index & supersymmetry

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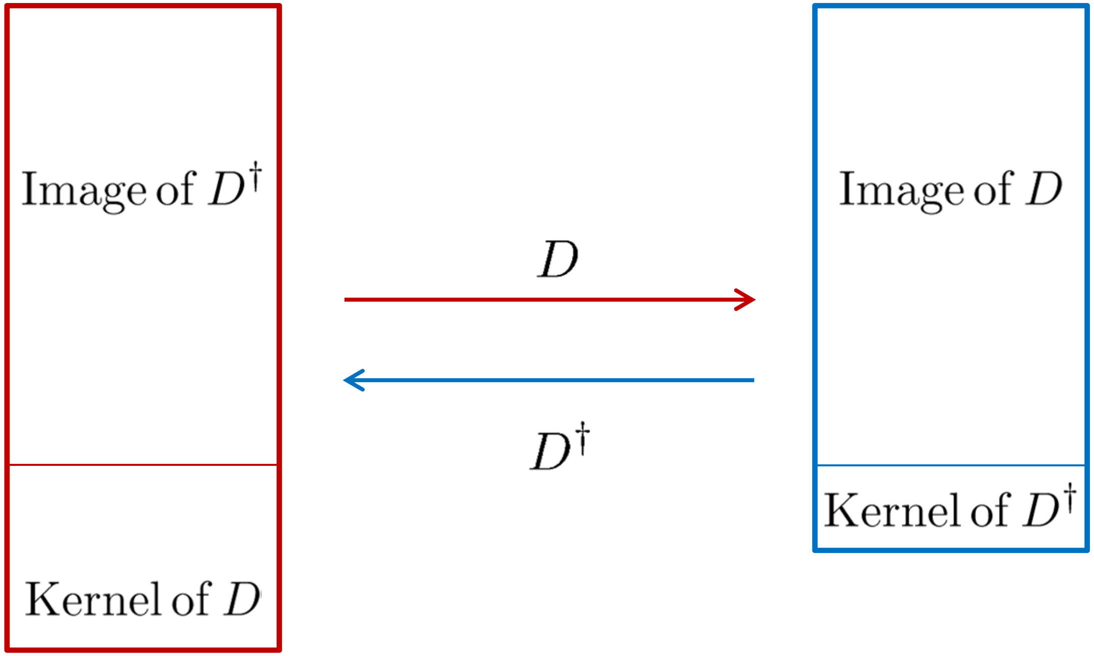
what is an index ?

index

$$\mathrm{Index}(D) = \dim[\text{Kernel of } D] - \dim[\text{Kernel of } D^\dagger]$$



prototype : supersymmetric harmonic oscillators

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1 \qquad\qquad b^2 = 0 = (b^\dagger)^2$$

$$H = \hbar w \left[\left(a^\dagger a + aa^\dagger\right)/2 + \left(b^\dagger b - bb^\dagger\right)/2\right] = \hbar w \left(a^\dagger a + 1/2\right) + \hbar w \left(b^\dagger b - 1/2\right)$$

$$= H_B + H_F$$

prototype : supersymmetric harmonic oscillators

$$\mathrm{Index}(D) = \#(\text{bosonic vacua}) - \#(\text{fermionic vacua}) = 1$$

bosonic

$$\left(a^\dagger\right)^{n+1} |0\rangle_B \otimes |0\rangle_F$$

$$|0\rangle_B \otimes |0\rangle_F$$

$$D \sim b^\dagger a$$

$$D^\dagger \sim a^\dagger b$$

fermionic

$$\left(a^\dagger\right)^{n} |0\rangle_B \otimes b^\dagger |0\rangle_F$$

prototype : supersymmetric harmonic oscillators

$$\mathrm{Index}(D) = \#(\text{bosonic vacua}) - \#(\text{fermionic vacua}) = -1$$

fermionic

$$\left(a^\dagger\right)^{n+1} |0\rangle_B \otimes |0\rangle_F$$

$$|0\rangle_B \otimes |0\rangle_F$$

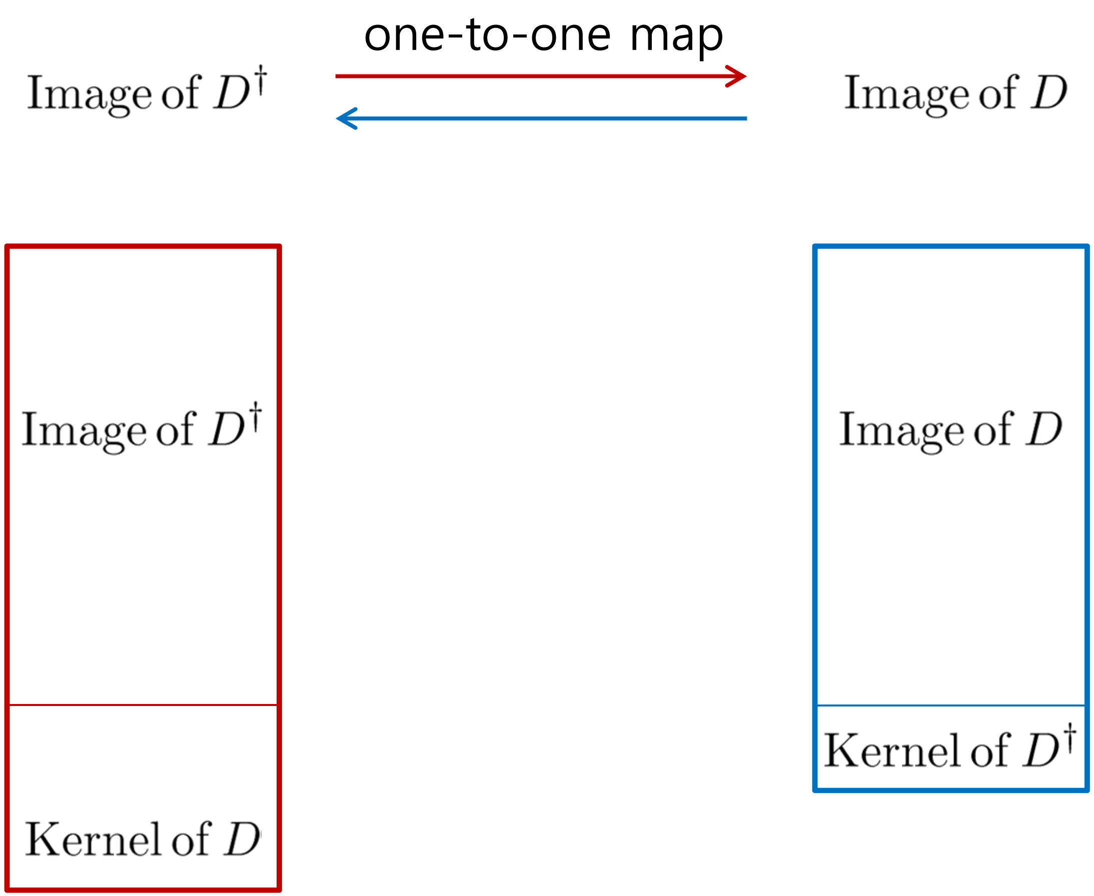
$$D \sim b^\dagger a$$

$$D^\dagger \sim a^\dagger b$$

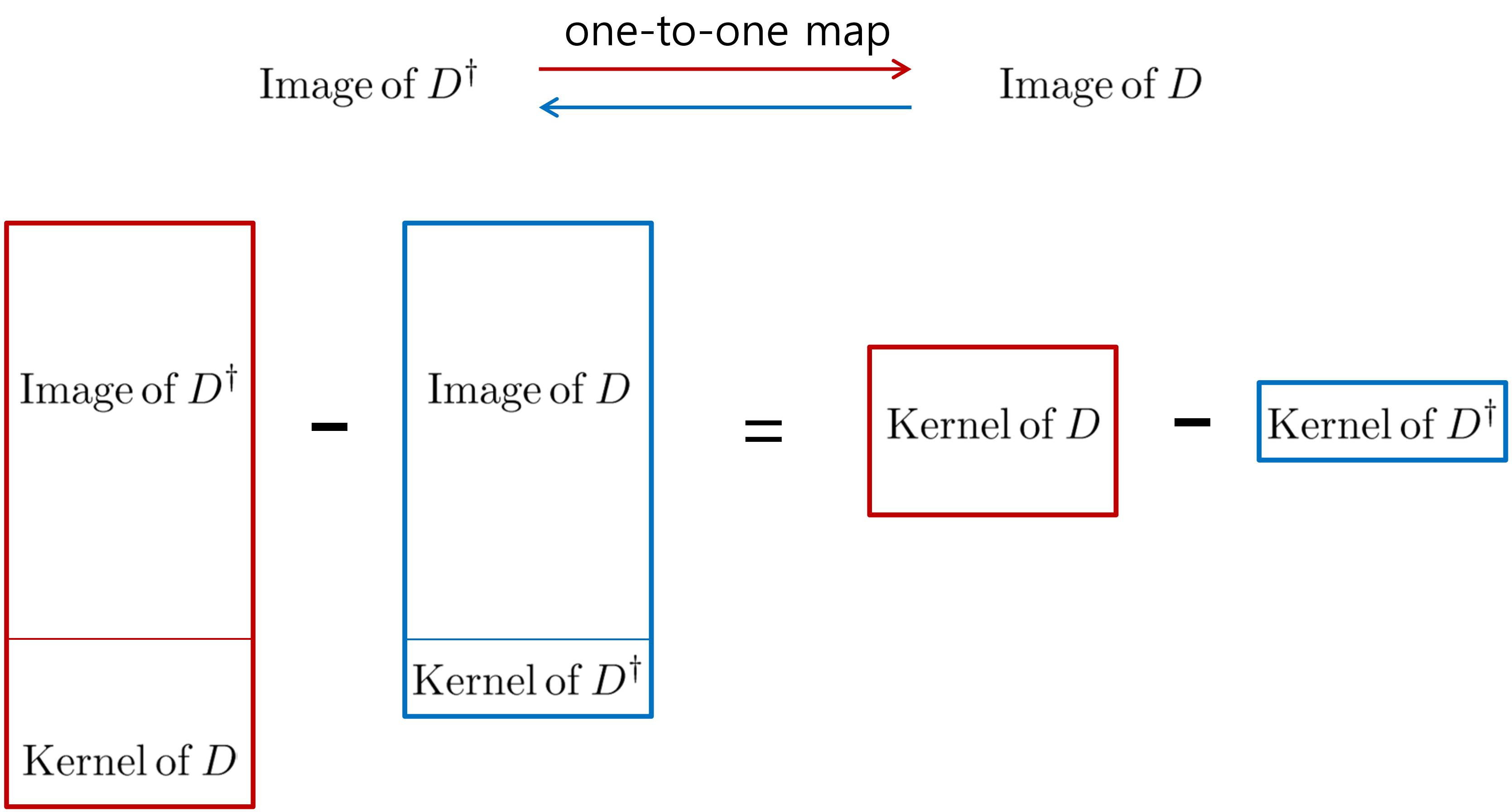
bosonic

$$\left(a^\dagger\right)^{n} |0\rangle_B \otimes b^\dagger |0\rangle_F$$

index


one-to-one map
Image of $D^\dagger$
Image of D
Image of $D^\dagger$
Kernel of D
Image of D
Kernel of $D^\dagger$

index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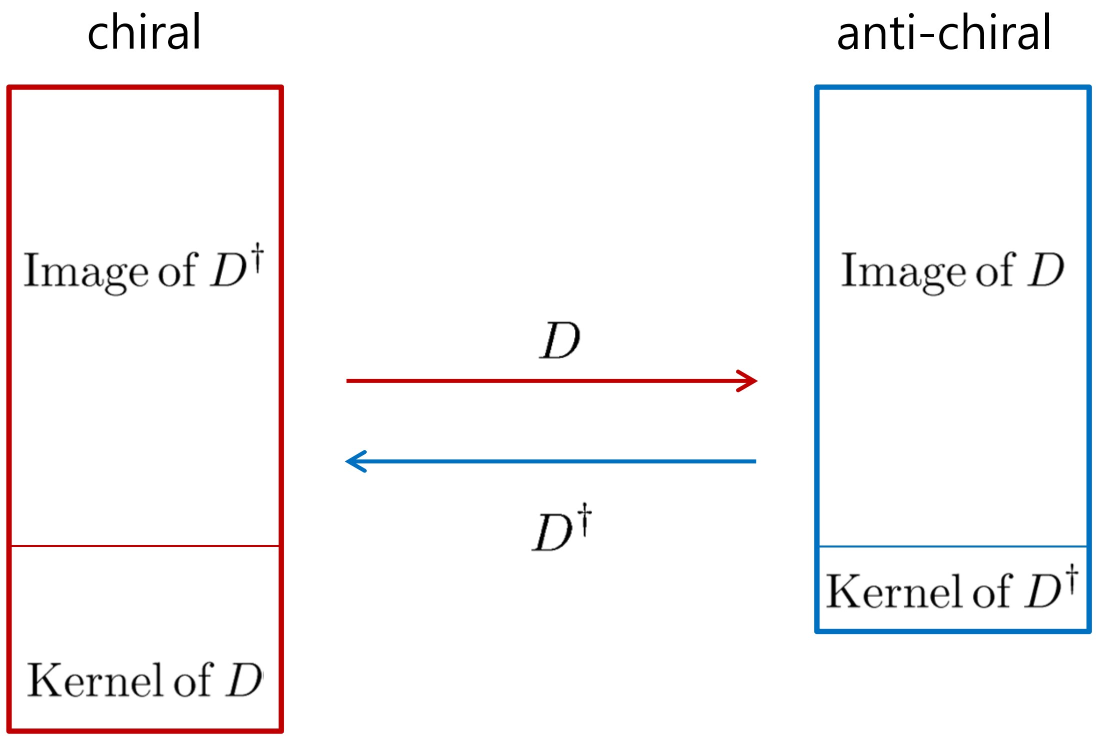
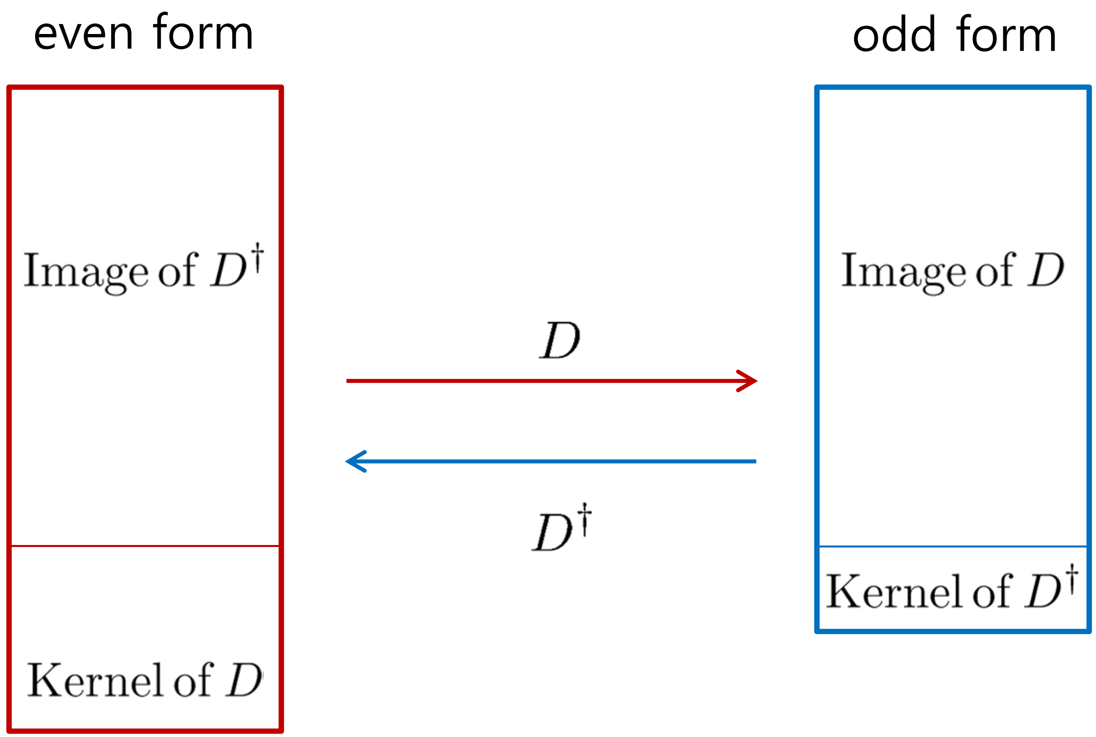
index theorems & elliptic operators

