algebra, geometry, and Schroedinger atoms

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Atiyah-Singer Index Theorem ~ 1963
Calabi-Yau ~ 1978
Calibrated Geometry ~ 1982
(Harvey & Lawson)
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Homological Mirror Symmetry ~ 1994
(Kontsevich)
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.
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Stability & Derived Category ~ 2000
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Wall-Crossing Conjecture~ 2008
(Conjecture by Kontsevich & Soibelman)
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.

1975 ~ Bogomolnyi-Prasad-Sommerfeld (BPS)
1977 ~ Supersymmetry
1983 ~ Superstring Theory
.
1985 ~ Calabi-Yau Compactification
1988 ~ Mirror Symmetry
.
1994 ~ Wall-Crossing Discovered
(Seiberg & Witten)
1995 ~ Dirichlet Branes
1998 ~ Wall-Crossing is Bound State
Dissociation (Lee & P.Y.)
.
2001 ~ Wall-Crossing for Black Holes
(Denef)
.
.
.
.
2008 ~ Konstevich-Soibelman Explained
(Gaiotto & Moore & Neitzke)
.
2011 ~ KS Wall-Crossing proved
via Quatum Mechanics
Manschot , Pioline & Sen /
Kim , Park, Wang & P.Y. / Sen

quantum is algebraic

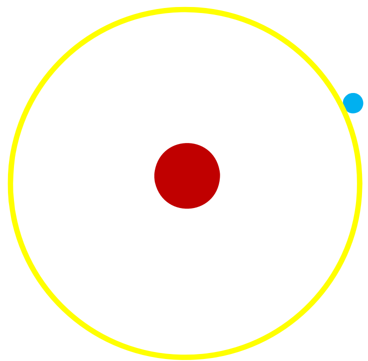
spacetime is geometric

a little bit of superstring theory

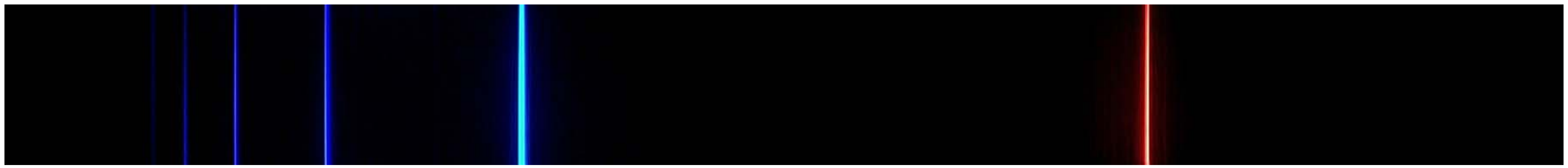
particles from geometry / geometry from particles

