the Dirac action is invariant except for the mass term

$$\int d^4x\bar{\psi}'(i\not{D} - m)\psi' = \int d^4x\bar{\psi}(i\not{D} - m - 2i\alpha m\gamma_5)\psi \tag{4.19}$$

by noting $\gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 0$.

To derive Ward-Takahashi identity, we use a localized parameter

$$\begin{aligned}\psi(x) &\to \psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x),\\ \bar{\psi}(x) &\to \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha(x)\gamma_5},\end{aligned} \tag{4.20}$$

and start with an identity

$$
\begin{aligned}
&\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu] \qquad (4.21)\\
&\times \exp\{\int d^4x[\bar{\psi}(i\not{D}-m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\}\\
&= \int \mathcal{D}\bar{\psi}'\mathcal{D}\psi'[\mathcal{D}A_\mu]\\
&\times \exp\{\int d^4x[\bar{\psi}'(i\not{D}-m)\psi' - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi' + \bar{\psi}'\eta]\}
\end{aligned}
$$

The action then changes

$$
\begin{aligned}
&\int d^4x[\bar{\psi}'(i\not{D}-m)\psi' - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi' + \bar{\psi}'\eta]\\
&= \int d^4x[\bar{\psi}(i\not{D}-m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta \qquad (4.22)\\
&-\partial_\mu\alpha(x)\bar{\psi}\gamma^\mu\gamma_5\psi - 2i\alpha(x)m\bar{\psi}\psi + i\alpha(x)\bar{\eta}\gamma_5\psi + i\alpha(x)\bar{\psi}\gamma_5\eta]
\end{aligned}
$$

By considering terms linear in $\alpha(x)$ and assuming the absence of the Jacobian in the path integral measure,

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi, \qquad (4.23)$$

we obtain the (naive) identity

$$\int d^4x\langle -\partial_\mu\alpha(x)\bar{\psi}\gamma^\mu\gamma_5\psi - 2i\alpha(x)m\bar{\psi}\psi + i\alpha(x)\bar{\eta}\gamma_5\psi + i\alpha(x)\bar{\psi}\gamma_5\eta\rangle_\eta = 0 \qquad (4.24)$$

or

$$\langle\partial_\mu[\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)] - 2im\bar{\psi}(x)\gamma_5\psi(x) + i\bar{\eta}(x)\gamma_5\psi(x) + i\bar{\psi}(x)\gamma_5\eta(x)\rangle_\eta = 0 \qquad (4.25)$$

where the path integral symbol is implicit. By taking the derivative of this relation with respect to $\frac{\delta}{\delta\bar{\eta}(y)}\frac{\delta}{\delta\eta(z)}$ and setting all the source terms to be zero, we obtain the well-known naive chiral WT identity

$$\langle T^{\star}\{\partial_{\mu}[\bar{\psi}(x)\gamma^{\mu}\gamma_{5}\psi(x)]\psi(y)\bar{\psi}(z)-2im\bar{\psi}(x)\gamma_{5}\psi(x)\psi(y)\bar{\psi}(z) \\ +i\delta(x-y)\gamma_{5}\psi(x)\bar{\psi}(z)+i\psi(y)\bar{\psi}(x)\gamma_{5}\delta(x-z)\rangle=0 \tag{4.26}$$

for the two-point Green's function

$$\langle T^{\star}\psi(y)\bar{\psi}(z)\rangle \tag{4.27}$$

Now we evaluate the Jacobian carefully. For this purpose, we define the path integral measure more precisely by expanding the path integral variables in terms of the complete set of eigenfunctions

$$\begin{aligned} &\not{D}\phi_{n}(x)=\lambda_{n}\phi_{n}(x), \\ &\int d^{4}x\phi_{n}^{\dagger}(x)\phi_{m}(x)=\delta_{n,m}, \end{aligned} \tag{4.28}$$

as

$$\begin{aligned} \psi(x)&=\sum_{n}a_{n}\phi_{n}(x)=\sum_{n}\langle x|n\rangle a_{n}, \\ \bar{\psi}(x)&=\sum_{n}\bar{b}_{n}\phi_{n}^{\dagger}(x)=\sum_{n}\bar{b}_{n}\langle n|x\rangle \end{aligned} \tag{4.29}$$