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = \gamma^a \nabla_a \qquad\qquad \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a^\dagger & 0 \end{pmatrix}$$



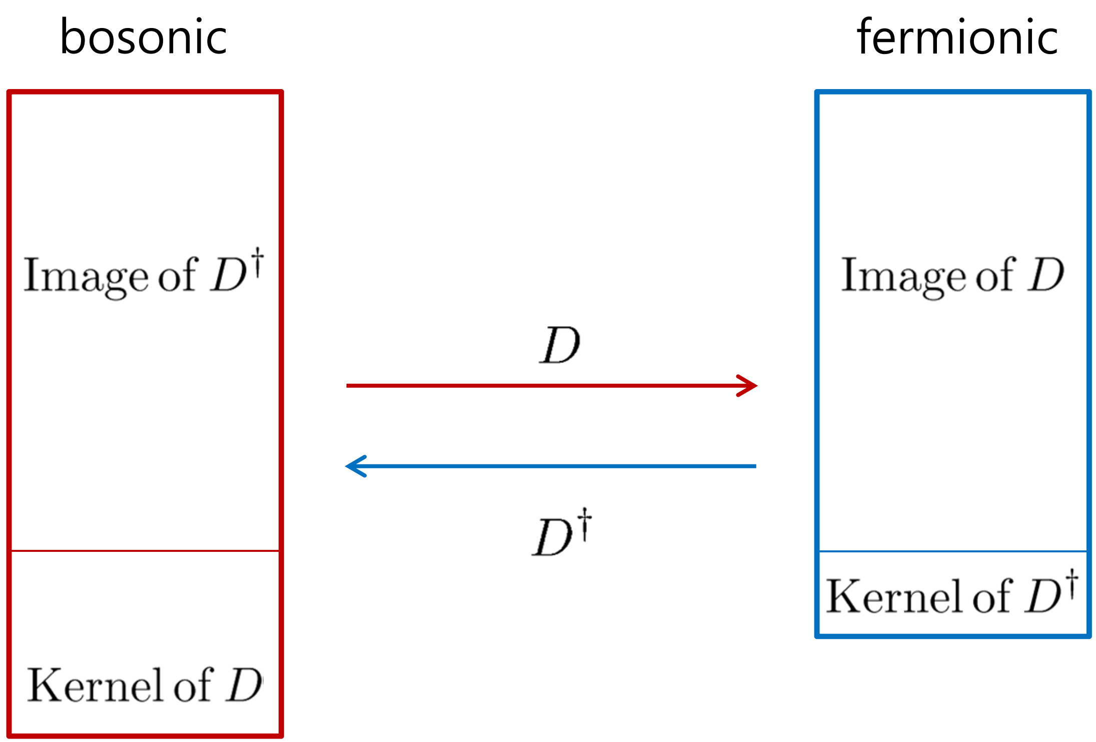
index theorems & elliptic operators

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = d + d^\dagger \qquad\qquad d^\dagger = (-1)^{\#} * d *$$



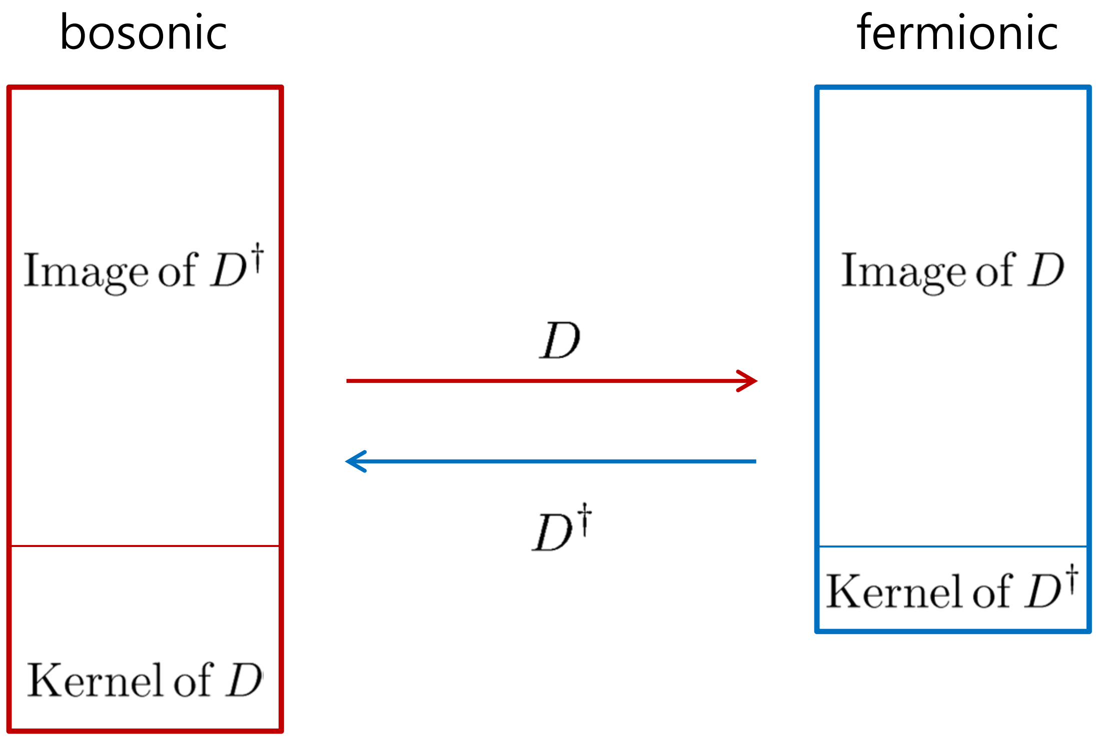
index theorems & supersymmetry

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = e^{i\alpha}\sqrt{i}Q + e^{-i\alpha}\sqrt{-i}Q^\dagger$$



index theorems & supersymmetry

$$D^\dagger D = \{Q, Q^\dagger\} - e^{2i\alpha} QQ - e^{-2i\alpha} Q^\dagger Q^\dagger = 2\left(H - \mathrm{Re}(e^{2i\alpha} Z)\right)\Big|_{\text{on bosonic}}$$



index theorems & supersymmetry

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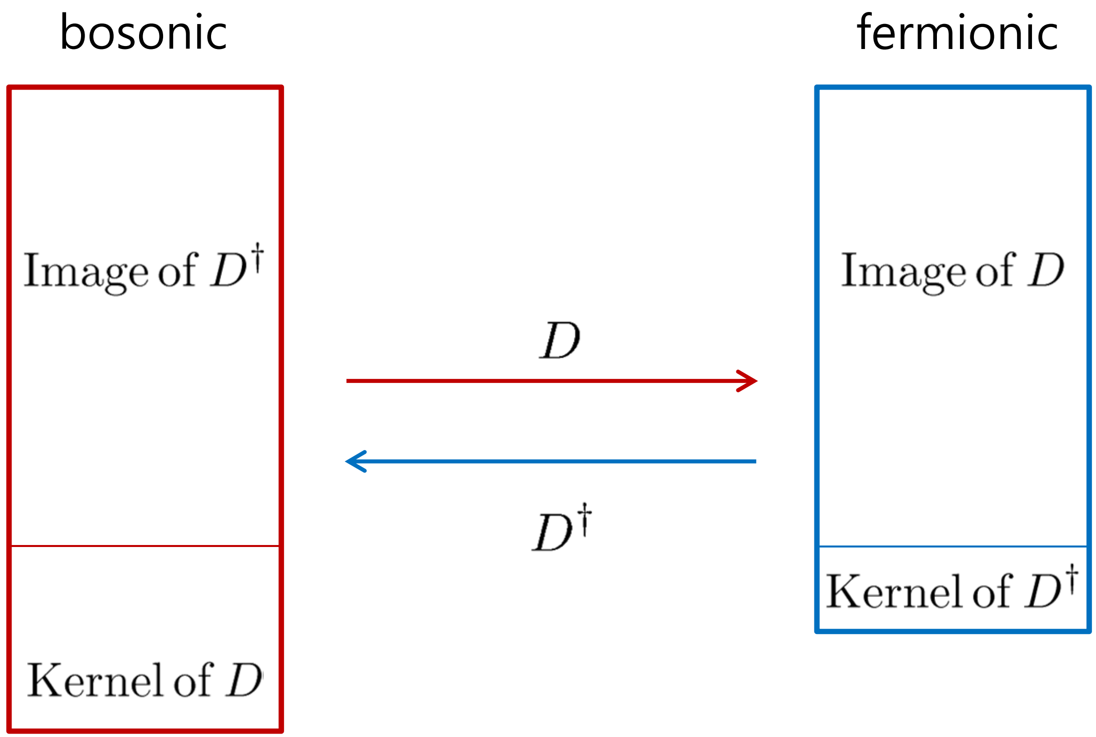
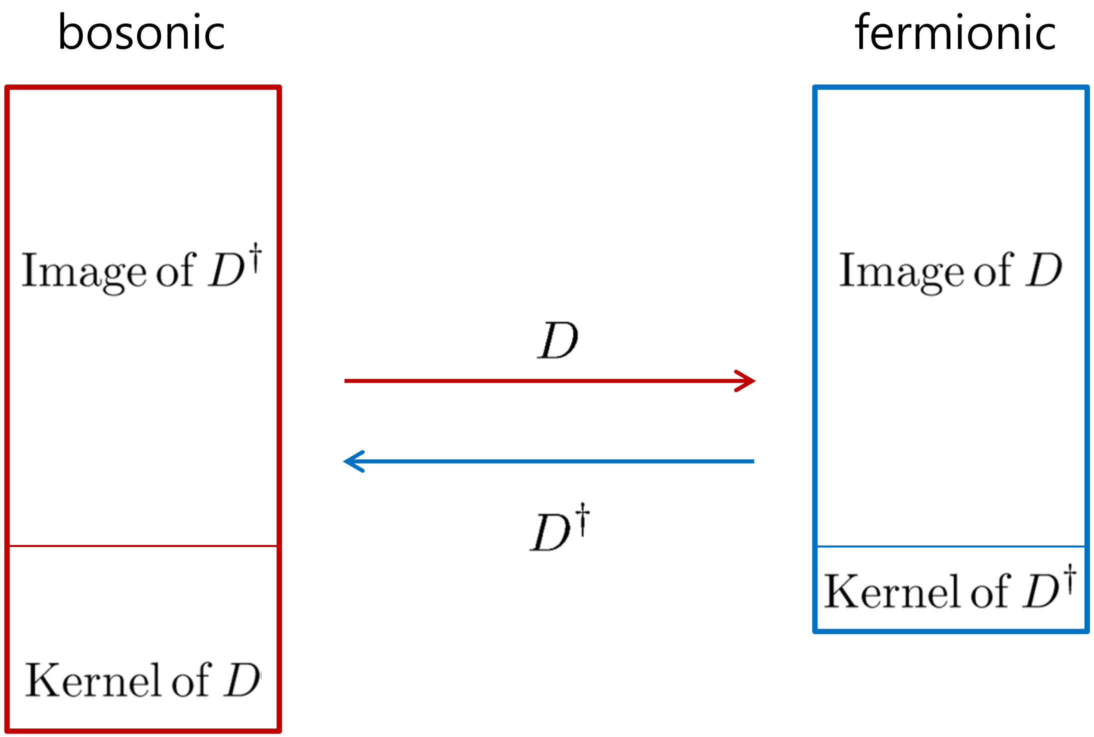


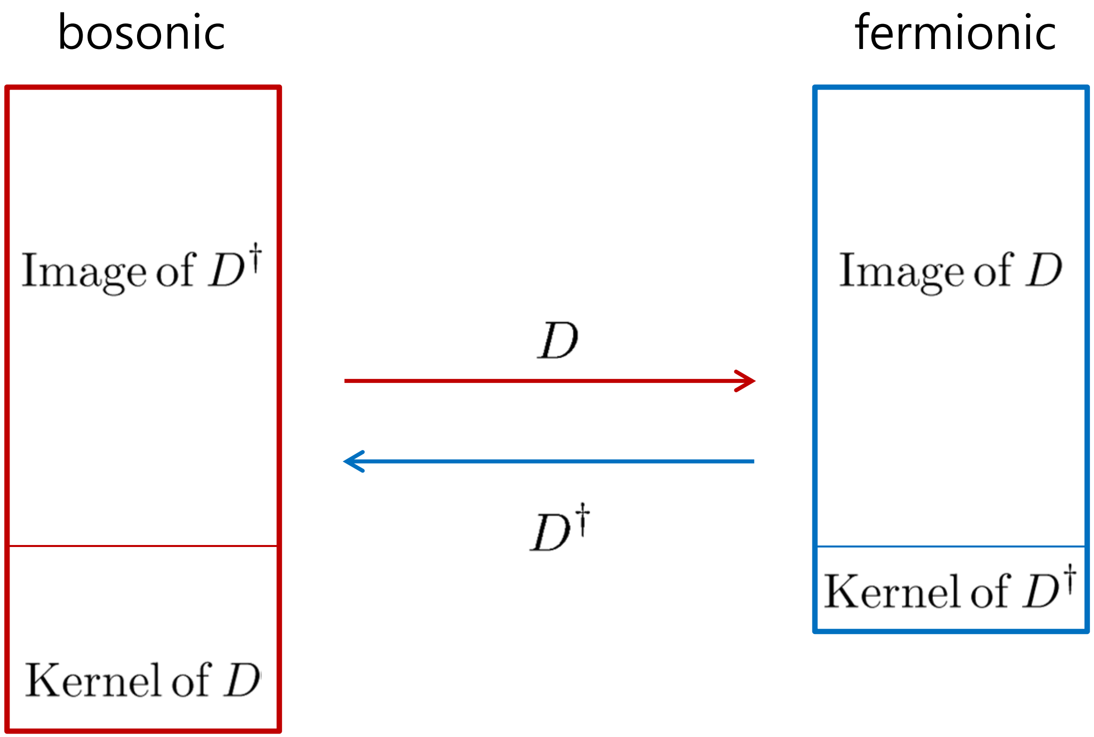
index theorems & supersymmetry

$$\mathrm{Index}(D) = \mathrm{tr}_{\mathrm{bosonic}} 1 - \mathrm{tr}_{\mathrm{fermionic}} 1 = \mathrm{tr}\left[(-1)^F\right]$$



index theorems & supersymmetry

$$\mathrm{Index}(D) = \mathrm{tr}\left[(-1)^F\right] = \#(\text{bosonic vacua}) - \#(\text{fermionic vacua})$$



so, how does one computes such things ?