how quantum mechanics solved modern geometry quantum is algebraic

Balmer / Rydberg ~ 1880's

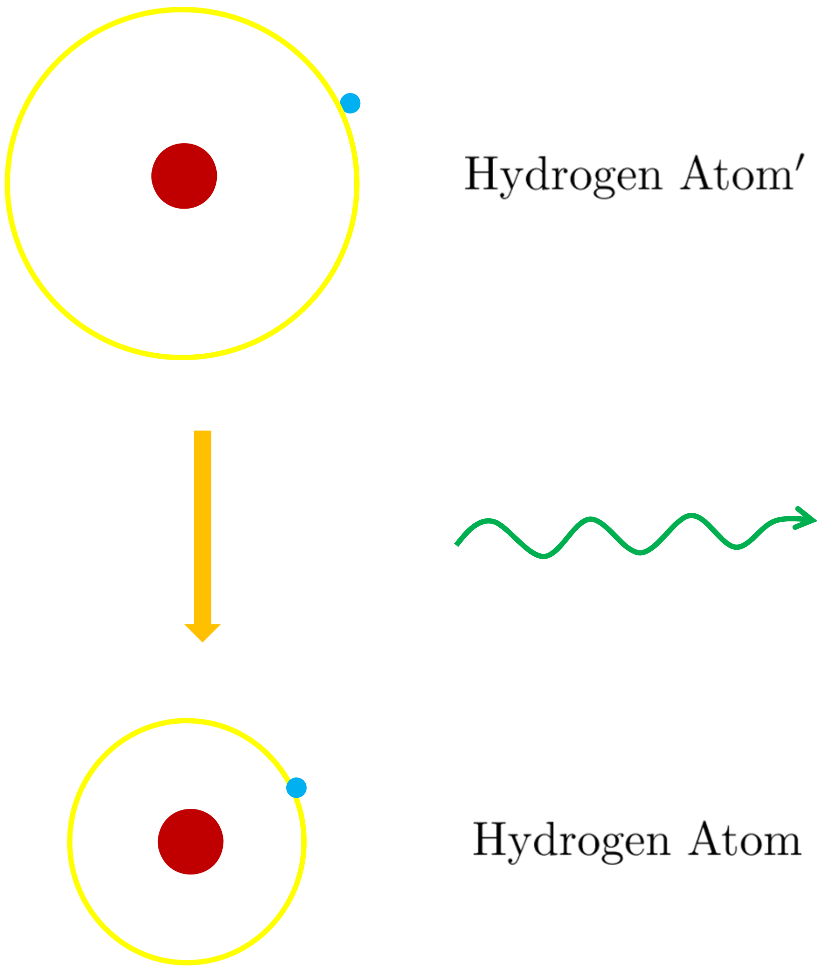


Hydrogen Atom



$$\lambda \sim 10^{-4}cm - 10^{-5}cm \;\; = 10000\text{Å}{-}1000\text{Å}$$

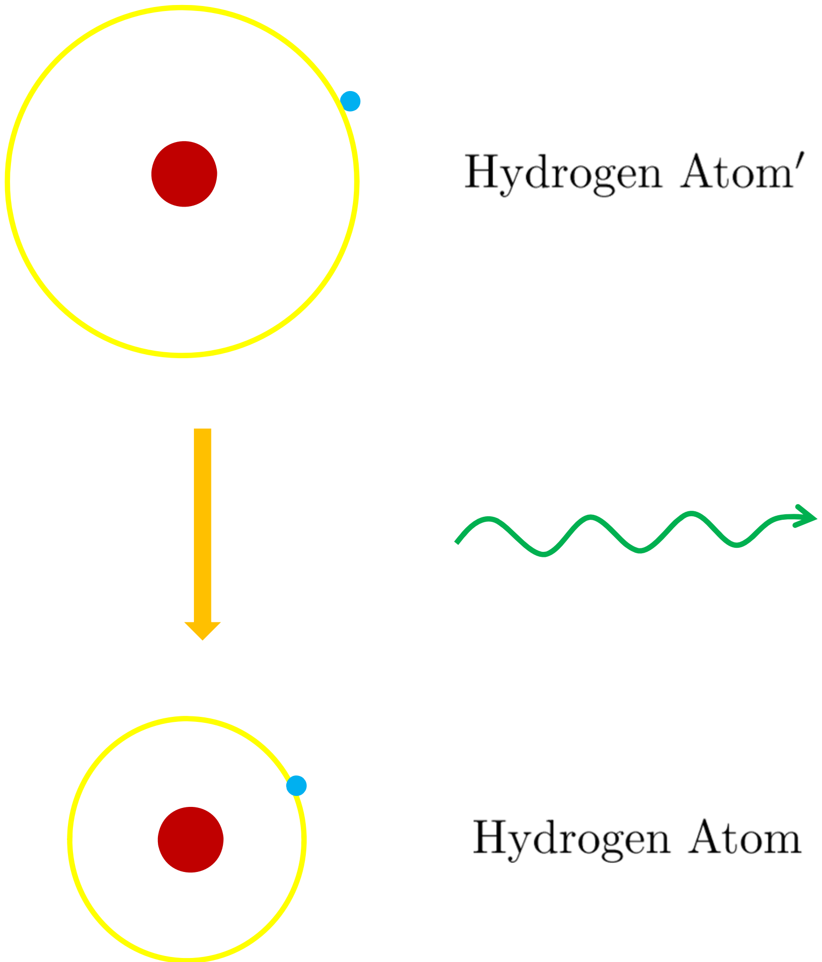
Balmer / Rydberg



$$\Delta E \ : \ 1 \sim 10 \ eV$$

$$1 \ eV \simeq 1.6 \times 10^{-19} Joule$$

Balmer / Rydberg



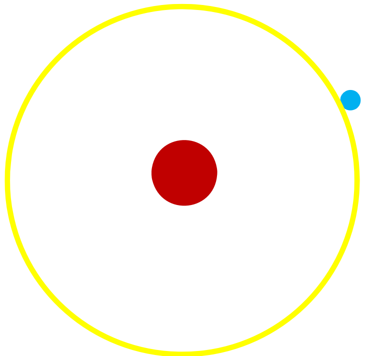
$$n, k = 1, 2, 3, \ldots$$

$$\Delta E = E_0 \times \left(\frac{1}{n^2} - \frac{1}{k^2} \right)$$

$$E_0 \simeq 13.6 \; eV$$

$$1 \; eV \simeq 1.6 \times 10^{-19} Joule$$

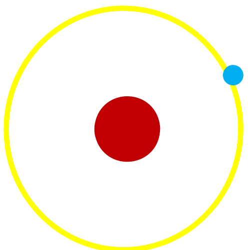
Balmer / Rydberg



$$\text{Hydrogen Atom}\Big|_k \qquad n, k = 1, 2, 3, \ldots$$

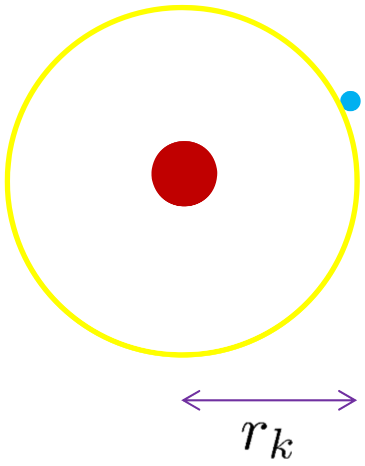
$$\Delta E = E_0 \times \left(\frac{1}{n^2} - \frac{1}{k^2} \right)$$

$$= E_k - E_n$$



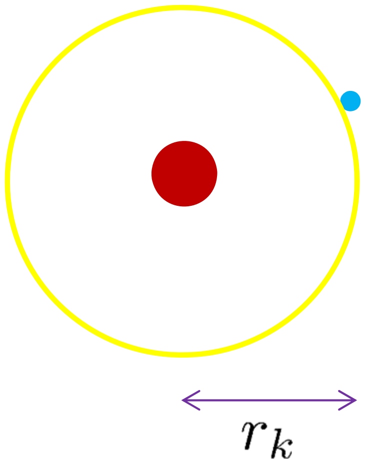
$$\text{Hydrogen Atom}\Big|_n$$

Bohr Atom ~ 1913



$$E_k = -E_0 \times \frac{1}{k^2} \qquad k = 1, 2, 3, \ldots$$

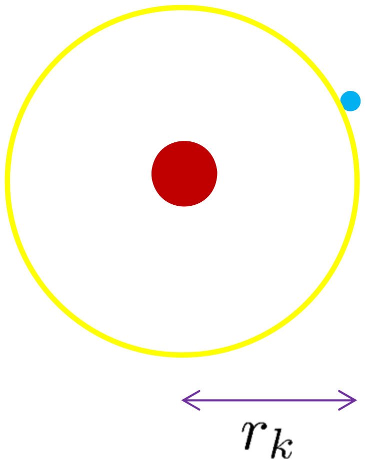
Bohr Atom



$$E_k = -E_0 \times \frac{1}{k^2} \qquad k = 1, 2, 3, \ldots$$

$$E(r)_{classical} = \frac{m_e v^2}{2} - \frac{e^2}{r}$$

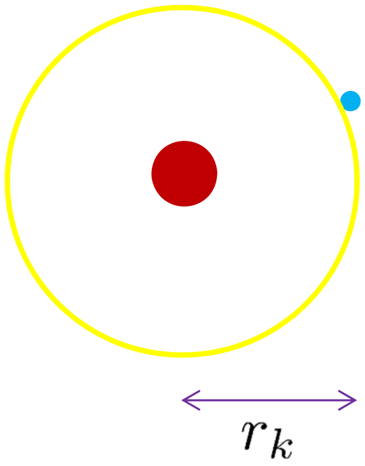
Bohr Atom



$$E_k = -E_0 \times \frac{1}{k^2} \qquad k = 1, 2, 3, \ldots$$

$$E(r) = \frac{m_e v^2}{2} - \frac{e^2}{r} \qquad \left.\frac{m_e v^2}{r}\right|_{orbit} = \left.\frac{e^2}{r^2}\right|_{orbit}$$