To be precise, we may write $\langle x,\alpha|n\rangle$ by including spinor indices. Now the dynamical variables are a_n and $\bar{b}_n$, which are Grassmann numbers. Note also that the space-time symmetry in Eucliden theory is $SO(4)$ instead of $SO(1,3)$ of Minkowski space. The above change of the integration variables is unitary in the sense of Dirac, and we have

$$\mathcal{D}\psi = \prod_x \frac{\delta}{\delta\psi(x)} = \det|\langle x|n\rangle|^{-1} \prod_n \frac{\delta}{\delta a_n} = \det|\langle x|n\rangle|^{-1} \prod_n da_n \tag{4.30}$$

and similarly

$$\mathcal{D}\bar{\psi} = \prod_x \frac{\delta}{\delta\bar{\psi}(x)} = \det|\langle n|x\rangle|^{-1} \prod_n \frac{\delta}{\delta\bar{b}_n} = \det|\langle n|x\rangle|^{-1} \prod_n d\bar{b}_n \tag{4.31}$$

Note the appearance of the inverse of the ordinary Jacobian for complex numbers, since we define the integral by left-derivative for Grassmann numbers.

We thus have

$$\begin{aligned}
\mathcal{D}\bar{\psi}\mathcal{D}\psi &= \det|\langle n|x\rangle|^{-1}\det|\langle x|n\rangle|^{-1} \prod_n d\bar{b}_n \prod_n da_n \\
&= \det|\textstyle\int d^4x\langle n|x\rangle\langle x|m\rangle|^{-1} \prod_n d\bar{b}_n \prod_n da_n \\
&= \det|\delta_{n,m}| \prod_n d\bar{b}_n \prod_n da_n \\
&= \prod_n d\bar{b}_n \prod_n da_n
\end{aligned} \tag{4.32}$$

and the Dirac action becomes

$$\int d^4x\bar{\psi}(i\not{D} - m)\psi = \sum_n (i\lambda_n - m)\bar{b}_n a_n \tag{4.33}$$

and the fermionic path integral

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{\textstyle\int d^4x[\bar{\psi}(i\not{D} - m)\psi]\}$$

$$
\begin{aligned}
&= \prod_n d\bar{b}_n \prod_n da_n \exp\{\sum_n (i\lambda_n - m)\bar{b}_n a_n\} \\
&= \prod_n (i\lambda_n - m) \\
&= \det|i \not{D} - m| \qquad (4.34)
\end{aligned}
$$

The definition of $\det|i \not{D} - m|$ is given as a product of all the eigenvalues of the operator. Our evaluation of the quadratic (Gaussian) fermionic path integral is exact.

We can use this definition to evaluate the possible Jacobian of the (infinitesimal) chiral transformation:

$$
\psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x) = [1 + i\alpha(x)\gamma_5]\psi(x) \qquad (4.35)
$$

is written as

$$
\psi'(x) = \sum_n a'_n \phi_n(x) = \sum_n a_m [1 + i\alpha(x)\gamma_5]\phi_m(x) \qquad (4.36)
$$

namely,

$$
a'_n = \sum_m a_m \int d^4x \phi^\dagger_n(x)[1 + i\alpha(x)\gamma_5]\phi_m(x) \qquad (4.37)
$$

and

$$
\begin{aligned}
\prod_n da'_n &= \det|\int d^4x \phi^\dagger_n(x)[1 + i\alpha(x)\gamma_5]\phi_m(x)|^{-1} \prod_m da_m \\
&= \exp[-i \sum_n \int d^4x \phi^\dagger_n(x)\alpha(x)\gamma_5\phi_n(x)] \prod_m da_m \qquad (4.38)
\end{aligned}
$$