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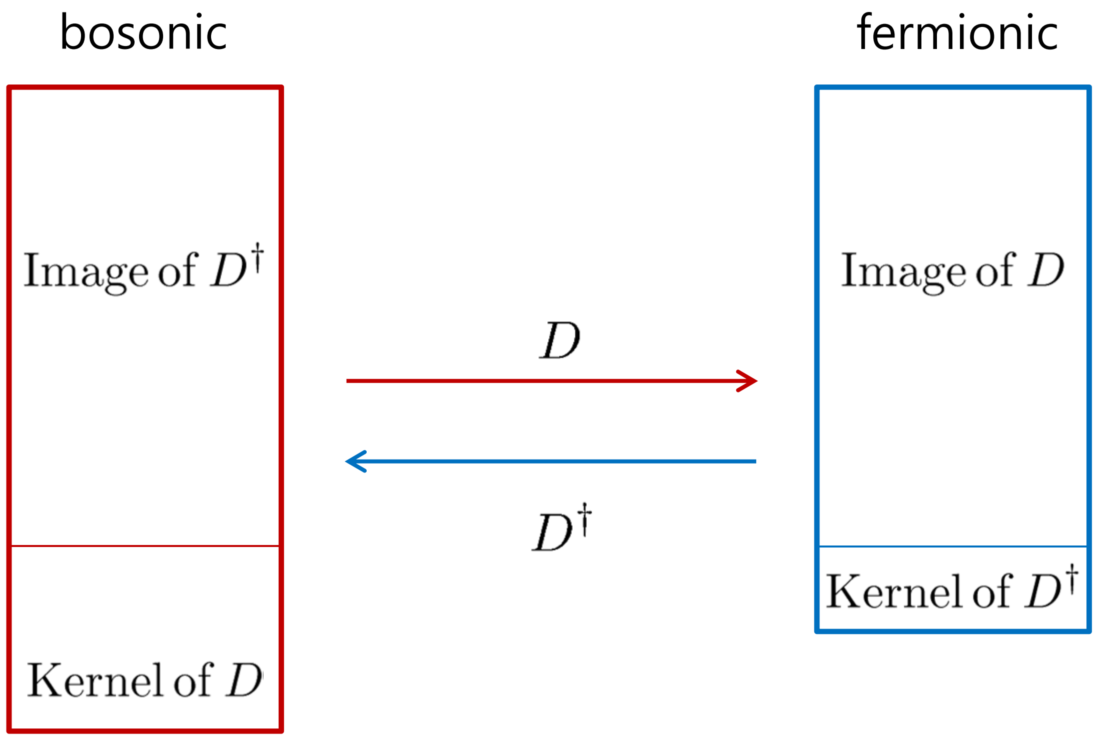

back to the supersymmetric harmonic oscillators

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1$$

$$H = \hbar w \left[\left(a^\dagger a + aa^\dagger \right) /2 + \left(b^\dagger b - bb^\dagger \right) /2 \right] = \hbar w \left(a^\dagger a + 1/2 \right) + \hbar w \left(b^\dagger b - 1/2 \right)$$

$$= H_B + H_F$$

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$$= H_B + H_F$$

$$Z = \mathrm{tr}\left[e^{-\beta H} \right] = \mathrm{tr}_B e^{-\beta H_B} \times \mathrm{tr}_F e^{-\beta H_F}$$

$$= (1/2sinh(\beta\hbar w/2)) \times 2\cosh(\beta\hbar w/2)$$

$$= 1/\tanh(\beta\hbar w/2)$$

so, how does one compute such things ?

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$$= H_B + H_F$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr} e^{-\beta H_B} \times \mathrm{tr}(-1)^F e^{-\beta H_F}$$

$$= (1/2sinh(\beta\hbar w/2)) \times 2\sinh(\beta\hbar w/2)$$

$$= 1 \qquad = \mathrm{tr}\left[(-1)^F\right]$$

so, how does one compute such things ?

$$\mathrm{tr}\left[(-1)^F\right] \quad = \quad \lim_{\beta\to 0} \boxed{\begin{aligned} &\mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \\ &= Z_{twisted}(\beta) \end{aligned}}$$

path integrals versus (twisted) partition functions

so, how does one compute such things without knowing the spectrum ?

$$H_B \;\leftarrow\quad L_B = \frac{1}{2}\left(\dot{x}^2 - w^2 x^2\right)$$

$$H_F \;\leftarrow\quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

so, how does one compute such things without knowing the spectrum ?

$$H_B \;\leftarrow\quad L_B = \tfrac{1}{2}\left(\dot{x}^2 - w^2 x^2\right)$$

$$\mathrm{tr}\left[e^{-\beta H_B}\right] = \int [dx]_{\mathrm{periodic\,BC}}\, e^{-\int_0^\beta L_B^{Euclidean} d\tau}$$

$$x = \hat{x}_0/\sqrt{\beta} + \textstyle\sum_n \hat{x}_n e^{2\pi i n\tau/\beta}/\sqrt{\beta}$$

$$\sim \int d\hat{x}_0 e^{-\frac{1}{2}w^2\hat{x}_0^2} \textstyle\prod_{n>0} \int d\hat{x}_n d\hat{x}_{-n}\, e^{-\frac{1}{2}((2\pi n/\beta)^2 + w^2)\hat{x}_n\hat{x}_{-n}}$$

$$\sim 1/w \times 1/\left(\textstyle\prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right)\right)$$

so, how does one compute such things without knowing the spectrum ?

$$H_B \;\leftarrow\quad L_B = \frac{1}{2}\left(\dot{x}^2 - w^2 x^2\right)$$

$$\mathrm{tr}\left[e^{-\beta H_B}\right] = \int [dx]_{\text{periodic BC}}\, e^{-\int_0^\beta L_B^{Euclidean}\, d\tau}$$

$$= 1/w \times 1/\prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right)$$

$$= 1/\left(\prod_{n\in\mathbf{Z}}\left((2\pi n/\beta)^2 + w^2\right)\right)^{1/2}$$

$$= 1/\sqrt{Det\left(-\partial_\tau^2 + w^2\right)}$$

so, how does one compute such things without knowing the spectrum ?

$$H_B \;\leftarrow\; \quad L_B = \frac{1}{2}\left(\dot{x}^2 - w^2 x^2\right)$$

$$\mathrm{tr}\left[e^{-\beta H_B}\right] = 1/w \times 1/\underbrace{\boxed{\prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right)}}_{=\sqrt{Det'\left(-\partial_\tau^2 + w^2\right)}}$$

$$= 1/w \times \left[1/\prod_{n>0}(2\pi n/\beta)^2\right]_{\zeta\, regularized}$$

$$\times \underbrace{\boxed{\prod_{n>0}(2\pi n/\beta)^2}}_{=\sqrt{Det'\left(-\partial_\tau^2\right)}} / \underbrace{\boxed{\prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right)}}_{=\sqrt{Det'\left(-\partial_\tau^2 + w^2\right)}}$$

so, how does one compute such things without knowing the spectrum ?

$$\zeta(s) = \sum_{n\geq 1} n^{-s} \quad \rightarrow$$

$$\zeta(0) = -1/2$$

$$\zeta'(0) = -\log(\sqrt{2\pi})$$

$$\mathrm{tr}\left[e^{-\beta H_B}\right] = 1/w \times 1/\prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right)$$

$$= 1/w \times \boxed{\left[1/\prod_{n>0}(2\pi n/\beta)^2\right]_{\zeta\, regularized}}$$

$$\times \boxed{\prod_{n>0}(2\pi n/\beta)^2/\prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right)}$$

$$= 1/w \times \boxed{1/\beta} \times \boxed{\frac{\beta w/2}{\sinh(\beta w/2)}} = 1/2\sinh(\beta w/2) \quad = \mathrm{tr}\left[e^{-\beta H_B}\right]!$$

so, how does one compute such things without knowing the spectrum ?

$$\zeta(s,q) = \sum_{n\geq 0}(n+q)^{-s} \quad \to \quad \begin{array}{l} \zeta(0,q) = 1/2 - q \\ \\ \partial_s\zeta(0,q) = \log(\Gamma(q)) - \log(\sqrt{2\pi}) \end{array}$$

$$\left[\prod_{n\geq 0}(2\pi(n+q)/\beta)^2\right]_\zeta = \exp\left(2\sum_{n\geq 0}\log(2\pi(n+q)/\beta\right)$$

$$\to \lim_{s\to 0}\exp\left(2\sum_{n\geq 0}\log(2\pi(n+q)/\beta)(n+q)^{-s}\right)$$

$$= \lim_{s\to 0}\exp\left(2\log(2\pi/\beta)\zeta(0,q) - 2\partial_s\zeta(0,q)\right)$$

$$= (2\pi)^{2(1-q)}\beta^{2q-1}/\Gamma(q)^2 \quad = \quad \left(\begin{array}{ll} \boxed{\beta \qquad\qquad q=1} \\ \\ 2 \qquad\qquad q=1/2 \end{array}\right.$$

so, how does one compute such things without knowing the spectrum ?

$$H_F \;\leftarrow\; \quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

$$\int [d\psi^\dagger d\psi]_{\mathrm{which\,BC\,?}}\, e^{-\int_0^\beta L_F^{Euclidean} d\tau}$$

$$= Det\,(\partial_\tau + w)_{\mathrm{which\,BC\,?}}$$

$$= \left(\begin{array}{ll} \prod_{n\in\mathbf{Z}} (2\pi i n/\beta + w) & \mathrm{periodic\,BC} \\ \prod_{n\in\mathbf{Z}} (2\pi i(n+1/2)/\beta + w) & \mathrm{antiperiodic\,BC} \end{array} \right.$$

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$$H_F \;\leftarrow\quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

$$\int [d\psi^\dagger d\psi]_{\mathrm{which\,BC\,?}}\, e^{-\int_0^\beta L_F^{Euclidean} d\tau}$$

$$= Det\,(\partial_\tau + w)_{\mathrm{which\,BC\,?}}$$

$$= \left(\begin{array}{ll} \omega \times \prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right) & \mathrm{periodic\,BC} \\ \prod_{n\geq 0}\left((2\pi(n+1/2)/\beta)^2 + w^2\right) & \mathrm{antiperiodic\,BC} \end{array}\right.$$

so, how does one compute such things without knowing the spectrum ?

$$H_F \;\leftarrow\quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

$$\int [d\psi^\dagger d\psi]_{\text{which BC ?}}\, e^{-\int_0^\beta L_F^{Euclidean} d\tau}$$

$$= Det\,(\partial_\tau + w)_{\text{which BC ?}}$$

$$= \left(\begin{array}{l} \omega \times \prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right) / \prod_{n>0}(2\pi n/\beta)^2 \times \prod_{n>0}(2\pi n/\beta)^2 \\ \boxed{\prod_{n\geq 0}\left((2\pi(n+1/2)/\beta)^2 + w^2\right) / \prod_{n\geq 0}(2\pi(n+1/2)/\beta)^2} \times \boxed{\prod_{n\geq 0}(2\pi(n+1/2)/\beta)^2} \end{array} \right.$$

so, how does one compute such things without knowing the spectrum ?

$$\zeta(s,q) = \sum_{n\geq 0}(n+q)^{-s} \quad \rightarrow \quad \begin{array}{l} \zeta(0,q) = 1/2 - q \\ \partial_s\zeta(0,q) = \log(\Gamma(q)) - \log(\sqrt{2\pi}) \end{array}$$

$$\left[\prod_{n\geq 0}(2\pi(n+q)/\beta)^2\right]_\zeta = \exp\left(2\sum_{n\geq 0}\log(2\pi(n+q)/\beta\right)$$

$$\rightarrow \lim_{s\rightarrow 0}\exp\left(2\sum_{n\geq 0}\log(2\pi(n+q)/\beta)(n+q)^{-s}\right)$$

$$= \lim_{s\rightarrow 0}\exp\left(2\log(2\pi/\beta)\zeta(0,q) - 2\partial_s\zeta(0,q)\right)$$

$$= (2\pi)^{2(1-q)}\beta^{2q-1}/\Gamma(q)^2 \quad = \quad \left(\begin{array}{ll} \beta & q=1 \\ \boxed{2 \qquad q=1/2} \end{array}\right.$$

so, how does one compute such things without knowing the spectrum ?

$$H_F \;\leftarrow\quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

$$\int [d\psi^\dagger d\psi]_{\text{which BC ?}}\, e^{-\int_0^\beta L_F^{Euclidean} d\tau}$$

$$= Det\,(\partial_\tau + w)_{\text{which BC ?}}$$

$$= \left\{ \begin{array}{ll} 2\sinh(\beta w/2) & \text{periodic BC} \\ \boxed{2}\,\boxed{\cosh(\beta w/2)} & \text{antiperiodic BC} \end{array} \right\}$$

so, how does one compute such things without knowing the spectrum ?

$$H_F \;\leftarrow\quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

$$\int [d\psi^\dagger d\psi]_{\text{which BC ?}}\, e^{-\int_0^\beta L_F^{Euclidean} d\tau}$$

$$= Det\,(\partial_\tau + w)_{\text{which BC ?}}$$

$$= \left\{\begin{array}{ll} 2\sinh(\beta w/2) & \text{periodic BC} \\ \\ 2\cosh(\beta w/2) & \text{antiperiodic BC} \end{array}\right\} = \left\{\begin{array}{l} \mathrm{tr}(-1)^F e^{-\beta H_F} \\ \\ \mathrm{tr} e^{-\beta H_F} \end{array}\right\}$$

therefore,

$$H_B + H_F \;\leftarrow\; \quad L_B + L_F$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr} e^{-\beta H_B} \times \mathrm{tr}(-1)^F e^{-\beta H_F}$$

$$= \int [dx\, d\psi^\dagger d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

generally,

$$H_{SUSY} \ \leftarrow \quad L_{SUSY}$$

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generally,

$$H_{SUSY} \leftarrow \quad L_{SUSY}$$

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$$= \int [dx\, d\psi^\dagger d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \mathrm{tr}(-1)^F$$?

but what happens if some fields are massless ?

$$H_{SUSY} \leftarrow \quad L_{SUSY} = \frac{1}{2}\dot{x}^2 + i\psi^\dagger\dot{\psi} + \cdots$$

$$\int d\tau \; L_{SUSY}^{Euclidean} = \frac{1}{2}\sum_{n\neq 0}\lambda_n^2 x_n x_n + \sum_{n\neq 0}\lambda_n c_n^\dagger c_n + \cdots$$

$$x = x_0 + \sum_n \hat{x}_n f_n(\tau)$$

$$\psi = \psi_0 + \sum_n c_n \phi_n(\tau) \qquad [dx\, d\psi^\dagger d\psi] \sim dx_0\, d\psi_0^\dagger\, d\psi_0 \times \prod_n d\hat{x}_n\, dc_n^\dagger dc_n$$

$$\psi^\dagger = \psi_0^\dagger + \sum_n c_n^\dagger \phi_n^*(\tau)$$

but what happens if some fields are massless ?

$$H_{SUSY} \leftarrow \quad L_{SUSY}(x, \psi, \psi^\dagger)$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \quad = \int [dx\, d\psi^\dagger\, d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$\sim \int dx_0\, d\psi_0^\dagger d\psi_0 \left[\prod_n d\hat{x}_n\, dc_n^\dagger dc_n\; e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$\sim \int dx_0\, d\psi_0^\dagger d\psi_0 \left[f(x_0) + \psi_0 g(x_0) + \psi_0^\dagger g^*(x_0) + \boxed{\psi_0 \psi_0^\dagger h(x_0)}\right]$$

$$\mathrm{tr} \qquad (-1)^F \qquad\qquad \langle x_0, \psi_0, \psi_0^\dagger | e^{-\beta H} | x_0, \psi_0, \psi_0^\dagger \rangle$$

compute the fermion-zero-mode-saturated piece !