Bohr Atom



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Bohr's rule →

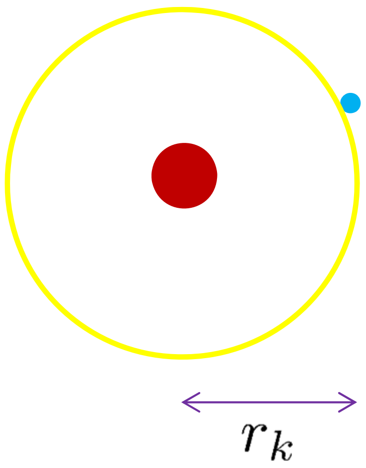
$$m_e v_k \times 2\pi r_k = k \times h$$

$$\Rightarrow m_e v_k \times r_k = k \times \hbar$$

→

$$\sqrt{\frac{e^2 r_k}{m_e}} = k \times \hbar$$

Bohr Atom



$$E_k = -E_0 \times \frac{1}{k^2} \qquad k = 1, 2, 3, \ldots$$

$$= E(r_k) = \frac{m_e v_k^2}{2} - \frac{e^2}{r_k} = -\underbrace{\boxed{\frac{e^4 m_e}{\hbar^2}}}_{\simeq 13.6\ eV = E_0} \frac{1}{k^2}$$

Bohr's rule →

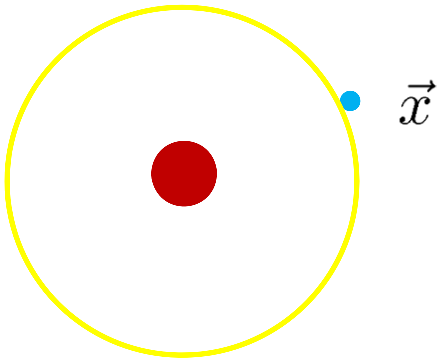
$$m_e v_k \times 2\pi r_k = k \times h$$

$$\Rightarrow m_e v_k \times r_k = k \times \hbar$$

→

$$\sqrt{\frac{e^2 r_k}{m_e}} = k \times \hbar$$

Schroedinger ~ 1925

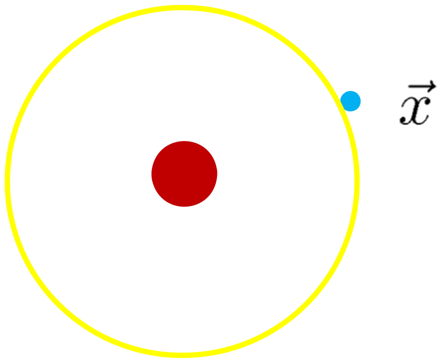


$$H|\Psi_{k,\ldots}\rangle = E_k|\Psi_{k,\ldots}\rangle$$

$$H = \frac{\vec{p}^{\,2}}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

$$\vec{p}_{classical} = m_e \frac{d\vec{x}_{classical}}{dt}$$

Heisenberg : Quantum is Algebraic



$$H|\Psi_{k,\dots}\rangle = E_k|\Psi_{k,\dots}\rangle$$

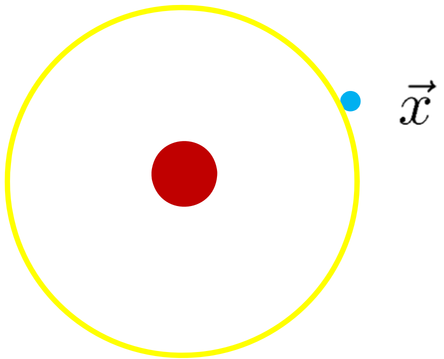
$$H = \frac{\vec{p}^2}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

$$[p_1, x^1] \equiv p_1 x^1 - x^1 p_1 = i\hbar$$

$$[p_2, x^2] = i\hbar$$

$$[p_3, x^3] = i\hbar$$

Heisenberg : Quantum is Algebraic



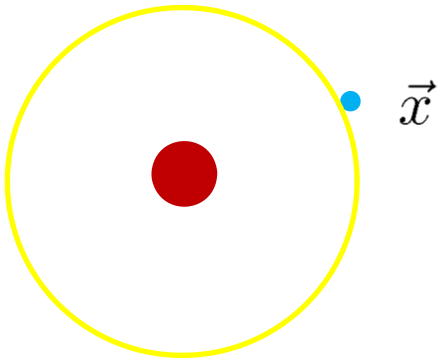
$$H|\Psi_{k,\ldots}\rangle = E_k|\Psi_{k,\ldots}\rangle$$

$$H = \frac{\vec{p}^{\,2}}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

$$\vec{L} = \vec{x} \times \vec{p}$$

$$[H, \vec{L}] = 0 \quad \Longrightarrow \quad SU(2) \sim \begin{pmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{pmatrix}$$