where we used the fact that the variables $\{a_n\}$ are Grassmann numbers and also the general relation

$$
\det|M| = \exp[\mathrm{Tr} \ln M] \qquad (4.39)
$$

namely,

$$
\begin{aligned}
& \det|\int d^4x\phi_n^\dagger(x)(1+i\alpha(x)\gamma_5)\phi_m(x)|^{-1} \\
= & \exp\{-\mathrm{Tr}\ln[\int d^4x\phi_n^\dagger(x)(1+i\alpha(x)\gamma_5)\phi_m(x)]\} \\
= & \exp\{-\mathrm{Tr}\ln[\delta_{n,m}+i\int d^4x\phi_n^\dagger(x)\alpha(x)\gamma_5\phi_m(x)]\} \\
= & \exp\{-i\int d^4x\phi_n^\dagger(x)\alpha(x)\gamma_5\phi_m(x)]\} \qquad (4.40)
\end{aligned}
$$

for an infinitesimal $\alpha(x)$.

Similarly we have

$$\bar{b}'_n=\sum_m \bar{b}_m\int d^4x\phi_m^\dagger(x)[1+i\alpha(x)\gamma_5]\phi_n(x) \qquad (4.41)$$

and

$$
\begin{aligned}
\prod_n d\bar{b}'_n &= \det|\int d^4x\phi_m^\dagger(x)[1+i\alpha(x)\gamma_5]\phi_n(x)|^{-1}\prod_m d\bar{b}_m \\
&= \exp[-i\sum_n\int d^4x\phi_n^\dagger(x)\alpha(x)\gamma_5\phi_n(x)]\prod_m d\bar{b}_m \qquad (4.42)
\end{aligned}
$$

We thus have

$$\prod_n d\bar{b}'_n\prod_n da'_n=J(\alpha)\prod_n d\bar{b}_n\prod_n da_n \qquad (4.43)$$

with the Jacobian factor

$$J(\alpha)=\exp[-2i\sum_n\int d^4x\phi_n^\dagger(x)\alpha(x)\gamma_5\phi_n(x)] \qquad (4.44)$$

This Jacobian contains the information about the anomaly, as is seen by choosing $\alpha(x) =$ *constant* and using the index relation for Yang-Mills field

$$
\begin{aligned}
J(\alpha) &= \exp[-2i\sum_n\alpha\int d^4x\phi_n^\dagger(x)\gamma_5\phi_n(x)] \\
&= \exp[-2i\alpha(n_+-n_-)] \\
&= \exp[-2i\alpha\int d^4x\frac{g^2}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}] \qquad (4.45)
\end{aligned}
$$

For general case, we evaluate the Jacobian using our previous calculation as

$$J(\alpha) = \lim_{M\to\infty} \exp[-2i \sum_n \int d^4x \alpha(x) \phi_n^\dagger(x)\gamma_5 f(\lambda_n^2/M^2)\phi_n(x)]$$
$$= \exp[-2i \int d^4x \alpha(x) \frac{g^2}{32\pi^2} \mathrm{tr}\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}F_{\alpha\beta}] \qquad (4.46)$$

which is valid for the Abelian case also if one removes the trace over Yang-Mills indices.

The WT identity is derived from

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu] \qquad (4.47)$$
$$\times \exp\{\int d^4x[\bar{\psi}(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi + \bar{\psi}\eta]\}$$
$$= \int \mathcal{D}\bar{\psi}'\mathcal{D}\psi'[\mathcal{D}A_\mu]$$
$$\times \exp\{\int d^4x[\bar{\psi}'(i \not{D} - m)\psi' - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\eta}\psi' + \bar{\psi}'\eta]\}$$

with

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = J(\alpha)\mathcal{D}\bar{\psi}\mathcal{D}\psi \qquad (4.48)$$