$$H_{SUSY} \leftarrow \quad L_{SUSY}(x, \psi, \psi^\dagger)$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \quad = \int [dx\, d\psi^\dagger\, d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$\sim \int dx_0\, d\psi_0^\dagger d\psi_0 \left[\prod_n d\hat{x}_n\, dc_n^\dagger dc_n\; e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

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$$= \int dx_0\; h(x_0)$$

compute the fermion-zero-mode-saturated piece !

$$\int d\psi_0\ \psi_0 = 1, \qquad \int d\psi_0\ 1 = 0$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \ = \int [dx\, d\psi^\dagger\, d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$\sim \int dx_0\, d\psi_0^\dagger d\psi_0 \left[\prod_n d\hat{x}_n\, dc_n^\dagger dc_n\ e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$\sim \int dx_0\, d\psi_0^\dagger d\psi_0 \ \left[f(x_0) + \psi_0 g(x_0) + \psi_0^\dagger g^*(x_0) + \boxed{\psi_0\psi_0^\dagger h(x_0)}\right]$$

$$= \int dx_0\ h(x_0)$$

note: normalization of the path integral measure

the path integral measure should be normalized to reproduce correctly-normalized partition functions of harmonic oscillators

$$x = x_0 + \sum_n \hat{x}_n f_n(\tau)$$

$$= \hat{x}_0\, f_0(\tau) + \sum_n \hat{x}_n f_n(\tau)$$

$$\int_0^\beta d\tau \;\, f_k(\tau) f_n(\tau) = \delta_{nk}$$

$$f_0(\tau) = 1/\sqrt{\beta}$$

integral over each $\hat{x}_n$ should produce the eigenvalue of $f_n(\tau)$ and nothing else

note: normalization of the path integral measure

$$\int dy\, e^{-\frac{1}{2}\lambda_n^2 y^2} = \boxed{1/\lambda_n} \times \sqrt{2\pi}$$

$$x = x_0 + \textstyle\sum_n \hat{x}_n f_n(\tau)$$

$$= \hat{x}_0\, f_0(\tau) + \textstyle\sum_n \hat{x}_n f_n(\tau)$$

$$\int_0^\beta d\tau \;\; f_k(\tau) f_n(\tau) = \delta_{nk}$$

$$f_0(\tau) = 1/\sqrt{\beta}$$

$$[dx] = d\hat{x}_0/\sqrt{2\pi}\, \textstyle\prod_n d\hat{x}_n/\sqrt{2\pi} = dx_0 \sqrt{\beta/2\pi} \times \prod_n d\hat{x}_n/\sqrt{2\pi}$$

note: normalization of the path integral measure

$$\int dc_n^\dagger dc_n \, e^{-\lambda_n c_n^\dagger c_n} = \lambda_n$$

$$\psi = \psi_0 + \sum_n c_n \phi_n(\tau)$$

$$= c_0 \, \phi_0(\tau) + \sum_n c_n \phi_n(\tau)$$

$$\int_0^\beta d\tau \;\; \phi_k^*(\tau)\phi_n(\tau) = \delta_{nk}$$

$$\phi_0(\tau) = 1/\sqrt{\beta}$$

$$[d\psi^\dagger d\psi] = dc_0^\dagger \prod_n dc_n^\dagger \times dc_0 \prod_n dc_n \simeq d\psi_0^\dagger d\psi_0/\beta \times \prod_n dc_n^\dagger dc_n$$

with the normalization explicit

$$H_{SUSY} \;\leftarrow\; L_{SUSY}(x,\psi,\psi^\dagger)$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \;=\int [dx\, d\psi^\dagger d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \frac{1}{\sqrt{2\pi\beta}}\int dx_0\, d\psi_0^\dagger d\psi_0 \left[\prod_n \frac{1}{\sqrt{2\pi}} d\hat{x}_n\, dc_n^\dagger dc_n\;\, e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$= \frac{1}{\sqrt{2\pi\beta}}\int dx_0\, d\psi_0^\dagger d\psi_0 \left[f(x_0) + \psi_0 g(x_0) + \psi_0^\dagger g^*(x_0) + \boxed{\psi_0\psi_0^\dagger h(x_0)}\right]$$

$$= \frac{1}{\sqrt{2\pi\beta}}\int dx_0\; h(x_0)$$

for general dimensions with complex susy

$$H_{SUSY} \leftarrow L_{SUSY}(x^\mu, \psi^\mu, \psi^\dagger_\mu)$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \int [dx\, d\psi^\dagger d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \frac{1}{\sqrt{2\pi\beta}^d} \prod_{\mu=1}^d \int dx_0^\mu\, d\psi^\dagger_{\mu 0}\, d(\psi_0^\mu) \left[\prod_n \frac{1}{\sqrt{2\pi}} d\hat{x}_n\, dc_n^\dagger dc_n\; e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$= \frac{1}{\sqrt{2\pi\beta}^d} \prod_{\mu=1}^d \int dx_0^\mu\, d\psi^\dagger_{\mu 0} d\psi_0^\mu \; \left[\;\cdots + \left(\prod_\mu \psi_{\mu 0} \prod_\mu \psi_0^{\mu\dagger}\right) \times h(x_0^\mu)\;\right]$$

$$= \frac{1}{\sqrt{2\pi\beta}^d} \int h$$

for general dimensions with complex susy

for computation of h, all zero-mode-excised one-loop determinants, if any, are understood to be divided by their regularizing counterpart

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \;\; = \int [dx\, d\psi^\dagger d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \frac{1}{\sqrt{2\pi\beta}^d} \prod_{\mu=1}^{d} \int dx_0^\mu \, d\psi_{\mu 0}^\dagger \, d(\psi_0^\mu) \left[\prod_n \frac{1}{\sqrt{2\pi}} d\hat{x}_n \, dc_n^\dagger dc_n \;\, e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$= \frac{1}{\sqrt{2\pi\beta}^d} \prod_{\mu=1}^{d} \int dx_0^\mu \, d\psi_{\mu 0}^\dagger d\psi_0^\mu \;\left[\;\cdots + \left(\prod_\mu \psi_{\mu 0} \prod_\mu \psi_0^{\mu\dagger}\right) \times h(x_0^\mu)\;\right]$$

$$= \frac{1}{\sqrt{2\pi\beta}^d} \int h$$

for cases with real fermions

$$H_{SUSY} \;\leftarrow\; L_{SUSY}(x^\mu, \lambda^\mu)$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \;=\; \int [dx\, d\psi\,]_{\text{periodic BC for all !}}\; e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \frac{1}{\sqrt{2\pi}} \int dx_0\, d\lambda_0 \left[\prod_n \frac{1}{\sqrt{2\pi}} d\hat{x}_n\, dc_n\; e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int dx_0\, d\lambda_0\; [f(x_0) + \boxed{\lambda_0 g(x_0)}] \,/\, \sqrt{\beta}$$

$$= \frac{1}{\sqrt{2\pi\beta}} \int dx_0\; g(x_0)$$

from mismatch btw bosonic & fermionic regularizing determinants, the zero-mode excised and zeta-function regularized

for general dimensions with real susy

$$H_{SUSY} \;\leftarrow\; L_{SUSY}(x^\mu, \lambda^\mu)$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \int [dx\, d\lambda]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \frac{1}{\sqrt{2\pi}^d} \prod_\mu^d \int dx_0^\mu\, d\lambda_0^\mu \left[\prod_n \frac{1}{\sqrt{2\pi}} d\hat{x}_n\, dc_n\; e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$= \frac{1}{\sqrt{2\pi}^d} \prod_\mu \int dx_0^\mu\, d\lambda_0^\mu \left[\;\cdots + \left(\prod_{\mu=1}^d \lambda_{\mu 0}\right) \times g(x_0^\mu)\;\right] / \sqrt{\beta}^d$$

$$= \frac{1}{\sqrt{2\pi\beta}^d} \int g$$

from mismatch btw bosonic & fermionic regularizing determinants, the zero-mode excised and zeta-function regularized

for general dimensions with real susy

for computation of g, all zero-mode-excised one-loop determinants are understood to be divided by their regularizing counterpart

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$$= \frac{1}{\sqrt{2\pi\beta}^d} \int g$$

from mismatch btw bosonic & fermionic regularizing determinants, the zero-mode excised and zeta-function regularized

for general dimensions with real susy

a final subtlety is an overall factor of i's associated with integrating real fermions

$$d\psi^\dagger d\psi = i d\lambda^1 d\lambda^2 \quad \leftarrow \quad \psi = (\lambda^1 + i\lambda^2)/\sqrt{2}$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \;= \int [dx\, d\lambda\,]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \frac{i^{d/2}}{\sqrt{2\pi}^d} \prod_\mu^d \int dx_0^\mu\, d\lambda_0^\mu \left[\prod_n \frac{1}{\sqrt{2\pi}} d\hat{x}_n\, dc_n\; e^{-\int_0^\beta L^{Euclidean} d\tau}\right]$$

$$= \frac{i^{d/2}}{\sqrt{2\pi}^d} \prod_\mu \int dx_0^\mu\, d\lambda_0^\mu \left[\;\cdots + \left(\prod_{\mu=1}^d \lambda_{\mu 0}\right) \times g(x_0^\mu)\;\right] / \sqrt{\beta}^d$$

$$= \frac{i^{d/2}}{\sqrt{2\pi\beta}^d} \int g$$

supersymmetric quantum mechanics and related index theorems

simplest, nontrivial susy QM with index

$$L_1(x^i,\lambda^i) \;=\; \tfrac{1}{2}\dot{x}_i\dot{x}^i + \tfrac{i}{2}\lambda_i\dot{\lambda}^i - A(x)_i\dot{x}^i + \tfrac{i}{2}F_{ik}(x)\lambda^i\lambda^k$$

simplest, nontrivial susy QM with index

$$L_1(x^i, \lambda^i) \;=\; \tfrac{1}{2}\dot{x}_i\dot{x}^i + \tfrac{i}{2}\lambda_i\dot{\lambda}^i - A(x)_i\dot{x}^i + \tfrac{i}{2}F_{ik}(x)\lambda^i\lambda^k$$

$$\delta_{SUSY}\lambda^i = \epsilon\dot{x}^i$$

$$\delta_{SUSY}x^i = -i\epsilon\lambda^i$$

$$\int d\tau\ \delta_{SUSY}L_1(x^i, \lambda^i) = 0$$

simplest, nontrivial susy QM with index

$$L_1(x^i, \lambda^i) \;=\; \tfrac{1}{2}\dot{x}_i\dot{x}^i + \tfrac{i}{2}\lambda_i\dot{\lambda}^i - A(x)_i\dot{x}^i + \tfrac{i}{2}F_{ik}(x)\lambda^i\lambda^k$$

$$[p_j, x^k] = -i\delta_j^k$$

$$\delta_{SUSY}\lambda^i = \epsilon\dot{x}^i$$

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$$\delta_{SUSY}\lambda^i = \epsilon\dot{x}^i$$

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$$H = \tfrac{1}{2}(p_i + A_i)^2 - \tfrac{i}{2}F_{ik}\lambda^i\lambda^k$$

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simplest, nontrivial susy QM with index

$$L_1(x^i,\lambda^i) \;=\; \tfrac{1}{2}\dot{x}_i\dot{x}^i + \tfrac{i}{2}\lambda_i\dot{\lambda}^i - A(x)_i\dot{x}^i + \tfrac{i}{2}F_{ik}(x)\lambda^i\lambda^k$$

$$[p_j, x^k] = -i\delta_j^k$$

$$\{\lambda^i,\lambda^k\} = \delta^{ik}$$

$$Q = \lambda^i(p_i + A_i) \longrightarrow$$

$$\delta_{SUSY}\lambda^i = \epsilon\dot{x}^i$$

$$\delta_{SUSY}x^i = -i\epsilon\lambda^i$$

$$H = \tfrac{1}{2}(p_i + A_i)^2 - \tfrac{i}{2}F_{ik}\lambda^i\lambda^k$$

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simplest, nontrivial susy QM with index

$$L_1(x^i, \lambda^i) \;=\; \tfrac{1}{2}\dot{x}_i\dot{x}^i + \tfrac{i}{2}\lambda_i\dot{\lambda}^i - A(x)_i\dot{x}^i + \tfrac{i}{2}F_{ik}(x)\lambda^i\lambda^k$$

$$[p_j, x^k] = -i\delta_j^k$$

$$\{\lambda^i, \lambda^k\} = \delta^{ik} \longleftarrow \lambda^i \simeq \gamma^i/\sqrt{2}$$

$$Q = \lambda^i(p_i + A_i)$$

$$H = \tfrac{1}{2}(p_i + A_i)^2 - \tfrac{i}{2}F_{ik}\lambda^i\lambda^k$$

simplest, nontrivial susy QM with index

$$L_1(x^i, \lambda^i) \;=\; \tfrac{1}{2}\dot{x}_i\dot{x}^i + \tfrac{i}{2}\lambda_i\dot{\lambda}^i - A(x)_i\dot{x}^i + \tfrac{i}{2}F_{ik}(x)\lambda^i\lambda^k$$

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$$H = \tfrac{1}{2}(p_i + A_i)^2 - \tfrac{i}{2}F_{ik}\lambda^i\lambda^k = \tfrac{1}{2}Q^2 \longleftarrow H \simeq \left[\gamma^i(p_i + A_i)\right]^2/4$$

simplest, nontrivial susy QM with index

$$L_1(x^i, \lambda^i) \;=\; \tfrac{1}{2}\dot{x}_i\dot{x}^i + \tfrac{i}{2}\lambda_i\dot{\lambda}^i - A(x)_i\dot{x}^i + \tfrac{i}{2}F_{ik}(x)\lambda^i\lambda^k$$

$$[p_j, x^k] = -i\delta_j^k$$

$$\{\lambda^i, \lambda^k\} = \delta^{ik} \longleftarrow \lambda^i \simeq \gamma^i/\sqrt{2}$$

$$(-1)^F \simeq \textstyle\prod_{k=1}^d(\sqrt{2i}\lambda^k) \longleftarrow (-1)^F \simeq \textstyle\prod_k \sqrt{i}\gamma^k = \gamma^{d+1}$$

$$Q = \lambda^i(p_i + A_i) \longleftarrow Q \simeq \gamma^i(p_i + A_i)/\sqrt{2}$$

$$H = \tfrac{1}{2}(p_i + A_i)^2 - \tfrac{i}{2}F_{ik}\lambda^i\lambda^k = \tfrac{1}{2}Q^2 \longleftarrow H \simeq \left[\gamma^i(p_i + A_i)\right]^2/4$$