Heisenberg : Quantum is Algebraic



$$H|\Psi_{k,\ldots}\rangle = E_k|\Psi_{k,\ldots}\rangle$$

$$H = \frac{\vec{p}^{\,2}}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

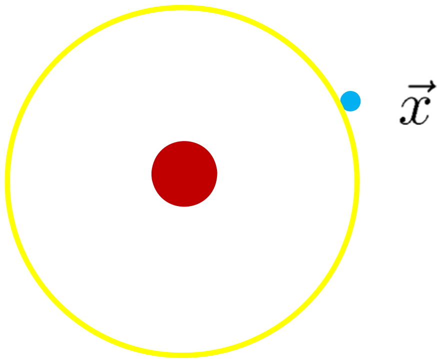
$$\vec{L} = \vec{x} \times \vec{p}$$

$$[H, \vec{L}] = 0 = [H, \vec{K}] \quad \Longrightarrow \quad SO(4) \sim \begin{pmatrix} 0 & \tilde{K}_1 & \tilde{K}_2 & \tilde{K}_3 \\ -\tilde{K}_1 & 0 & L_3 & -L_2 \\ -\tilde{K}_2 & -L_3 & 0 & L_1 \\ -\tilde{K}_3 & L_2 & -L_1 & 0 \end{pmatrix}$$

$$\vec{K} = \frac{1}{2m}\left(\vec{L} \times \vec{p} - \vec{p} \times \vec{L}\right) + e^2 \frac{\vec{x}}{|\vec{x}|}$$

$$\tilde{K}_i = K_i \sqrt{\frac{m}{-2H}}$$

Heisenberg : Quantum is Algebraic



$$H|\Psi_{k,\ldots}\rangle = E_k|\Psi_{k,\ldots}\rangle$$

$$H = \frac{\vec{p}^{\,2}}{2m_e} - \frac{e^2}{\sqrt{|\vec{x}|^2}}$$

$E_1 = -E_0 \simeq -13.6\ eV$ $\qquad$ 2×1 distinct $|\Psi_{1,\ldots}\rangle$'s

$E_2 = -E_0/4 \simeq -3.4\ eV$ $\qquad$ 2×4 distinct $|\Psi_{2,\ldots}\rangle$'s

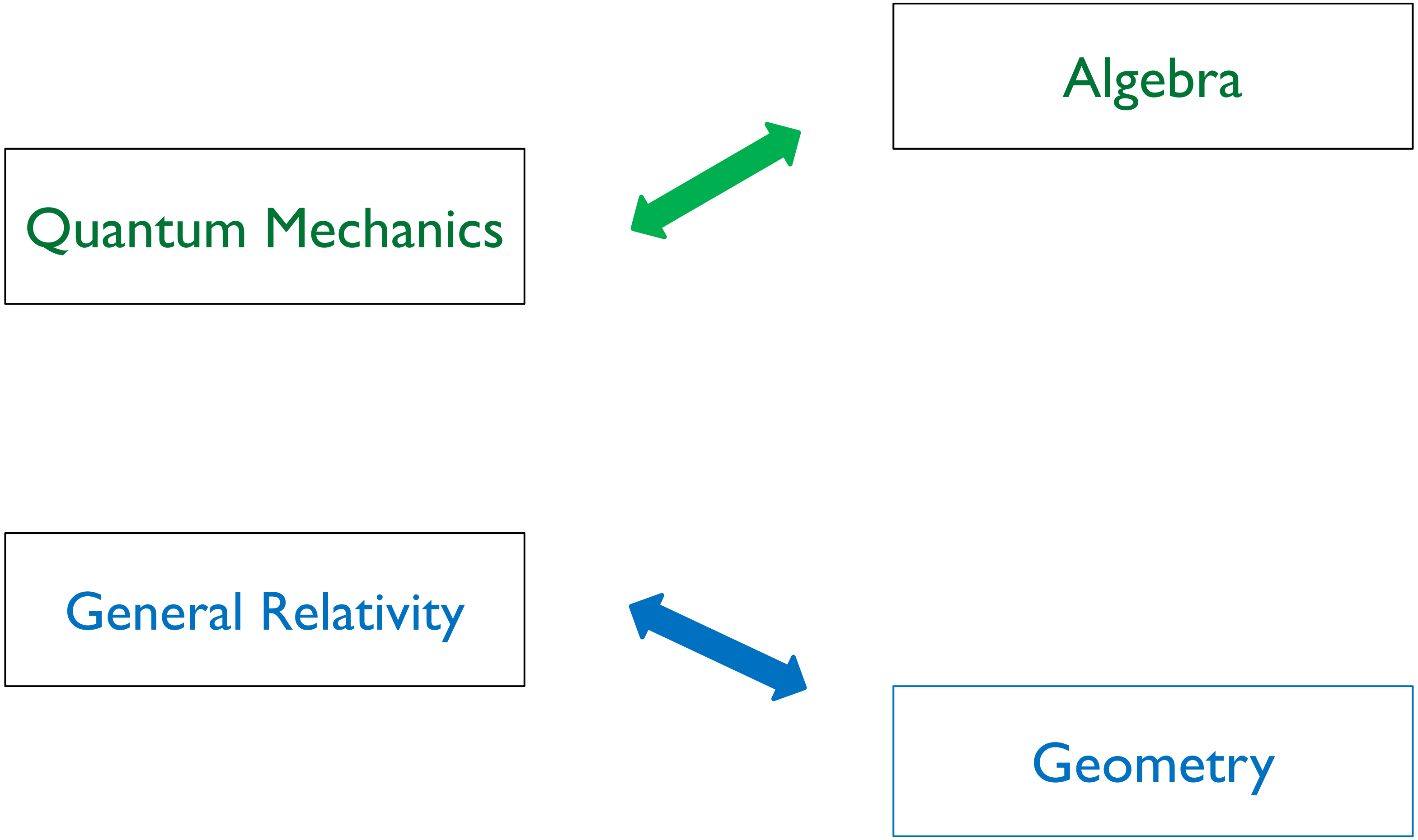
$E_3 = -E_0/9 \simeq -1.5\ eV$ $\qquad$ 2×9 distinct $|\Psi_{3,\ldots}\rangle$'s