and for example,

$$\langle T^\star\{\partial_\mu[\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)]\psi(y)\bar{\psi}(z) - 2im\bar{\psi}(x)\gamma_5\psi(x)\psi(y)\bar{\psi}(z)$$
$$-2i\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}(x)\psi(y)\bar{\psi}(z)$$
$$+i\delta(x-y)\gamma_5\psi(x)\bar{\psi}(z) + i\psi(y)\bar{\psi}(x)\gamma_5\delta(x-z)\}\rangle = 0$$
$$(4.49)$$

or in the operator notation

$$\partial_\mu[\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)] = 2im\bar{\psi}(x)\gamma_5\psi(x) + 2i\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}(x) \tag{4.50}$$

5 Anomalies in gauge symmetry

The formula for the Jacobian is valid for any gauge symmetry including the general coordinate transformation. For example, one may consider

$$S = \int d^4x\bar{\psi}(x)i\gamma^\mu D_\mu\psi(x) \tag{5.1}$$

with

$$D_\mu = \partial_\mu - iV_\mu^a(x)T^a - iA_\mu^a(x)T^a\gamma_5 \tag{5.2}$$

and

$$[T^a, T^b] = if^{abc}T^c. \tag{5.3}$$

This theory is invariant under the gauge symmetry

$$\begin{aligned}
\psi'(x) &= U(\alpha,\beta)\psi(x) = \exp[i\alpha^a(x)T^a + i\gamma_5\beta^a(x)T^a]\psi(x), \\
\bar{\psi}'(x) &= \bar{\psi}(x)\tilde{U}(\alpha,\beta) = \bar{\psi}(x)\exp[-i\alpha^a(x)T^a + i\gamma_5\beta^a(x)T^a],
\end{aligned}$$

$$\begin{aligned}
&\gamma^\mu(\partial_\mu - iV_\mu^a(x)'T^a - iA_\mu^a(x)'T^a\gamma_5) \\
&= \gamma^\mu\exp[i\alpha^a(x)T^a + i\gamma_5\beta^a(x)T^a](\partial_\mu - iV_\mu^a(x)T^a - iA_\mu^a(x)T^a\gamma_5) \\
&\times\exp[-i\alpha^a(x)T^a - i\gamma_5\beta^a(x)T^a]
\end{aligned} \tag{5.4}$$

Since $\not{D}$ is not hermitian in Euclidean sense

$$\begin{aligned}(\Psi, \not{D}\Psi) &= \int d^4x \Psi^\dagger(x) \not{D}\Psi(x) \\ &= \int d^4x [\gamma^\mu(\partial_\mu - iV_\mu^a(x)T^a + iA_\mu^a(x)T^a\gamma_5)\Psi(x)]^\dagger\Psi(x) \\ &\neq (\not{D}\Psi, \Psi) \end{aligned} \qquad (5.5)$$

we may define the hermitian

$$\not{D} = \gamma^\mu((\partial_\mu - iV_\mu^a(x)T^a - A_\mu^a(x)T^a\gamma_5) = \not{D}^\dagger \qquad (5.6)$$

by rotating

$$A_\mu^a(x) \to -iA_\mu^a(x) \qquad (5.7)$$

and rotate back after the calculation. This rotation spoils axial gauge symmetry. We expand

$$\begin{aligned}&\not{D}\varphi_n(x) = \lambda_n\varphi_n(x), \\ &\psi(x) = \sum_n a_n\varphi_n(x), \\ &\bar{\psi}(x) = \sum_n \bar{b}_n\varphi_n^\dagger(x)\end{aligned} \qquad (5.8)$$

and the Jacobian for the gauge transformation is given by

$$J(\alpha, \beta) = \exp\{-2i\sum_n \varphi_n^\dagger(x)\beta^a(x)T^a\gamma_5\varphi_n(x)\} \qquad (5.9)$$

Namely, the vector gauge symmetry parameterized by $\alpha(x)$ is anomaly-free but axial gauge symmetry parameterized by $\beta(x)$ contains the anomaly, which is evaluated by

$$\begin{aligned}&\lim_{M\to\infty}\sum_n \varphi_n^\dagger(x)\beta^a(x)T^a\gamma_5 f(\lambda_n^2/M^2)\varphi_n(x) \\ &= \lim_{M\to\infty}\sum_n \varphi_n^\dagger(x)\beta^a(x)T^a\gamma_5 f(\not{D}^2/M^2)\varphi_n(x)\end{aligned} \qquad (5.10)$$

The calculation is straightforward and gives rise to the so-called *consistent form* of anomaly, but the calculation is very tedious.