→ charged spinor under the influence of magnetic field

$$\begin{pmatrix} 0 & D \\ D^{\dagger} & 0 \end{pmatrix} = Q = \gamma^k \left(-i\partial_k + A_k\right)/\sqrt{2}$$

$$H\Psi_{\pm} = E\Psi_{\pm}$$

$$\gamma^{d+1}\Psi_{+} = \Psi_{+} \qquad \underset{\longleftarrow}{\overset{\longrightarrow}{Q}} \qquad \gamma^{d+1}\Psi_{-} = -\Psi_{-}$$

$$Q\Psi_{+} = 0 = H\Psi_{+} \qquad\qquad Q\Psi_{-} = 0 = H\Psi_{-}$$

→ charged spinor under the influence of magnetic field

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = Q = \gamma^k \left(-i\partial_k + A_k\right)/\sqrt{2}$$

$$H\Psi_\pm = E\Psi_\pm$$

$$\gamma^{d+1}\Psi_+ = \Psi_+$$

$$\xrightarrow{\quad Q \quad}$$

$$\Psi_- = \frac{1}{\sqrt{E}} Q\,\Psi_+$$

$$Q\Psi_+ = 0 = H\Psi_+$$

$$Q\Psi_- = 0 = H\Psi_-$$

$$\gamma^{d+1}\,Q\,\Psi_+ = -Q\gamma^{d+1}\Psi_+ = -Q\Psi_+$$

→ charged spinor under the influence of magnetic field

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = Q = \gamma^k \left(-i\partial_k + A_k\right)/\sqrt{2}$$

$$H\Psi_\pm = E\Psi_\pm$$

$$\Psi_+ = \frac{1}{\sqrt{E}} Q\,\Psi_- \quad \xleftarrow{\;\;Q\;\;} \quad \gamma^{d+1}\Psi_- = -\Psi_-$$

$$Q\Psi_+ = 0 = H\Psi_+ \qquad\qquad Q\Psi_- = 0 = H\Psi_-$$

$$\gamma^{d+1}\,Q\,\Psi_- = -Q\gamma^{d+1}\Psi_- = Q\Psi_-$$

1) Abelian Dirac index

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = Q = \gamma^k \left(-i\partial_k + A_k\right)/\sqrt{2}$$

$$\begin{aligned}
L_1^{Euclidean} &= \frac{1}{2}\Delta\dot{x}_i\Delta\dot{x}^i - \frac{1}{2}\Delta\lambda_i\Delta\dot{\lambda}^i - iA(x_0+\Delta x)_i\delta\dot{x}^i \\
&- \frac{i}{2}F_{ik}(x_0+\Delta x)(\lambda_0^i+\Delta\lambda^i)(\lambda_0^k+\Delta\lambda^k) \\
&\simeq -\frac{i}{2}F_{ik}(x_0)\lambda_0^i\lambda_0^k \\
&+ \frac{1}{2}\Delta\dot{x}_i\Delta\dot{x}^i - i\partial_k A(x_0)_i\Delta x^k\Delta\dot{x}^i \\
&- \frac{1}{2}\Delta\lambda_i\Delta\dot{\lambda}^i - \frac{i}{2}F_{ik}(x_0)\Delta\lambda^i\Delta\lambda^k \\
&- i\partial_m F_{ik}(x_0)\lambda_0^i\Delta x^m\Delta\lambda^k + \cdots
\end{aligned}$$

$$x = x_0 + \Delta x$$

$$\lambda = \lambda_0 + \Delta\lambda$$

1) Abelian Dirac index

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = Q = \gamma^k \left(-i\partial_k + A_k\right)/\sqrt{2}$$

$$\begin{aligned} L_1^{Euclidean} &= \frac{1}{2}\Delta\dot{x}_i\Delta\dot{x}^i - \frac{1}{2}\Delta\lambda_i\Delta\dot{\lambda}^i - iA(x_0+\Delta x)_i\delta\dot{x}^i \\ &- \frac{i}{2}F_{ik}(x_0+\Delta x)(\lambda_0^i+\Delta\lambda^i)(\lambda_0^k+\Delta\lambda^k) \\ \\ &\simeq -\frac{i}{2}F_{ik}(x_0)\lambda_0^i\lambda_0^k \\ &- \frac{1}{2}\Delta x_i\Delta\ddot{x}^i - \frac{i}{2}F_{ik}(x_0)\Delta x^i\Delta\dot{x}^k \\ &- \frac{1}{2}\Delta\lambda_i\Delta\dot{\lambda}^i - \frac{i}{2}F_{ik}(x_0)\Delta\lambda^i\Delta\lambda^k \\ &\cancel{- i\partial_m F_{ik}(x_0)\lambda_0^i\Delta x^m\Delta\lambda^k +} \cdots \end{aligned}$$

$$x = x_0 + \Delta x$$

$$\lambda = \lambda_0 + \Delta\lambda$$

will ignore this term for simplicity, as it turns out to be ignorable

1) Abelian Dirac index

$$\lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$$

$$= \lim_{\beta\to 0} \frac{i^{d/2}}{\sqrt{2\pi\beta}^d} \prod_{i=1}^{d} \int dx_0^i \, d\lambda_0^i \; e^{i\beta/2\cdot F_{ik}(x_0)\lambda_0^i\lambda_0^k} \; \left[\frac{Det'(-\partial_\tau - iF)}{Det'(-\partial_\tau)} \cdot \frac{Det'(-\partial_\tau^2)}{Det'((-\partial_\tau - iF)\partial_\tau)}\right]^{1/2}$$

1) Abelian Dirac index

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$$= \lim_{\beta\to 0} \frac{i^{d/2}}{\sqrt{2\pi\beta}^d} \prod_{i=1}^{d} \int dx_0^i \, d\lambda_0^i \; e^{i\beta/2\cdot F_{ik}(x_0)\lambda_0^i\lambda_0^k} \left[\frac{Det'\cancel{(-\partial_\tau - iF)}}{Det'(-\cancel{\partial}_\tau)} \cdot \frac{Det'(-\cancel{\partial}^2_\tau)}{Det'(\cancel{(-\partial_\tau - iF)}\cancel{\partial}_\tau)}\right]^{1/2}$$

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$$= \int \prod_{i=1}^{d} dx_0^i \prod_{i=1}^{d} d\lambda_0^i \; e^{-F/2\pi} \qquad\qquad F \equiv \frac{1}{2} F_{ik}(x_0)\lambda_0^i \lambda_0^k$$

1) Abelian Dirac index

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$$= \lim_{\beta\to 0} \frac{i^{d/2}}{\sqrt{2\pi\beta}^{d}} \prod_{i=1}^{d} \int dx_0^i \, d\lambda_0^i \; e^{i\beta/2\cdot F_{ik}(x_0)\lambda_0^i\lambda_0^k}$$

$$= \int \prod_{i=1}^{d} dx_0^i \prod_{i=1}^{d} d\lambda_0^i \; e^{-F/2\pi} \qquad F \equiv \frac{1}{2} F_{ik}(x_0)\lambda_0^i\lambda_0^k$$

$$= \int e^{F_{ij}dx^i\wedge dx^k/4\pi} = \int e^{\mathcal{F}/2\pi} \qquad \mathcal{F} \equiv \frac{1}{2} F_{ik}(x) dx^i \wedge dx^k$$

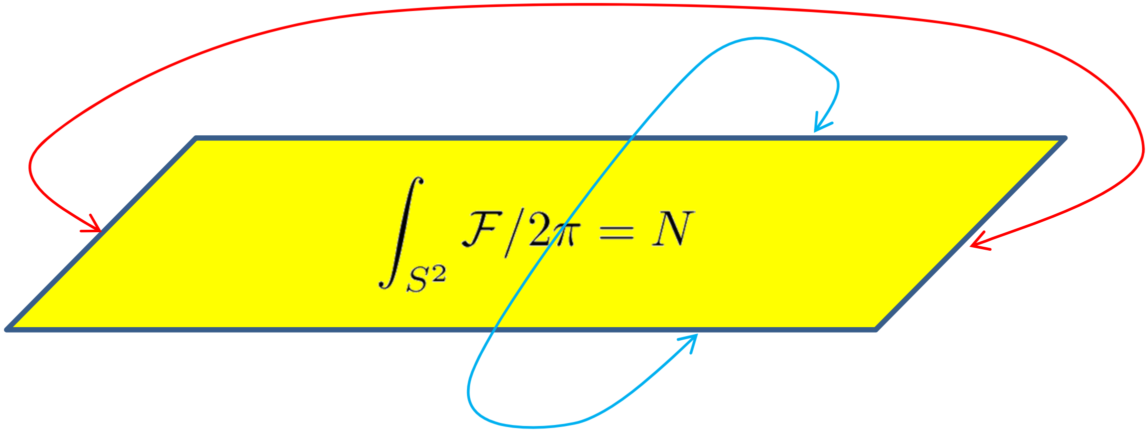
a simple Dirac index with Abelian gauge field

$$\lim_{\beta \to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H_1}\right]$$

$$= \int_{T^2} \mathrm{tr}\, e^{\mathcal{F}/2\pi}$$

$$= \int_{T^2} \mathcal{F}/2\pi$$

$$= N$$



a simple Dirac index with Abelian gauge field

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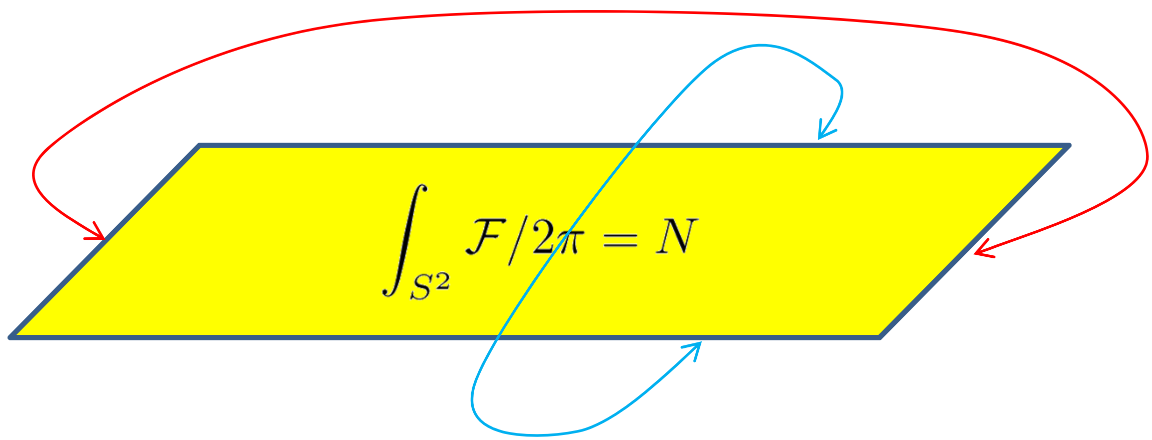

= #(lowest landau level states on torus with magnetic flux N

2) non-Abelian Dirac index

$$L_2(x^i,\lambda^i;\eta^a,\eta^*_a) = \frac{1}{2}\dot{x}_i\dot{x}^i + \frac{i}{2}\lambda_i\dot{\lambda}^i + i\eta^*_a\dot{\eta}^a - \dot{x}^i A(x)_i{}^a{}_b\eta^*_a\eta^b + \frac{i}{2}F_{ik}{}^a{}_b(x)\eta^*_a\eta^b\lambda^i\lambda^k$$

$$[p_j, x^k] = -i\delta^k_j$$

$$\{\lambda^i,\lambda^k\} = \delta^{ik}$$

$$\{\eta^a,\eta^*_b\} = \delta^a_b$$

$$Q = \lambda^i(p_i + A(x)_i{}^a{}_b\eta^*_a\eta^b)$$

$$\delta_{SUSY}\eta^a = -\lambda^i A(x)_i{}^a{}_b\eta^b$$

$$\delta_{SUSY}\lambda^i = \dot{x}^i$$

$$\delta_{SUSY}x^i = -i\lambda^i$$

$$\int d\tau\ \delta_{SUSY}L_2 = 0$$

2) non-Abelian Dirac index

$$L_2(x^i,\lambda^i;\eta^a,\eta^*_a) = \frac{1}{2}\dot{x}_i\dot{x}^i + \frac{i}{2}\lambda_i\dot{\lambda}^i + i\eta^*_a\dot{\eta}^a - \dot{x}^i A(x)_i{}^a{}_b\eta^*_a\eta^b + \frac{i}{2}F_{ik}{}^a{}_b(x)\eta^*_a\eta^b\lambda^i\lambda^k$$

1. η^*,η keep track only of gauge indices of wavefunctions, and have no superpartners

2. (-1)^F does not include these fermions
   → anti-periodic boundary condition for them

3. for traceless gauge field, η^*,η has no zero-point energy, and excitations by η^*_a cost no energy

4. path integral over η^*,η sector is to be restricted, so that one effectively traces over one-particle states, $\eta^*_a|0\rangle$, only.

$$Q = \lambda^i(p_i + A(x)_i{}^a{}_b\eta^*_a\eta^b)$$

$$\delta_{SUSY}\eta^a = -\epsilon\lambda^i A(x)_i{}^a{}_b\eta^b$$

$$\delta_{SUSY}\lambda^i = \epsilon\dot{x}^i$$

$$\delta_{SUSY}x^i = -i\epsilon\lambda^i$$

$$\int d\tau\ \delta_{SUSY}L_2 = 0$$

2) non-Abelian Dirac index

$$L_2(x^i, \lambda^i; \eta^a, \eta_a^*) = \frac{1}{2}\dot{x}_i\dot{x}^i + \frac{i}{2}\lambda_i\dot{\lambda}^i + i\eta_a^*\dot{\eta}^a - \dot{x}^i A(x)_i{}^a{}_b\eta_a^*\eta^b + \frac{i}{2}F_{ik}{}^a{}_b(x)\eta_a^*\eta^b\lambda^i\lambda^k$$

$$Q = \lambda^i(p_i + A(x)_i{}^a{}_b\eta_a^*\eta^b)$$

use a hybrid formulation where η^*, η sector is quantized first and one-particle subsector is traced over

$$\delta_{SUSY}\eta^a = -\epsilon\lambda^i A(x)_i{}^a{}_b\eta^b$$

$$\delta_{SUSY}\lambda^i = \epsilon\dot{x}^i$$

$$\delta_{SUSY}x^i = -i\epsilon\lambda^i$$

path integral over η^*, η sector is to be restricted, so that one effectively traces over one-particle states, $\eta_a^*|0\rangle$, only.

$$\int d\tau\ \delta_{SUSY}L_2 = 0$$

2) non-Abelian Dirac index

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = Q = \gamma^k \left(-i\partial_k + A_k{}^a{}_b \eta_a^* \eta^b\right) / \sqrt{2}$$

$$x = x_0 + \Delta x$$

$$\lambda = \lambda_0 + \Delta\lambda$$

$$\begin{aligned} L_2^{Euclidean} - p_\eta \dot{\eta} \quad &\simeq \quad -\frac{i}{2} F_{ik}(x_0)^a{}_b \eta_a^* \eta^b \lambda_0^i \lambda_0^k \\ &- \quad \frac{1}{2}\Delta x_i \Delta \ddot{x}^i - \frac{i}{2} F_{ik}(x_0)^a{}_b \eta_a^* \eta^b \Delta x^i \Delta \dot{x}^k \\ &- \quad \frac{1}{2}\Delta \lambda_i \Delta \dot{\lambda}^i - \frac{i}{2} F_{ik}(x_0)^a{}_b \eta_a^* \eta^b \Delta \lambda^i \Delta \lambda^k + \cdots \end{aligned}$$

2) non-Abelian Dirac index

$$\lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$$

$$= \lim_{\beta\to 0} \mathrm{tr}_{gauge} \frac{i^{d/2}}{\sqrt{2\pi\beta}^d} \prod_{i=1}^{d} \int dx_0^i \, d\lambda_0^i \; e^{i\beta/2 \cdot F_{ik}(x_0)\lambda_0^i \lambda_0^k} \left[\frac{Det'\cancel{(-\partial_\tau - iF)}}{Det'(-\cancel{\partial}_\tau)} \cdot \frac{Det'(-\cancel{\partial}_\tau^2)}{Det'((\cancel{-\partial_\tau - iF})\cancel{\partial}_\tau)} \right]^{1/2}$$

$$= \lim_{\beta\to 0} \mathrm{tr}_{gauge} \frac{i^{d/2}}{\sqrt{2\pi\beta}^d} \prod_{i=1}^{d} \int dx_0^i \, d\lambda_0^i \; e^{i\beta/2 \cdot F_{ik}(x_0)\lambda_0^i \lambda_0^k}$$

$$= \int \prod_{i=1}^{d} dx_0^i \prod_{i=1}^{d} d\lambda_0^i \; \mathrm{tr}_{gauge} e^{-F/2\pi} \qquad F^a{}_b \equiv \frac{1}{2} F_{ik}{}^a{}_b(x_0)\lambda_0^i \lambda_0^k$$

$$= \int \mathrm{tr}_{gauge} e^{F_{ij} dx^i \wedge dx^k / 4\pi} = \int \mathrm{tr}_{gauge} e^{\mathcal{F}/2\pi} \qquad \mathcal{F}^a{}_b \equiv \frac{1}{2} F_{ik}{}^a{}_b(x) dx^i \wedge dx^k$$