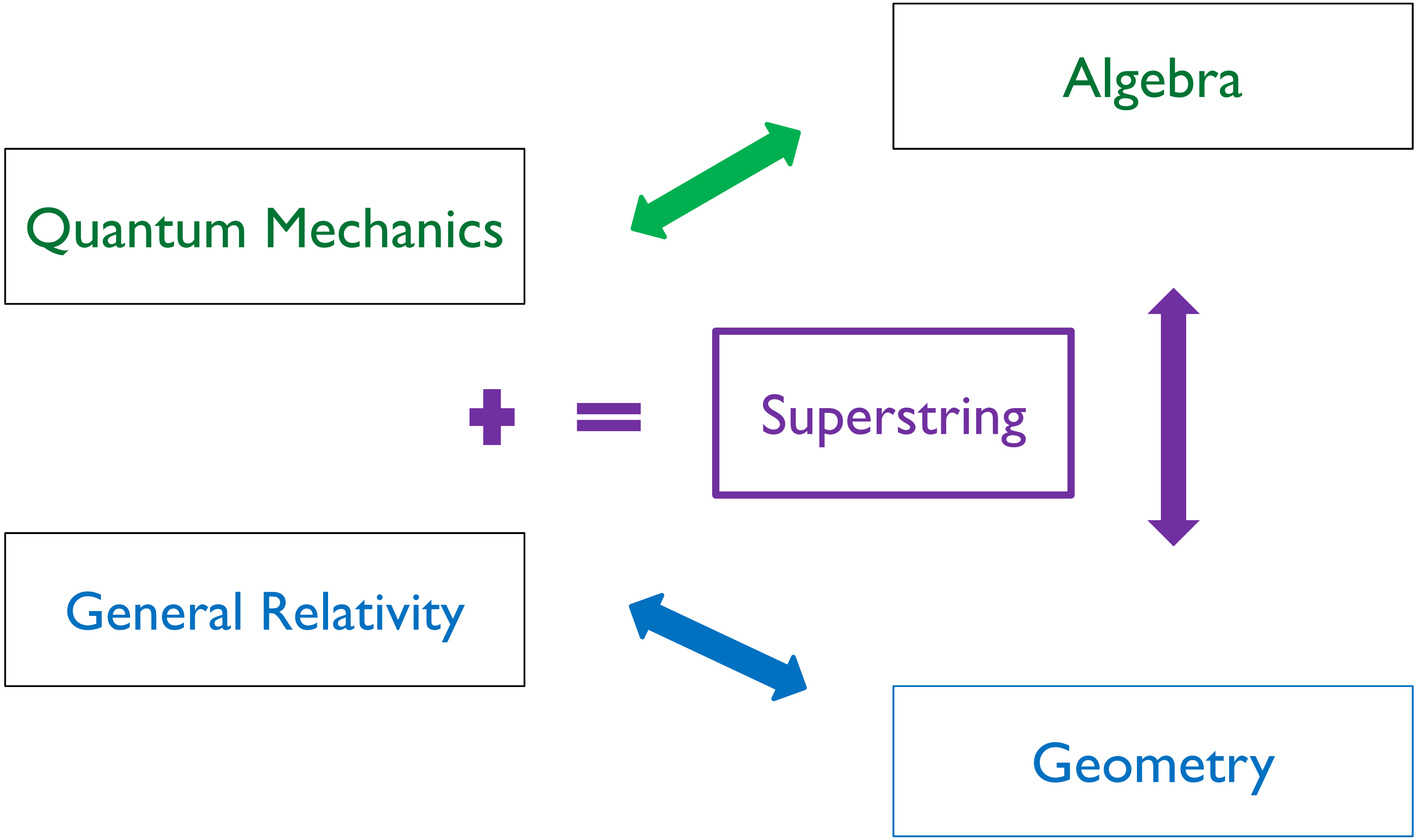
Periodic Table of the Elements

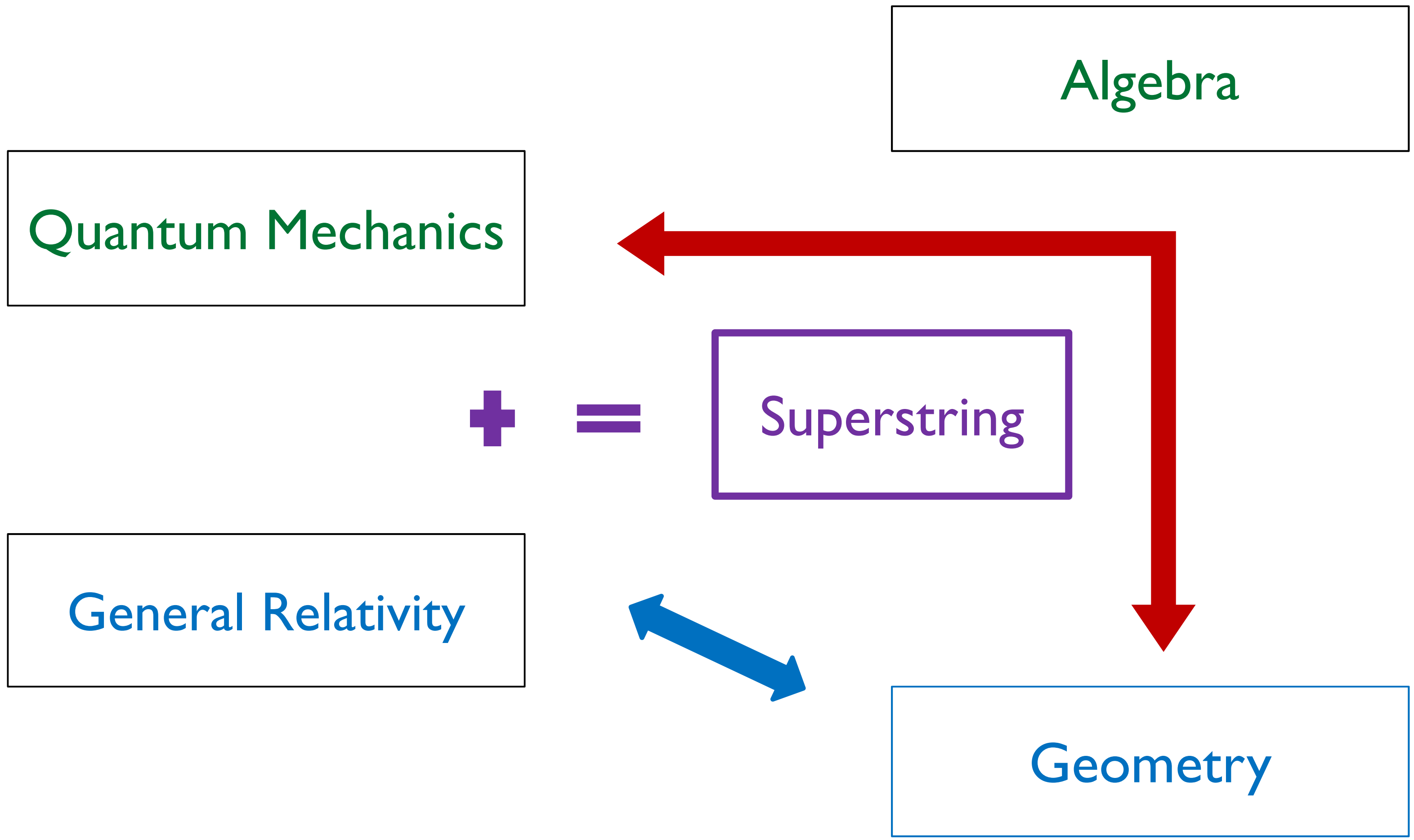
- hydrogen
- alkali metals
- alkali earth metals
- transition metals
- poor metals
- nonmetals
- noble gases
- rare earth metals