Alternative way is to write the covariant derivative as

$$\begin{aligned} \not{D} &= \gamma^\mu(\partial_\mu - iV^a_\mu(x)T^a - iA^a_\mu(x)T^a\gamma_5) \\ &\equiv [\gamma^\mu(\partial_\mu - iL^a_\mu(x)T^a)](\frac{1-\gamma_5}{2}) + [\gamma^\mu(\partial_\mu - iR^a_\mu(x)T^a)](\frac{1+\gamma_5}{2}) \end{aligned} \quad (5.11)$$

with

$$\begin{aligned} R^a_\mu(x)T^a) &= V^a_\mu(x)T^a + A^a_\mu(x)T^a, \\ L^a_\mu(x)T^a) &= V^a_\mu(x)T^a - A^a_\mu(x)T^a, \end{aligned} \quad (5.12)$$

and

$$S = \int d^4x\bar{\psi}(x)i\not{D}\psi(x). \quad (5.13)$$

The gauge transformation is

$$\begin{aligned} \psi'(x) &= \exp[i\alpha^a_L(x)T^a(\frac{1-\gamma_5}{2}) + i\alpha^a_R(x)T^a(\frac{1+\gamma_5}{2})]\psi(x), \\ \bar{\psi}'(x) &= \bar{\psi}(x)\exp[-i\alpha^a_L(x)T^a(\frac{1+\gamma_5}{2}) - i\alpha^a_R(x)T^a(\frac{1-\gamma_5}{2})] \end{aligned} \quad (5.14)$$

We define hermitian operators

$$\not{D}^\dagger\not{D} = [\gamma^\mu(\partial_\mu - iL^a_\mu(x)T^a)]^2(\frac{1-\gamma_5}{2}) + [\gamma^\mu(\partial_\mu - iR^a_\mu(x)T^a)]^2(\frac{1+\gamma_5}{2})$$

$$
\not{D}\,\not{D}^\dagger = [\gamma^\mu(\partial_\mu - iL^a_\mu(x)T^a)]^2(\frac{1+\gamma_5}{2}) + [\gamma^\mu(\partial_\mu - iR^a_\mu(x)T^a)]^2(\frac{1-\gamma_5}{2}) \tag{5.15}
$$

and expand

$$
\begin{aligned}
&\not{D}^\dagger\,\not{D}\phi_n = \lambda_n^2\phi_n, \\
&\psi(x) = \sum_n a_n\phi_n(x), \\
&\not{D}\,\not{D}^\dagger\varphi_n = \lambda_n^2\varphi_n, \\
&\bar{\psi}(x) = \sum_n \bar{b}_n\varphi_n^\dagger(x)
\end{aligned} \tag{5.16}
$$

with

$$
\int d^4x\bar{\psi}(x)i\,\not{D}\psi(x) = \sum_n i\lambda_n\bar{b}_n a_n \tag{5.17}
$$

by noting $\not{D}\phi_n = \lambda_n\varphi_n(x)$, namely, the action is exactly diagonalized.