3) Atiyah-Singer index

$$L_3(x^i, \lambda^i; \eta^a, \eta^*_a)$$

$$= \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + \frac{i}{2} g_{ij}(x)\lambda^i \nabla_\tau \lambda^j + i\eta^*_a \dot{\eta}^a - \dot{x}^i A_i{}^a{}_b(x)\eta^*_a \eta^b + \frac{i}{2} F_{ij}{}^a{}_b \lambda^i \lambda^j \eta^*_a \eta^b$$

$$\nabla_\tau \lambda^i = \dot{\lambda}^i + \Gamma^i_{jk} \dot{x}^j \lambda^k$$

$$[p_j, x^k] = -i\delta^k_j$$

$$[p_j, \lambda^A] = 0 \qquad \lambda^A \equiv e^A_i \lambda^i$$

$$\{\lambda^A, \lambda^B\} = \delta^{AB}$$

$$\{\eta^a, \eta^*_b\} = \delta^a_b$$

3) Atiyah-Singer index

$$L_3(x^i, \lambda^i; \eta^a, \eta^*_a)$$

$$= \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + \frac{i}{2} g_{ij}(x)\lambda^i \nabla_\tau \lambda^j + i\eta^*_a \dot{\eta}^a - \dot{x}^i A_i{}^a{}_b(x)\eta^*_a \eta^b + \frac{i}{2} F_{ij}{}^a{}_b \lambda^i \lambda^j \eta^*_a \eta^b$$

$$\nabla_\tau \lambda^i = \dot{\lambda}^i + \Gamma^i_{jk}\dot{x}^j \lambda^k$$

$$Q = \lambda^i (p_i - \frac{i}{4} w_{iAB}\lambda^A \lambda^B + A(x)_i{}^a{}_b \eta^*_a \eta^b)$$

$$[p_j, x^k] = -i\delta^k_j$$

$$[p_j, \lambda^A] = 0 \qquad \lambda^A \equiv e^A_i \lambda^i$$

$$\{\lambda^A, \lambda^B\} = \delta^{AB}$$

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3) Atiyah-Singer index

$$L_3(x^i, \lambda^i; \eta^a, \eta^*_a)$$

$$= \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + \frac{i}{2} g_{ij}(x)\lambda^i\nabla_\tau\lambda^j + i\eta^*_a\dot{\eta}^a - \dot{x}^i A_i{}^a{}_b(x)\eta^*_a\eta^b + \frac{i}{2} F_{ij}{}^a{}_b\lambda^i\lambda^j\eta^*_a\eta^b$$

$$\nabla_\tau\lambda^i = \dot{\lambda}^i + \Gamma^i_{jk}\dot{x}^j\lambda^k$$

$$Q = \lambda^i(p_i - \frac{i}{4} w_{iAB}\lambda^A\lambda^B + A(x)_i{}^a{}_b\eta^*_a\eta^b)$$

$$[p_j, x^k] = -i\delta^k_j$$

$$[p_j, \lambda^A] = 0 \qquad \lambda^A \equiv e^A_i\lambda^i$$

$$\{\lambda^A, \lambda^B\} = \delta^{AB}$$

$$\{\eta^a, \eta^*_b\} = \delta^a_b$$

$$\delta_{SUSY}\eta^a = -\epsilon\lambda^i A(x)_i{}^a{}_b\eta^b$$

$$\delta_{SUSY}\lambda^i = \epsilon\dot{x}^i + i\epsilon\Gamma^i_{jk}\lambda^j\lambda^k$$

$$\delta_{SUSY}x^i = -i\epsilon\lambda^i$$

$$\int d\tau\ \delta_{SUSY}L_3 = 0$$

3) Atiyah-Singer index

$$\begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} = Q = \gamma^k \left(-i\partial_k + A_k{}^a{}_b \eta_a^* \eta^b\right)/\sqrt{2}$$

$$x^i = x_0^i + \Delta x^i$$

$$\lambda^A = \lambda_0^A + \Delta\lambda^A$$

$$\begin{aligned}
L_3^{Euclidean} - p_\eta \dot{\eta} \quad \simeq \quad & -\frac{i}{2} F_{AB}(x_0)\lambda_0^A \lambda_0^B \\
- \quad & \frac{1}{2}\Delta x_i \Delta \ddot{x}^i - \frac{i}{2} F_{ik}(x_0)\Delta x^i \Delta \dot{x}^k \\
- \quad & \frac{1}{4} R_{ABik}(x_0)\lambda_0^A \lambda_0^B \Delta x^i \Delta \dot{x}^k \\
- \quad & \frac{1}{2}\Delta\lambda_A \Delta\dot{\lambda}^A - \frac{i}{2} F_{AB}(x_0)\Delta\lambda^A \Delta\lambda^B \\
+ \quad & \cdots
\end{aligned}$$

3) Atiyah-Singer index

$$\lim_{\beta\to 0}\mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \lim_{\beta\to 0}\mathrm{tr}_{gauge}\frac{i^{d/2}}{\sqrt{2\pi\beta}^d}\prod_i\int dx_0^i \prod_A d\lambda_0^A \; e^{i\beta/2\cdot F_{AB}(x_0)\lambda_0^A\lambda_0^B}$$

$$\times\left[\frac{Det'(-\partial_\tau - iF_{ij})}{Det'(-\partial_\tau)}\cdot\frac{Det'(-\partial_\tau^2)}{Det'((-\partial_\tau - iF_{ij} - R_{ABij}\lambda_0^A\lambda_0^B/2)\partial_\tau)}\right]^{1/2}$$

$$= \lim_{\beta\to 0}\mathrm{tr}_{gauge}\frac{i^{d/2}}{\sqrt{2\pi\beta}^d}\prod_{i=1}^d\int dx_0^i\, d\lambda_0^i\; e^{i\beta/2\cdot F_{AB}(x_0)\lambda_0^A\lambda_0^B}\times\left[\det\frac{\beta R_{ABCD}\lambda_0^A\lambda_0^B/4}{\sinh(\beta R_{ABCD}\lambda_0^A\lambda_0^B/4)}\right]^{1/2}$$

$$= \int \mathrm{tr}\, e^{\mathcal{F}/2\pi}\wedge\left[\det\frac{\mathcal{R}/4\pi}{\sinh(\mathcal{R}/4\pi)}\right]^{1/2}$$

$$\mathcal{F}^a{}_b \equiv \frac{1}{2}F_{ik}{}^a{}_b(x)dx^i\wedge dx^k$$

$$\mathcal{R}_{AB} \equiv \frac{1}{2}R_{ikAB}(x)dx^i\wedge dx^k$$

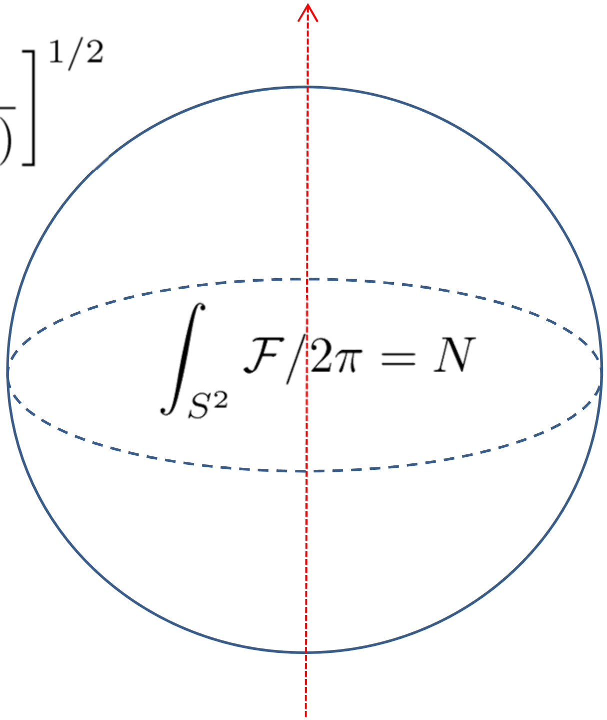
a simple Atiyah-Singer index with Abelian gauge field

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H_3}\right]$$

$$= \int_{S^2} \mathrm{tr}\, e^{\mathcal{F}/2\pi} \wedge \det\left[\frac{\mathcal{R}/4\pi}{\sinh(\mathcal{R}/4\pi)}\right]^{1/2}$$

$$= \int_{S^2} \mathcal{F}/2\pi$$

$$= N$$



4) Euler index

$$L_4(x^i, \psi^i, \psi^{*i}) \;\; = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + i g_{ij} \psi^{*i} \nabla_\tau \psi^j + \frac{1}{4} R_{ijkl} \psi^{*i} \psi^{*i} \psi^k \psi^l$$

$$\nabla_\tau \psi^i = \frac{d}{dt} \psi^i + \Gamma^i_{jk} \dot{x}^j \psi^k$$

$$\delta_{SUSY} x^i = \epsilon \psi^{*i} - \epsilon^* \psi^i$$

$$\delta_{SUSY} \psi^i = i \epsilon \dot{x}^i - \Gamma^i_{jk} \epsilon \psi^{*j} \psi^k$$

$$\delta_{SUSY} \psi^{*i} = -i \epsilon^* \dot{x}^i + \Gamma^i_{jk} \epsilon^* \psi^j \psi^{*k}$$

$$\int d\tau \; \delta_{SUSY} L_4 = 0$$

4) Euler index

$$L_4(x^i, \psi^i, \psi^{*i}) \ = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + i g_{ij}\psi^{*i}\nabla_\tau \psi^j + \frac{1}{4} R_{ijkl}\psi^{*i}\psi^{*i}\psi^k\psi^l$$

$$\nabla_\tau \psi^i = \frac{d}{dt}\psi^i + \Gamma^i_{jk}\dot{x}^j\psi^k$$

$$[p_j, x^k] = -i\delta^k_j$$

$$[p_j, \psi^A] = 0 \qquad \psi^A \equiv e^A_i \psi^i$$

$$[p_j, \psi^{*A}] = 0 \qquad \psi^{*A} \equiv e^A_i \psi^{*i}$$

$$\{\psi^A, \psi^{*B}\} = \delta^{AB} \qquad \{\psi^A, \psi^B\} = 0 = \{\psi^{*A}, \psi^{*B}\}$$

4) Euler index

$$L_4(x^i,\psi^i,\psi^{*i}) \;=\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j+ig_{ij}\psi^{*i}\nabla_\tau\psi^j+\frac{1}{4}R_{ijkl}\psi^{*i}\psi^{*i}\psi^k\psi^l$$

$$\nabla_\tau\psi^i=\frac{d}{dt}\psi^i+\Gamma^i_{jk}\dot{x}^j\psi^k \qquad\qquad \delta_{SUSY}=i\epsilon Q^*-i\epsilon^*Q$$

$$Q=\psi^A e^i_A(p_i+w_{iAB}\psi^{*A}\psi^B) \qquad\qquad Q^*=\psi^{*A}e^i_A(p_i+w_{iAB}\psi^{*A}\psi^B)$$

$$\psi^{*i} \;\rightarrow\; dx^i\wedge$$

$$\psi_i \;\rightarrow\; \left\langle \frac{\partial}{\partial x^i},\ \cdot\ \right\rangle$$

$$(-1)^F \;\rightarrow\; (-1)^{rank}$$

$$d^\dagger \qquad\qquad\qquad d$$

4) Euler index

$$L_4(x^i, \psi^i, \psi^{*i}) \;\; = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + i g_{ij}\psi^{*i}\nabla_\tau\psi^j + \frac{1}{4}R_{ijkl}\psi^{*i}\psi^{*i}\psi^k\psi^l$$

$$\boxed{\mathrm{Index}(Q + Q^\dagger) = \mathrm{Index}(d^\dagger + d) \simeq \sum_{r=1}^{\dim M} (-1)^r H^r(M) = \chi(M)}$$

$$Q = \psi^A e^i_A(p_i + w_{iAB}\psi^{*A}\psi^B) \qquad Q^* = \psi^{*A} e^i_A(p_i + w_{iAB}\psi^{*A}\psi^B)$$

$$\psi^{*i} \quad \rightarrow \quad dx^i \wedge$$

$$\psi_i \quad \rightarrow \quad \left\langle \frac{\partial}{\partial x^i}, \; \cdot \; \right\rangle$$

$$(-1)^F \quad \rightarrow \quad (-1)^{rank}$$

$$d^\dagger \qquad\qquad\qquad\qquad d$$

4) Euler index

$$\chi = \lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$$

$$= \lim_{\beta\to 0} \frac{1}{\sqrt{2\pi\beta}^d} \int \prod_i dx_0^i \prod_A (d\psi_A^* d\psi_0^A)\; e^{\beta/4\cdot R_{ABCD}(x_0)\psi_0^{*A}\psi_0^{*B}\psi_0^C\psi_0^D} \left[\frac{Det'\cancel{(-\partial_\tau^2 - \cdots)}}{Det'\cancel{(-\partial_\tau^2 - \cdots)}}\right]^{1/2}$$

$$= \lim_{\beta\to 0} \frac{1}{\sqrt{2\pi\beta}^d} \prod_{i=1}^d \int \prod_i dx_0^i \prod_A (d\psi_A^* d\psi_0^A)\; e^{\beta/4\cdot R_{ABCD}(x_0)\psi_0^{*A}\psi_0^{*B}\psi_0^C\psi_0^D}$$

$$= \frac{1}{(2\pi)^{d/2}} \int \frac{1}{2^{d/2}(d/2)!} \epsilon^{A_1 A_2\cdots A_d} R_{A_1 A_2} \wedge \cdots \wedge R_{A_{d-1}A_d} = \frac{1}{(2\pi)^{d/2}} \int \mathrm{Pf}(R_{AB})$$

$$R_{AB} \equiv \frac{1}{2} R_{ABik}(x) dx^i \wedge dx^k$$

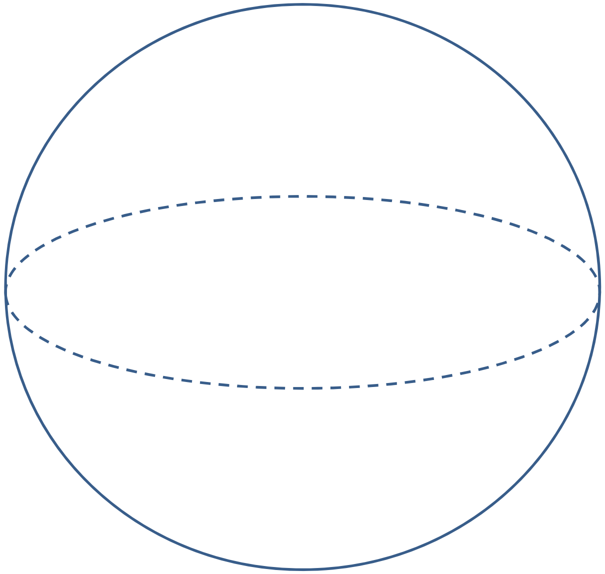
Euler index in 2d

$$\lim_{\beta\to 0} \operatorname{tr}\left[(-1)^F e^{-\beta H}\right]$$

$$= \frac{1}{2\pi}\int R_{1212}$$

$$= \frac{1}{4\pi}\int R_{scalar}$$

$$= 2 - 2g$$



Euler index in 2d

$$\lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$$

$$= \frac{1}{2\pi}\int R_{1212}$$

$$= \frac{1}{4\pi}\int R_{scalar}$$

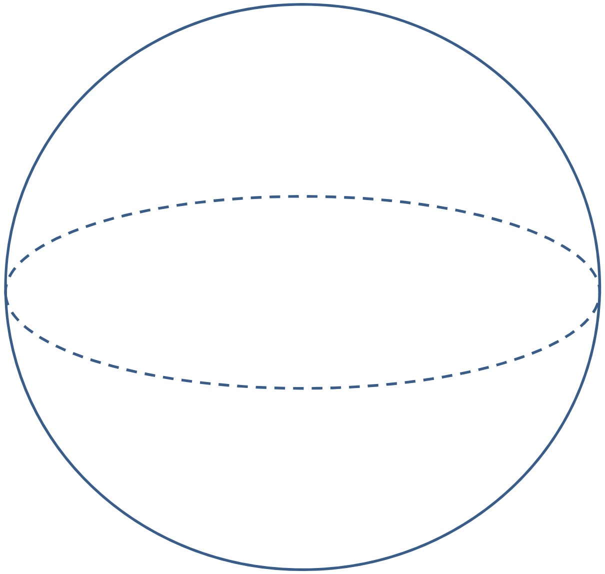
$$= 2-2g$$

$$= 1-2g+1$$

$$\dim H^0(\Sigma_g) = \dim H_0(\Sigma_g) = 1$$

$$\dim H^1(\Sigma_g) = \dim H_1(\Sigma_g) = 2g$$

$$\dim H^2(\Sigma_g) = \dim H_2(\Sigma_g) = 1$$



Hamiltonian view and heat kernel expansion

with the simplest case of Euler index

back to the Hamiltonian viewpoint

1. conceptually straightforward
2. trivial to normalize
3. sign ambiguity issue more transparent
4. easier to deal with gauge symmetry
5. perhaps more model-dependent computationally
6. less flexible for localization procedure