1 H																	2 He
3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	89 Ac	104 Unq	105 Unp	106 Unh	107 Uns	108 Uno	109 Une	110 Unn								

58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu
90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr


Algebra
Quantum Mechanics
General Relativity
Geometry


Algebra
Quantum Mechanics
Superstring
General Relativity
Geometry


Algebra
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Superstring
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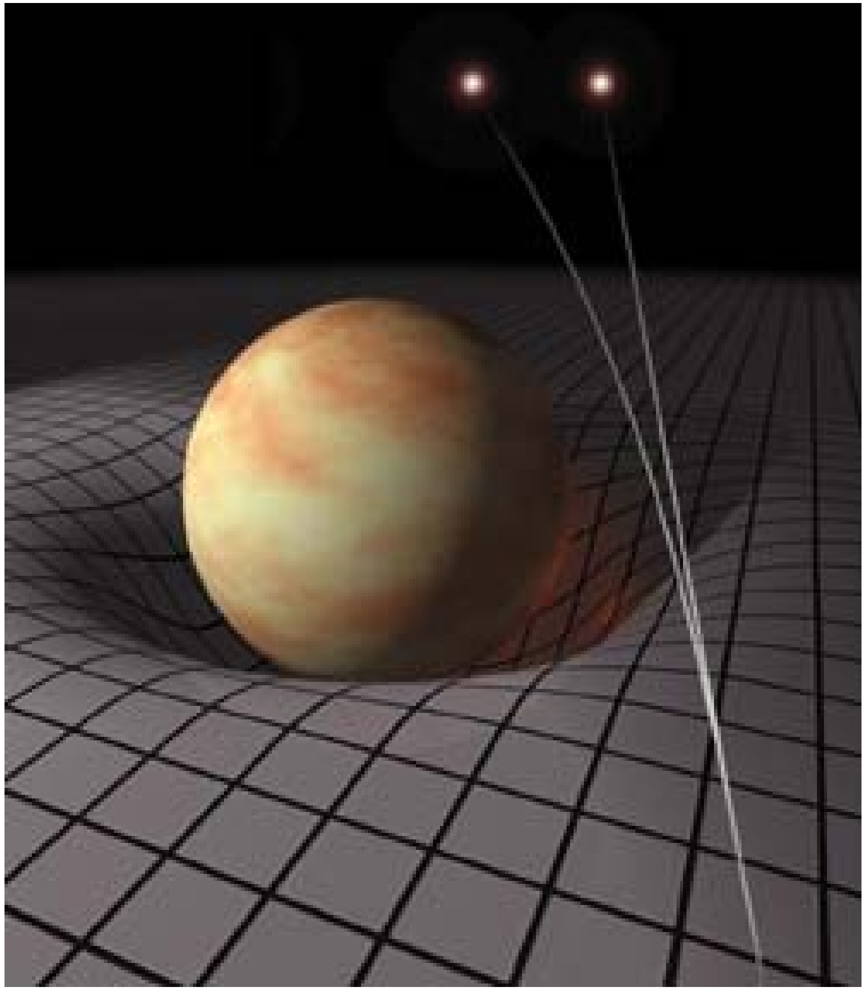
Geometry