The Jacobian of the gauge transformation is

$$
\begin{aligned}
\ln J &= -\sum_n\int d^4x\{\phi_n^\dagger[i\alpha^a_L(x)T^a(\frac{1-\gamma_5}{2}) + i\alpha^a_R(x)T^a(\frac{1+\gamma_5}{2})]\exp[-\frac{\not{D}^\dagger\,\not{D}}{M^2}] \\
&\quad -\varphi_n^\dagger[i\alpha^a_L(x)T^a(\frac{1+\gamma_5}{2}) + i\alpha^a_R(x)T^a(\frac{1-\gamma_5}{2})]\exp[-\frac{\not{D}\,\not{D}^\dagger}{M^2}]\varphi_n\} \\
&= -\sum_n\int d^4x\{\phi_n^\dagger[i\alpha^a_L(x)T^a(\frac{1-\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iL^a_\mu(x)T^a)]^2}{M^2}} \\
&\quad +i\alpha^a_R(x)T^a(\frac{1+\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iR^a_\mu(x)T^a)]^2}{M^2}}]\phi_n^\dagger \\
&\quad -\varphi_n^\dagger[i\alpha^a_L(x)T^a(\frac{1+\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iL^a_\mu(x)T^a)]^2}{M^2}}]
\end{aligned}
$$

$$
\begin{aligned}
& -i\alpha_R^a(x)T^a(\frac{1-\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iR_\mu^a(x)T^a)]^2}{M^2}}]\varphi_n\} \\
= \ & -\mathrm{Tr}\int d^4x \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\{[i\alpha_L^a(x)T^a(\frac{1-\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iL_\mu^a(x)T^a)]^2}{M^2}} \\
& +i\alpha_R^a(x)T^a(\frac{1+\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iR_\mu^a(x)T^a)]^2}{M^2}}] \\
& -[i\alpha_L^a(x)T^a(\frac{1+\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iL_\mu^a(x)T^a)]^2}{M^2}}] \\
& -i\alpha_R^a(x)T^a(\frac{1-\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iR_\mu^a(x)T^a)]^2}{M^2}}]e^{ikx}\} \\
= \ & -\mathrm{Tr}\int d^4x \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\{[-i\alpha_L^a(x)T^a\gamma_5 e^{-\frac{[\gamma^\mu(\partial_\mu - iL_\mu^a(x)T^a)]^2}{M^2}} \\
& +i\alpha_R^a(x)T^a\gamma_5 e^{-\frac{[\gamma^\mu(\partial_\mu - iR_\mu^a(x)T^a)]^2}{M^2}}]e^{ikx}\} \\
= \ & -\int d^4x \frac{1}{32\pi^2}\mathrm{tr}[-i\alpha_L^a(x)T^a\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(L)F_{\alpha\beta}(L) \\
& +i\alpha_R^a(x)T^a\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(R)F_{\alpha\beta}(R)] \\
= \ & -\int d^4x \frac{1}{32\pi^2}\frac{1}{2}[-i\alpha_L^a(x)\mathrm{tr}T^a\{T^b,T^c\}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}^b(L)F_{\alpha\beta}^c(L) \\
& \qquad +i\alpha_R^a(x)\mathrm{tr}T^a\{T^b,T^c\}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}^b(R)F_{\alpha\beta}^c(R)] \qquad (5.1
\end{aligned}
$$

where we used $f(x) = e^{-x}$ and the ralation, for example,

$$
(\frac{1+\gamma_5}{2})\exp[-\frac{\not{D}^\dagger \not{D}}{M^2}] = (\frac{1+\gamma_5}{2})e^{-\frac{[\gamma^\mu(\partial_\mu - iR_\mu^a(x)T^a)]^2}{M^2}} \qquad (5.19)
$$

since $\frac{1\pm\gamma_5}{2}$ are projection operators. The anomaly relations are

$$
D_\mu[\bar{\psi}(x)T^a\gamma^\mu\frac{1+\gamma_5}{2}\psi(x)]
$$

$$
\begin{aligned}
&\equiv [\delta^{ac}\partial_\mu + f^{abc}R^b_\mu][\bar{\psi}(x)T^c\gamma^\mu\frac{1+\gamma_5}{2}\psi(x)] \\
&= i\frac{1}{32\pi^2}[\mathrm{tr}T^a\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(R)F_{\alpha\beta}(R)] \qquad (5.20)
\end{aligned}
$$