4)′ Euler index in the Hamiltonian view

$$L_4(x^i, \psi^i, \psi^{*i}) \;= \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + i g_{ij} \psi^{*i} \nabla_\tau \psi^j + \frac{1}{4} R_{ijkl} \psi^{*i} \psi^{*i} \psi^k \psi^l$$

$$H_4^{\text{classical}} \;= \frac{1}{2} g^{ij} (p_i + \cdots)(p_j + \cdots) - \frac{1}{4} R_{ijkl} \psi^{*i} \psi^{*i} \psi^k \psi^l$$

?

$$H_4^{\text{quantum}} \;= -\frac{1}{2} \nabla^2 - \frac{1}{4} R_{ABCD} \psi^{*A} \psi^{*B} \psi^C \psi^D$$

4)' Euler index in the Hamiltonian view

$$L_4(x^i, \psi^i, \psi^{*i}) \;\; = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + i g_{ij}\psi^{*i}\nabla_\tau \psi^j + \frac{1}{4} R_{ijkl}\psi^{*i}\psi^{*i}\psi^k\psi^l$$

$$H_4^{\text{classical}} \;\; = \frac{1}{2} g^{ij}(p_i + \cdots)(p_j + \cdots) - \frac{1}{4} R_{ijkl}\psi^{*i}\psi^{*i}\psi^k\psi^l$$

covariant Laplacian on differential forms

$$H_4^{\text{quantum}} \;\; = -\frac{1}{2}\nabla^2 - \frac{1}{4} R_{ABCD}\psi^{*A}\psi^{*B}\psi^C\psi^D$$

4)′ Euler index in the Hamiltonian view

$$L_4(x^i, \psi^i, \psi^{*i}) \;\; = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + i g_{ij} \psi^{*i} \nabla_\tau \psi^j + \frac{1}{4} R_{ijkl} \psi^{*i} \psi^{*i} \psi^k \psi^l$$

$$H_4^{\text{classical}} \quad = \frac{1}{2} g^{ij}(p_i + \cdots)(p_j + \cdots) - \frac{1}{4} R_{ijkl} \psi^{*i} \psi^{*i} \psi^k \psi^l$$

$$H_4^{\text{quantum}} \quad = -\frac{1}{2} \nabla^2_{scalar} + \cdots - \frac{1}{4} R_{ABCD} \psi^{*A} \psi^{*B} \psi^C \psi^D$$

4)′ Euler index in the Hamiltonian view

$$H_4^{\text{quantum}} \quad = -\frac{1}{2}\partial_i\partial_i + \cdots - \frac{1}{4}R_{ABCD}\psi^{*A}\psi^{*B}\psi^C\psi^D$$

in geodesic normal coordinates,
near at any given point

$$\text{tr}\left[(-1)^F e^{-\beta H}\right] \; = \int dx^d \sqrt{g}\, \text{tr}_F \left[(-1)^F \langle x|e^{-\beta H}|x\rangle\right]$$

4)′ Euler index in the Hamiltonian view

$$H_4^{\text{quantum}} \quad = -\frac{1}{2}\partial_i\partial_i + \cdots - \frac{1}{4}R_{ABCD}\psi^{*A}\psi^{*B}\psi^{C}\psi^{D}$$

in geodesic normal coordinates,
near at any given point

$$\mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \;= \int dx^d \sqrt{g}\;\mathrm{tr}_F\left[(-1)^F \langle x|e^{-\beta H}|x\rangle\right]$$

$$= \int dx^d \sqrt{g}\;\mathrm{tr}_F\left[(-1)^F G_\beta(x;x)\right]$$

$$G_\beta(x;y) \equiv \langle x|e^{-\beta H}|y\rangle$$

index theorems in the Hamiltonian view → heat kernel

$$\mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \quad = \int dx^d \sqrt{g}\, \mathrm{tr}_F \left[(-1)^F G_\beta(x;x)\right]$$

$$G_\beta(x;y) \equiv \langle x|e^{-\beta H}|y\rangle \qquad G_{\beta\to 0}(x;y) \to \langle x|y\rangle = \delta(x;y)$$

$$-\frac{\partial}{\partial\beta} G_\beta(x;y) = H G_\beta(x;y) \;= \left(H^{(0)} + H^{(1)}\right) G_\beta(x;y)$$

$$= \left(-\nabla^2_{scalar}/2 + H^{(1)}\right) G_\beta(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = HG_\beta(x;y) \; = \left(H^{(0)} + H^{(1)}\right) G_\beta(x;y)$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$G_\beta^{(0)}(x;y) \equiv \langle x|e^{\beta\nabla^2_{scalar}/2}|y\rangle \otimes 1_{\mathrm{fermion}}$$

$$\lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y) \otimes 1_{\mathrm{fermion}}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = HG_\beta(x;y) \; = \left(H^{(0)} + H^{(1)}\right)G_\beta(x;y)$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$G_\beta^{(0)}(x;y) \equiv \langle x|e^{\beta\nabla^2_{scalar}/2}|y\rangle \otimes 1_{\text{fermion}}$$

$$\langle x|e^{\beta\nabla^2_{scalar}/2}|y\rangle = \frac{1}{(2\pi\beta)^{d/2}}e^{-d(x;y)/2\beta} \;\to\; \frac{1}{(2\pi\beta)^{d/2}}e^{-(x-y)^2/2\beta} \quad \text{for } R^d$$

$$\langle x|e^{\beta\nabla^2_{scalar}/2}|x\rangle = \frac{1}{(2\pi\beta)^{d/2}}e^{-0^2/2\beta} = \frac{1}{(2\pi\beta)^{d/2}}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = HG_\beta(x;y) \; = \left(H^{(0)} + H^{(1)}\right)G_\beta(x;y)$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)} = H^{(0)}G_\beta^{(n+1)} + H^{(1)}G_\beta^{(n)}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)} = H^{(0)}G_\beta^{(n+1)} + H^{(1)}G_\beta^{(n)}$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)} = H^{(0)}G_\beta^{(n+1)} + H^{(1)}G_\beta^{(n)}$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = \int_0^\beta ds \int_z \frac{\partial}{\partial\beta}G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

$$+\lim_{s\to\beta}\int_z G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

$$\lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y)\otimes 1_{\mathrm{fermion}}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)} = H^{(0)}G_\beta^{(n+1)} + H^{(1)}G_\beta^{(n)}$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = -H^{(0)}\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

$$+H^{(1)}(x)G_\beta^{(n)}(x;y)$$

$$\lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y)\otimes 1_{\text{fermion}}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)} = H^{(0)}G_\beta^{(n+1)} + H^{(1)}G_\beta^{(n)}$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

$$-\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = -H^{(0)}\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)H^{(1)}(z)G_s^{(n)}(z;y)$$

$$+H^{(1)}(x)G_\beta^{(n)}(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = HG_\beta(x;y) \ = \left(H^{(0)} + H^{(1)}\right)G_\beta(x;y)$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$G_\beta^{(n)}(x;y) = -\int_0^\beta ds \int_z \ G^{(0)}_{\beta-s}(x;z)H^{(1)}G_s^{(n-1)}(z;y)$$

$$= (-1)^n \int_0^\beta ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \int_{z_1} \cdots \int_{z_n}$$

$$G^{(0)}_{\beta-s_1}(x;z_1)H^{(1)}G_{s_1-s_2}(z_1;z_2)\cdots H^{(1)}G^{(0)}_{s_n}(z_n;y)$$

heat kernel expansion : β power counting

1. each $G^{(0)}$ → $\beta^{-d/2}$
2. each x-integral → $\beta^{d/2}$
3. each s-integral → β
4. each derivative of x in $H^{(1)}$ → $\beta^{-1/2}$
5. each x in $H^{(1)}$ → $\beta^{1/2}$

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$$H_4^{\text{quantum}} \sim \underbrace{\boxed{-\frac{1}{2}\nabla^2_{scalar}}}_{=H^{(0)}} + \underbrace{\boxed{\Gamma\partial + \Gamma\Gamma - \frac{1}{4}R_{ABCD}\psi^{*A}\psi^{*B}\psi^{C}\psi^{D}}}_{=H^{(1)}}$$

$$\Gamma \sim \psi^* \psi R \Delta x$$

heat kernel expansion : β power counting

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$$H_4^{\text{quantum}} \sim \underbrace{\boxed{-\frac{1}{2}\nabla^2_{scalar}}}_{=H^{(0)}} + \underbrace{\boxed{\underset{\downarrow\,\beta}{\Gamma\partial} + \underset{\downarrow\,\beta^2}{\Gamma\Gamma} - \frac{1}{4}\underset{\downarrow\,\beta}{R_{ABCD}}\psi^{*A}\psi^{*B}\psi^{C}\psi^{D}}}_{=H^{(1)}}$$

$$\Gamma \sim \psi^*\psi R\Delta x$$

heat kernel expansion : β power counting

$$G_\beta(x;x) = \langle x | e^{-\beta H_4^{(0)} - \beta H_4^{(1)}} | x \rangle = \frac{1}{(2\pi\beta)^{d/2}} + G_\beta^{(1)}(x;x) + \cdots$$

$$(2\pi\beta)^{d/2} G_\beta^{(n)}(x;x) \sim \beta^n (R\psi^*\psi^*\psi\psi)^n + \beta^n (R\psi^*\psi)^n + \beta^{n+1}(R\psi^*\psi)^{n+1} + \cdots$$

$$H_4^{\text{quantum}} \sim \underbrace{\boxed{-\frac{1}{2}\nabla^2_{scalar}}}_{= H^{(0)}} + \underbrace{\boxed{\Gamma\partial + \Gamma\Gamma - \frac{1}{4}R_{ABCD}\psi^{*A}\psi^{*B}\psi^C\psi^D}}_{= H^{(1)}}$$

$$\Gamma \sim \psi^*\psi R \Delta x$$

β $\quad\beta^2\quad$ β

4)′ Euler index in the Hamiltonian view

$$\lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H_4}\right] \;= \lim_{\beta\to 0}\int dx^d\sqrt{g}\,\mathrm{tr}_F\left[(-1)^F G_\beta(x;x)\right]$$

$$(2\pi\beta)^{d/2}G_\beta^{(n)}(x;x) \sim \beta^n(R\psi^*\psi^*\psi\psi)^n + \beta^n(R\psi^*\psi)^n + \beta^{n+1}(R\psi^*\psi)^{n+1} + \cdots$$

$$\{\psi^{*A},\psi^B\} = \delta^{AB} \quad\to\quad \psi^A \;\sim\; (\gamma^{2A-1}+i\gamma^{2A})/2$$

$$\{(-1)^F,\psi^B\} = 0$$

$$\to\quad (-1)^F \;\sim \prod_{a=1}^{2d}\left(\gamma^a/\sqrt{2}\right)$$

$$\{(-1)^F,\psi^{*B}\} = 0$$

4)′ Euler index in the Hamiltonian view

$$\lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H_4}\right] = \lim_{\beta\to 0} \int dx^d \sqrt{g}\, \mathrm{tr}_F\left[(-1)^F G_\beta(x;x)\right]$$

$$(2\pi\beta)^{d/2} G_\beta^{(n)}(x;x) \sim \beta^n (R\psi^*\psi^*\psi\psi)^n \boxed{+ \beta^n (R\psi^*\psi)^n + \beta^{n+1}(R\psi^*\psi)^{n+1} + \cdots}$$

irrelevant for the index computation

$$\mathrm{tr}_F\left[(-1)^F \psi^{*A_1}\cdots\psi^{*A_d}\psi^{B_1}\cdots\psi^{B_d}\right] = (-1)^{d/2}\epsilon^{A_1\cdots A_d}\epsilon^{B_1\cdots B_d}$$

$$\mathrm{tr}_F\left[(-1)^F \psi^{*A_1}\cdots\psi^{*A_l}\psi^{B_1}\cdots\psi^{B_k}\right] = 0 \qquad \text{if } l < d \text{ or } k < d$$

4)′ Euler index in the Hamiltonian view

$$\lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H_4}\right] = \lim_{\beta\to 0} \int dx^d \sqrt{g}\, \mathrm{tr}_F\left[(-1)^F G_\beta(x;x)\right]$$

$$(2\pi\beta)^{d/2} G_\beta^{(n)}(x;x) \sim \beta^n (R\psi^*\psi^*\psi\psi)^n \boxed{+ \beta^n (R\psi^*\psi)^n + \beta^{n+1}(R\psi^*\psi)^{n+1} + \cdots}$$

irrelevant for the index computation

$$\lim_{\beta\to 0} \mathrm{tr}_F\left[(-1)^F G_\beta(x;x)\right] = \lim_{\beta\to 0} \mathrm{tr}_F\left[(-1)^F \frac{1}{(2\pi\beta)^{d/2}} e^{(\beta/4)R_{ABCD}(x)\psi^{*A}\psi^{*B}\psi^C\psi^D}\right]$$

$$= \lim_{\beta\to 0} \mathrm{tr}_F\left[(-1)^F \frac{\left(R_{ABCD}(x)\psi^{*A}\psi^{*B}\psi^C\psi^D/4\right)^{d/2}}{(2\pi)^{d/2}(d/2)!}\right]$$