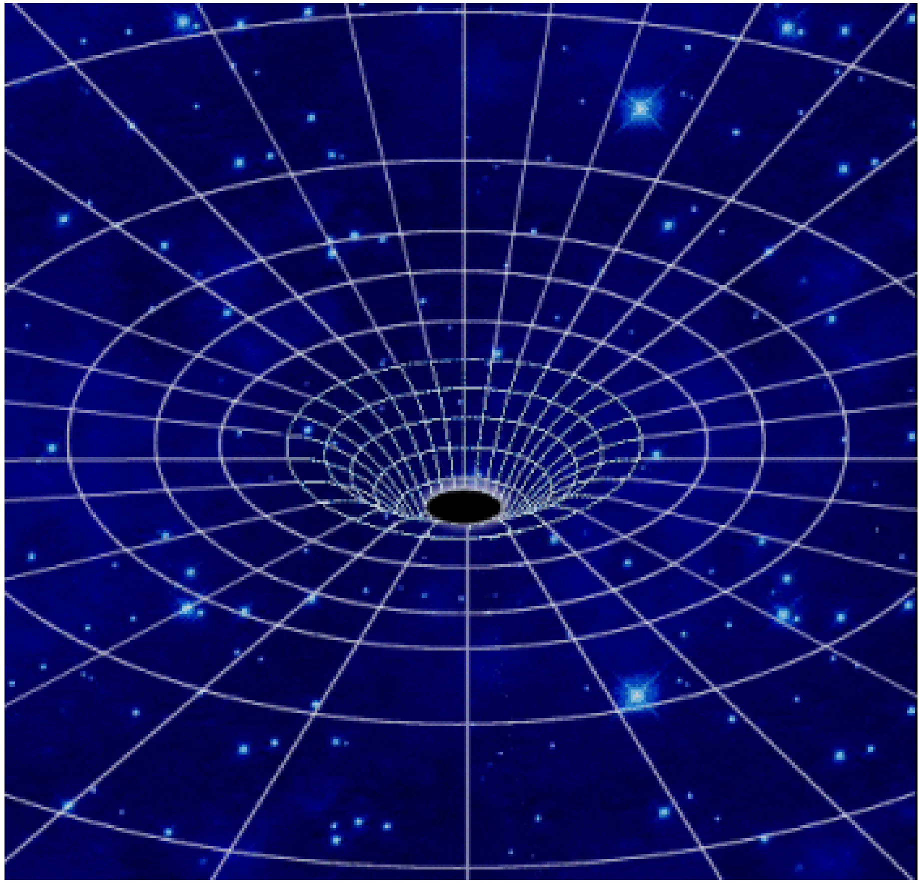
spacetime is geometric

Einstein ~ 1915



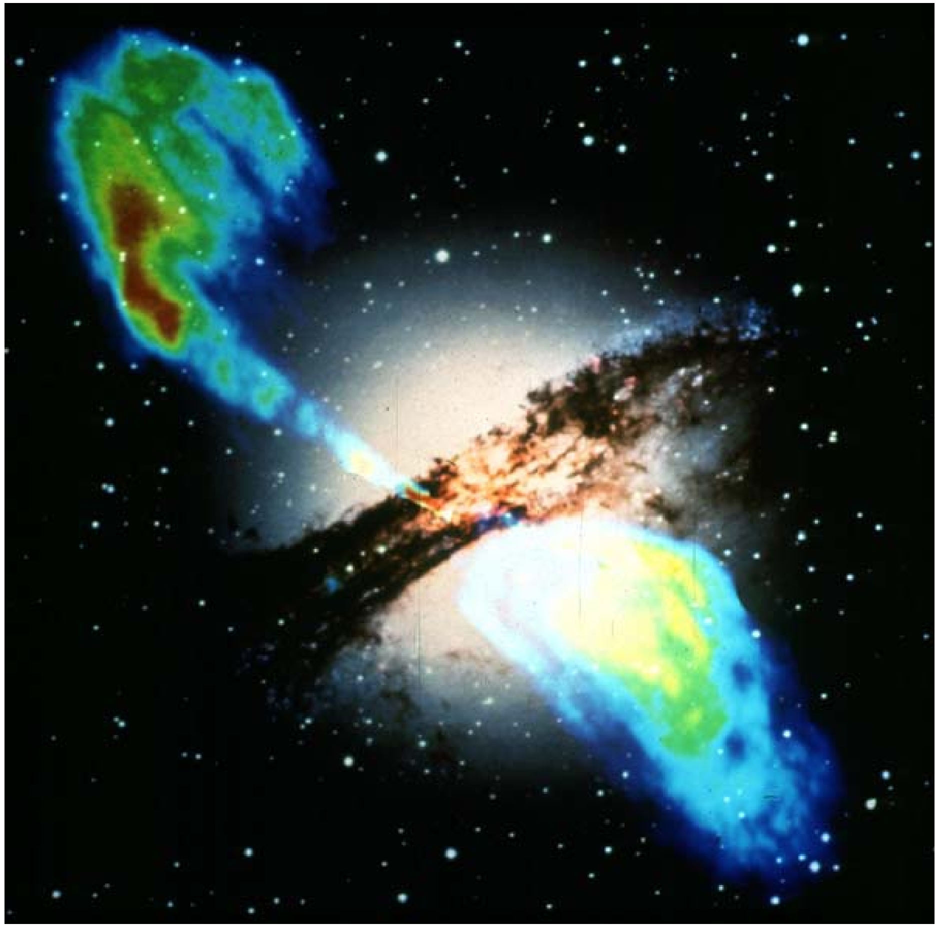
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

Black Holes ~ 1915



$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

Real Black Holes in Galactic Centers

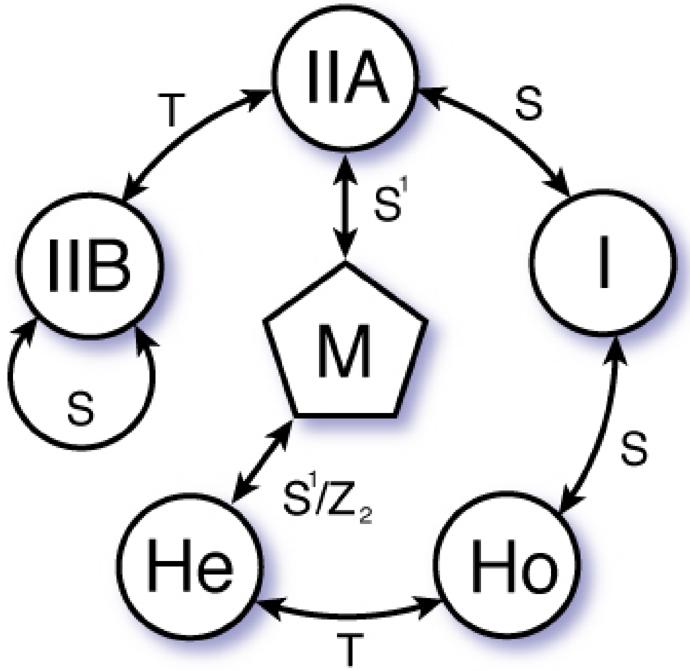


$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

Kaluza & Klein (~1921) asked :

can the world has more than
3 spatial dimensions or 4 spacetime dimensions
with the extra directions curled up so small to be practically invisible
and, if so, what are the physical consequences ?