and

$$
\begin{aligned}
&D_\mu[\bar{\psi}(x)T^a\gamma^\mu\frac{1-\gamma_5}{2}\psi(x)] \\
&\equiv [\delta^{ac}\partial_\mu + f^{abc}L^b_\mu][\bar{\psi}(x)T^c\gamma^\mu\frac{1-\gamma_5}{2}\psi(x)] \\
&= -i\frac{1}{32\pi^2}[\mathrm{tr}T^a\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(L)F_{\alpha\beta}(L)] \qquad (5.21)
\end{aligned}
$$

The resulting anomaly has a covariant form and for this reason it is called "covariant anomaly".

The *anomaly cancellation condition* is

$$
\mathrm{tr}T^a\{T^b, T^c\} = 0 \qquad (5.22)
$$

by considering the case $R^a_\mu(x) = 0$, for example, or if the same gauge field couples to both of the left- and right-handed currents with in general different representations, then the cancellation condition is

$$
\mathrm{tr}T^a_R\{T^b_R, T^c_R\} - \mathrm{tr}T^a_L\{T^b_L, T^c_L\} = 0. \qquad (5.23)
$$

An important application of the covariant anomaly is the fermion number anomaly in the Standard Model by considering

$$
\not{D} = [\gamma^\mu(\partial_\mu - iW^a_\mu(x)T^a)](\frac{1-\gamma_5}{2}) + [\gamma^\mu\partial_\mu](\frac{1+\gamma_5}{2})
$$

$$(5.24)$$

with the action for an $SU(2)$ doublet of fermions

$$S = \int d^4x \bar{\psi}\{i[\gamma^\mu(\partial_\mu - iW^a_\mu(x)T^a)](\frac{1-\gamma_5}{2}) + i[\gamma^\mu\partial_\mu](\frac{1+\gamma_5}{2})\}\psi, \tag{5.25}$$

and the fermion number transformation

$$\begin{aligned} \psi'(x) &= \exp[i\alpha(x)(\frac{1-\gamma_5}{2}) + i\alpha(x)(\frac{1+\gamma_5}{2})]\psi(x), \\ \bar{\psi}'(x) &= \bar{\psi}(x)\exp[-i\alpha(x)(\frac{1+\gamma_5}{2}) - i\alpha(x)(\frac{1-\gamma_5}{2})] \end{aligned} \tag{5.26}$$

The anomaly factor is

$$\ln J(\alpha) = i\int d^4x \frac{1}{32\pi^2}\alpha(x)\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(W)F_{\alpha\beta}(W) \tag{5.27}$$

and the fermion (quark) number non-conservation

$$\partial_\mu J^\mu_{quark}(x) = -i\frac{1}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(W)F_{\alpha\beta}(W) \tag{5.28}$$

and similarly, the lepton number non-conservation

$$\partial_\mu J^\mu_{lepton}(x) = -i\frac{1}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(W)F_{\alpha\beta}(W) \tag{5.29}$$

If one remembers that quark carries 1/3 of a baryon number with 3 color freedom, one concludes that

$$\Delta B - \Delta L = 0. \tag{5.30}$$

Namely, $B - L$ is anomaly-free and conserved.

This analysis is not simple in the consistent V_μ and A_μ form of anomaly.

6 Further aspects of anomalies

We have discussed the basic idea of quantum anomaly and its evaluation in the path intergral formulation. We covered only the basic aspects and we have not discussed most of interesting applications. Also we have not discussed issues related to:

1. The Weyl anomaly and renormalization group
2. Two-dimensional field theory and bosonization
3. Index theorem on the lattice and chiral anomalies
4. Gravitational anomalies

If you are ineterested in the detailed discussions of those issues, please be referred to the textbook in References, where many original references are found.

References

[1] K. Fujikawa and H. Suzuki, "Path Integrals and Quantum Anomalies" (Clarendon Press, Oxford, 2004).