4)′ Euler index in the Hamiltonian view

$$\lim_{\beta\to 0} \mathrm{tr}\left[(-1)^F e^{-\beta H_4}\right] = \lim_{\beta\to 0}\int dx^d \sqrt{g}\, \mathrm{tr}_F\left[(-1)^F G_\beta(x;x)\right]$$

$$= \frac{(-1)^{d/2}}{(2\pi)^{d/2}}\int \mathrm{Pf}(R_{AB})$$

$$R_{AB} \equiv \frac{1}{2} R_{ABik}(x) dx^i \wedge dx^k$$

$$\lim_{\beta\to 0} \mathrm{tr}_F\left[(-1)^F G_\beta(x;x)\right] = \lim_{\beta\to 0}\mathrm{tr}_F\left[(-1)^F \frac{1}{(2\pi\beta)^{d/2}} e^{(\beta/4)R_{ABCD}(x)\psi^{*A}\psi^{*B}\psi^C\psi^D}\right]$$

$$= \lim_{\beta\to 0}\mathrm{tr}_F\left[(-1)^F \frac{\left(R_{ABCD}(x)\psi^{*A}\psi^{*B}\psi^C\psi^D/4\right)^{d/2}}{(2\pi)^{d/2}(d/2)!}\right]$$

gauged quantum mechanics, or
how to rediscover the gauge field

gauged quantum mechanics

$$L_5(A_0, \phi^i, \psi^i) \quad = (D_\tau \phi)^* (D_\tau \phi) + i\psi^{*i} D_\tau \psi^j + \cdots$$

$$D_\tau = \partial_\tau - ieA_0$$

$$H_5 = \pi_\phi \dot{\phi} + \pi_{\phi *} \dot{\phi}^* + \pi_\psi \dot{\psi} - L_5 = \boxed{\pi_\phi \pi_\phi^* + \cdots} + A_0 e \boxed{G(\pi_\phi, \phi; \pi_{\phi^*}, \phi^*; \pi_\psi, \psi)}$$

Hamiltonian $= H_5'$

Gauss constraint

gauged quantum mechanics

$$L_5(A_0, \phi^i, \psi^i) \quad = (D_\tau \phi)^* (D_\tau \phi) + i\psi^* D_\tau \psi + \cdots$$

$$D_\tau = \partial_\tau - ieA_0$$

$$H_5 = \pi_\phi \dot{\phi} + \pi_{\phi *} \dot{\phi}^* + \pi_\psi \dot{\psi} - L_5 = \underbrace{\boxed{\pi_\phi \pi_\phi^* + \cdots}}_{\substack{\text{Hamiltonian} \\ = H_5'}} + A_0 e \underbrace{\boxed{G(\pi_\phi, \phi; \pi_{\phi^*}, \phi^*; \pi_\psi, \psi)}}_{\text{Gauss constraint}}$$

→ time evolution by H_5' with the constraint $G = 0$ imposed

gauged quantum mechanics

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H_5'}\delta(G)\right]$$

how in the world do we do such a computation ?

gauged quantum mechanics

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H_5'}\delta(G)\right]$$

how in the world do we do such a computation ?

consider the simple cases with Abelian gauge fields

$$G \;\rightarrow\; e \times \mathrm{integer}$$

$$\delta(G) \;=\; \delta_{G/e,0} \;\rightarrow\; \int_0^{2\pi} d\theta\, e^{i\theta G/e}$$

gauged quantum mechanics

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H_5'} \int_0^{2\pi} d\theta \, e^{i\theta G}\right]$$

how in the world do we do such a computation ?

consider the simple cases with Abelian gauge fields

$$eG \;\rightarrow\; e \times \text{integer}$$

$$\delta(G) \;=\; \delta_{G,0} \;\rightarrow\; \int_0^{2\pi} d\theta \, e^{i\theta G}$$

gauged quantum mechanics

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H_5'} \int_0^{2\pi} d\theta\, e^{i\theta G}\right]$$

again, how in the world do we do such a computation ?

$$= \int d\phi d\phi^* \ \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_5'} \int_0^{2\pi} d\theta e^{i\theta G} | \phi, \phi^* \rangle\right]$$

$$= \int_0^{2\pi} d\theta \int d\phi d\phi^* \ \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_5'} e^{i\theta G_F} | e^{i\theta}\phi, e^{-i\theta}\phi^* \rangle\right]$$

$$H_5' = \pi_\phi \pi_\phi^* + \cdots$$

gauged quantum mechanics

$$Z_{twisted} \;\; = \int_0^{2\pi} d\theta \int d\phi d\phi^* \; \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_5'} e^{i\theta G_F} | e^{i\theta}\phi, e^{-i\theta}\phi^* \rangle \right]$$

$$H_5' = \pi_\phi \pi_\phi^* + \cdots$$

which should be contrasted against previous ungauged cases

$$\mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \;\; = \int dx^d \sqrt{g}\, \mathrm{tr}_F \left[(-1)^F \langle x | e^{-\beta H} | x \rangle \right]$$

$$= \int dx^d \sqrt{g}\, \mathrm{tr}_F \left[(-1)^F G_\beta(x;x) \right]$$

$$G_\beta(x;y) \equiv \langle x | e^{-\beta H} | y \rangle$$

gauged quantum mechanics

$$Z_{twisted} = \int_0^{2\pi} d\theta \int d\phi d\phi^* \ \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_5'} e^{i\theta G_F/e} | e^{i\theta}\phi, e^{-i\theta}\phi^* \rangle\right]$$

$$H_5' = \pi_\phi \pi_\phi^* + \cdots$$

which should be contrasted against previous ungauged cases

$$\langle \phi, \phi^* | e^{-\beta H_5^{(0)\prime}} | e^{i\theta}\phi, e^{-i\theta}\phi^* \rangle \sim \frac{1}{(2\pi\beta)^m} e^{-|\phi - e^{i\theta}\phi|^2/\beta}$$

$$\sim \frac{1}{(2\pi\beta)^m} e^{-\theta^2|\phi|^2/\beta} \qquad \text{if } |\theta| \ll 1$$

$$\sim \frac{1}{(2\pi\beta)^m} e^{-\beta(\theta/\beta)^2|\phi|^2}$$

gauged quantum mechanics

$$\simeq \int_0^{2\pi/\beta} d(\theta/\beta) \int d\phi d\phi^* \ \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_5^{(1)'}} e^{-\beta(\theta/\beta)^2 |\phi|^2} e^{i\theta G_F} | \phi, \phi^* \rangle \right]$$

$$\uparrow$$

$$\langle \phi, \phi^* | e^{-\beta H_5^{(0)'}} | e^{i\theta}\phi, e^{-i\theta}\phi^* \rangle \sim \frac{1}{(2\pi\beta)^m} e^{-|\phi - e^{i\theta}\phi|^2/\beta}$$

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gauged quantum mechanics

$$\simeq \int_0^{2\pi/\beta} d(\theta/\beta) \int d\phi d\phi^* \; \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_5^{(1)\prime}} e^{-\beta(\theta/\beta)^2 |\phi|^2} e^{i\theta G_F} | \phi, \phi^* \rangle \right]$$

$$\simeq \int_0^{2\pi/e\beta} dA_0 \int d\phi d\phi^* \; \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_5^{(1)\prime}} \boxed{e^{-\beta e^2 A_0^2 |\phi|^2} e^{i\beta e A_0 G_F / e}} | \phi, \phi^* \rangle \right]$$

$$=$$

$$\boxed{\left. e^{-\int_0^\beta d\tau (D_\tau \phi)^* (D_\tau \phi) + i\psi^{*i} D_\tau \psi^j} \right|_{\partial_\tau \to 0; A_0 \to i A_0}}$$

index of gauged quantum mechanics

$$L_5(A_0, \phi^i, \psi^i) \quad = (D_\tau \phi)^* (D_\tau \phi) + i\psi^{*i} D_\tau \psi^j + \cdots$$

$$D_\tau = \partial_\tau - ieA_0$$

again, we are lead to the Euclidean path integral

with periodic boundary condition,

where A_0 is Euclideanized and frozen to be time-independent equivariant index and how it localizes the computation

equivariant index
→ quantum mechanics with global symmetry

$$L_6(\phi^i,\psi^i) = L_5\Big|_{A_0\to 0} = (\partial_\tau\phi)^*(\partial_\tau\phi) + i\psi^{*i}\partial_\tau\psi^j + \cdots$$

$e^{i\theta G(\pi_\phi,\phi;\pi_{\phi^*},\phi^*;\pi_\psi,\psi)}$: $\qquad \phi \to e^{i\theta}\phi$

is now a global symmetry $\qquad \psi \to e^{i\theta}\psi$

$$Z^{\rm equivariant}_{twisted} = {\rm tr}\left[(-1)^F e^{-\beta H_6} e^{i\mu G}\right]$$

equivariant index
→ quantum mechanics with global symmetry

$$Z_{twisted}^{\text{equivariant}} = \int d\phi d\phi^* \ \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_6'} e^{i\mu G_F} | e^{i\mu}\phi, e^{-i\mu}\phi^* \rangle \right]$$

$$\langle \phi, \phi^* | e^{-\beta H_6^{(0)'}} | e^{i\mu}\phi, e^{-i\mu}\phi^* \rangle \sim \frac{1}{(2\pi\beta)^m} e^{-|\phi - e^{i\mu}\phi|^2/\beta}$$

$$\sim \frac{1}{(2\pi\beta)^m} e^{-\mu^2|\phi|^2/\beta}$$

$$Z_{twisted}^{\text{equivariant}} = \mathrm{tr}\left[(-1)^F e^{-\beta H_6} e^{i\mu G} \right]$$

equivariant index
→ quantum mechanics with global symmetry

$$Z^{\rm equivariant}_{twisted} \; = \int d\phi d\phi^* \; {\rm tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_6'} e^{i\mu G_F} | e^{i\mu}\phi, e^{-i\mu}\phi^* \rangle \right]$$

$$\langle \phi, \phi^* | e^{-\beta H_6^{(0)'}} | e^{i\mu}\phi, e^{-i\mu}\phi^* \rangle \sim \frac{1}{(2\pi\beta)^m} e^{-|\phi - e^{i\mu}\phi|^2/\beta}$$

$$\sim \frac{1}{(2\pi\beta)^m} e^{-\mu^2|\phi|^2/\beta}$$

unlike θ of the gauged case, μ is not a dummy variable to be integrated over

→ in the small β limit, the computation received contribution

from saddle points (submanifold) invariant under the global symmetry

equivariant index
→ quantum mechanics with global symmetry

$$Z^{\text{equivariant}}_{twisted} = \int d\phi d\phi^* \ \mathrm{tr}_F \left[(-1)^F \langle \phi, \phi^* | e^{-\beta H_6'} e^{i\mu G_F} | e^{i\mu}\phi, e^{-i\mu}\phi^* \rangle \right]$$

$$\langle \phi, \phi^* | e^{-\beta H_6^{(0)'}} | e^{i\mu}\phi, e^{-i\mu}\phi^* \rangle \sim \frac{1}{(2\pi\beta)^m} e^{-|\phi - e^{i\mu}\phi|^2/\beta}$$

$$\sim \frac{1}{(2\pi\beta)^m} e^{-\mu^2|\phi|^2/\beta}$$

we are lead to the Euclidean path integral

with the global charge coupled to external gauge field

with the gauge field fixed at the value $\mu/e\beta$

equivariant Euler index

$$L_4(x^i, \psi^i, \psi^{*i})^{\text{equivariant}}$$

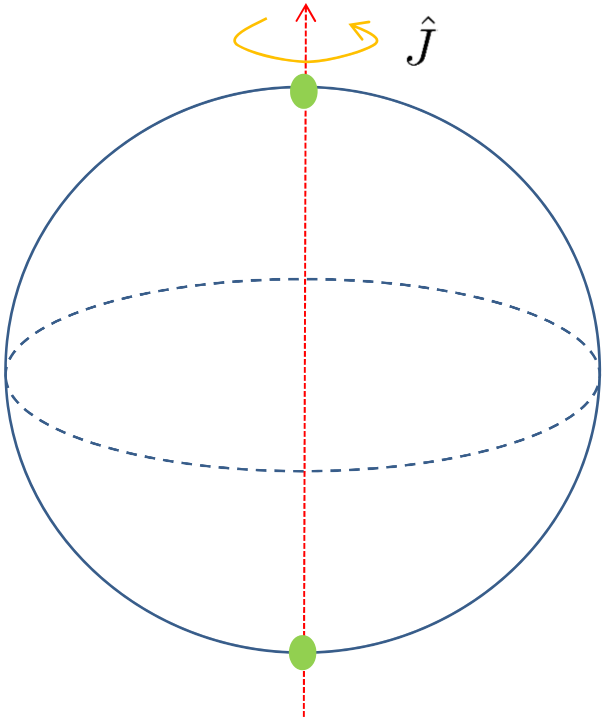
$$= \frac{1}{2} g_{ij}(x) D_\tau x^i D_\tau x^j + i g_{ij} \psi^{*i}(D_\tau \psi^j + \Gamma^j_{kl} D_\tau x^k \psi^l) + \frac{1}{4} R_{ijkl} \psi^{*i} \psi^{*i} \psi^k \psi^l$$

$$D_\tau = \partial_\tau + i(\mu/\beta)\hat{J}$$

$$\lim_{\beta \to 0} \operatorname{tr}\left[(-1)^F e^{-\beta H_4} e^{i\mu \hat{J}}\right]$$

$= \text{Gaussian Integral at North Pole}$

$+ \text{Gaussian Integral at South Pole}$

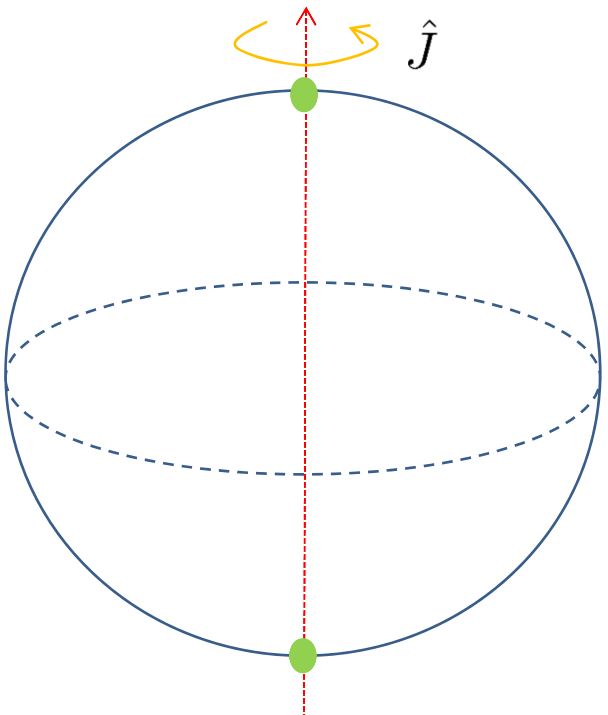


refined Euler index

$$L_4(x^i, \psi^i, \psi^{*i})^{\text{refined}}$$

$$= \frac{1}{2} g_{ij}(x) D_\tau x^i D_\tau x^j + i g_{ij} \psi^{*i} (D_\tau \psi^j + \Gamma^j_{kl} D_\tau x^k \psi^l) + \frac{1}{4} R_{ijkl} \psi^{*i} \psi^{*i} \psi^k \psi^l$$

$$D_\tau = \partial_\tau + i(\mu/\beta)\hat{J}$$



$$\lim_{\beta \to 0} \operatorname{tr}\left[(-1)^F e^{-\beta H_4} e^{i\mu \hat{J}}\right]$$

$$= \text{Gaussian Integral at North Pole}$$

$$+ \text{Gaussian Integral at South Pole}$$

$$= 1 + 1 = 2$$

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