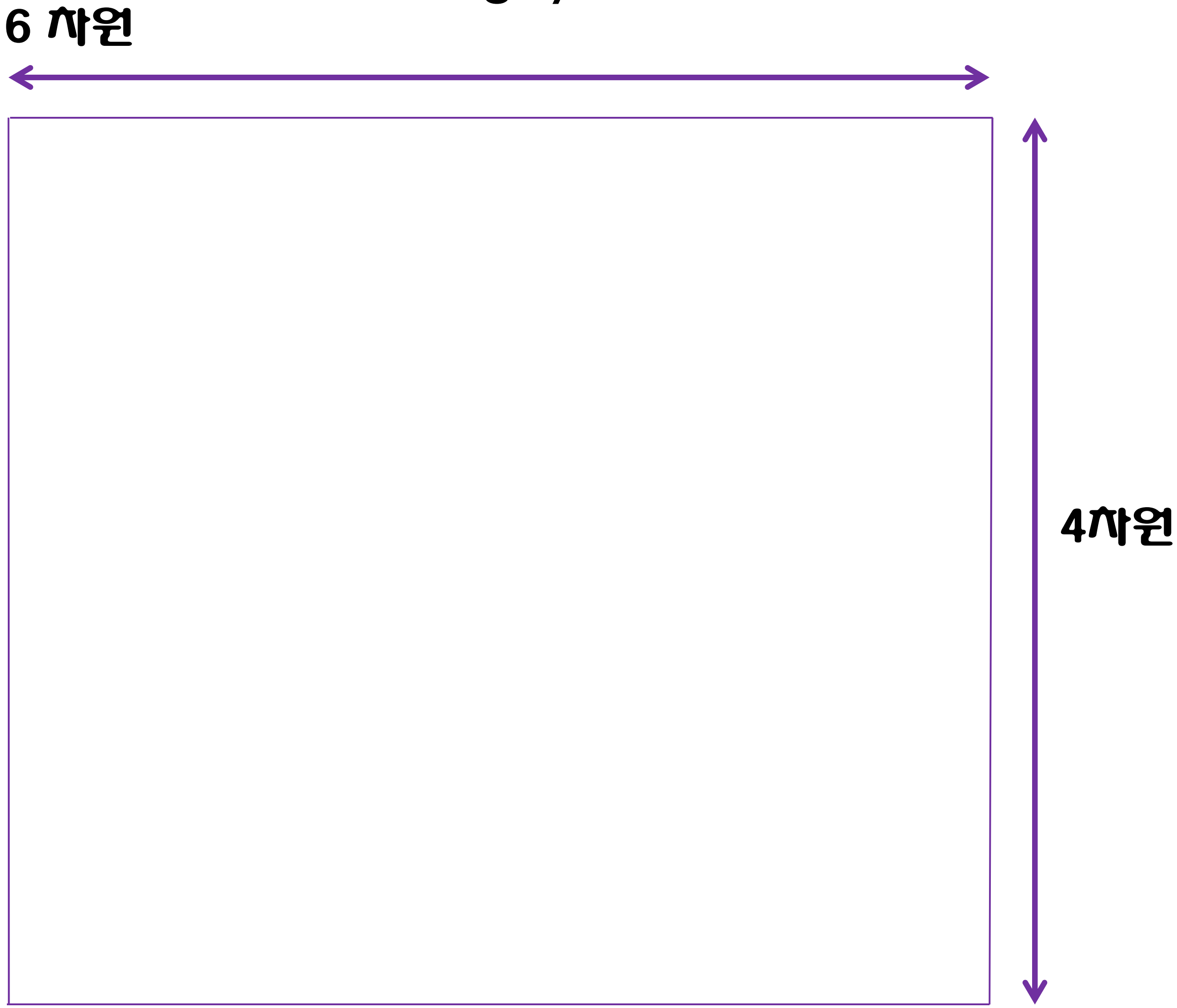
after all, space (& time) is supposed to curved five superstring theories live in 10 dimensional spacetime



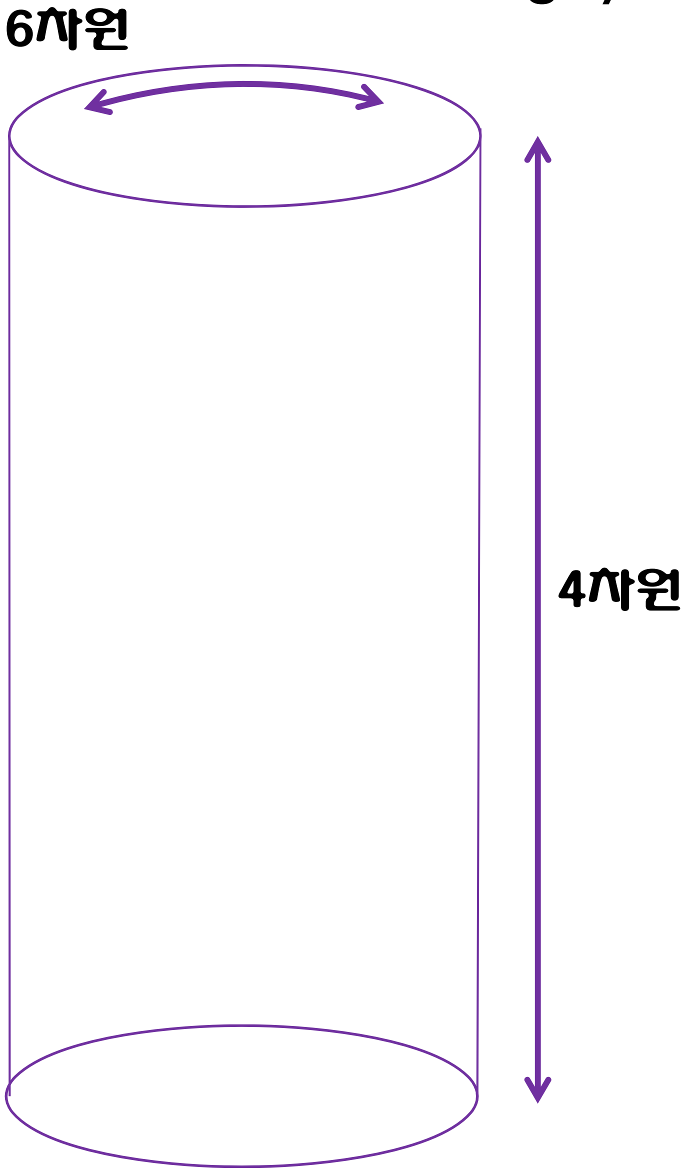
superstring theory says :

spacetime is composed of 4+6 dimensions with
very small & tightly-curved (say, Calabi-Yau) 6D manifold
sitting at each and every point of usual 3D space,

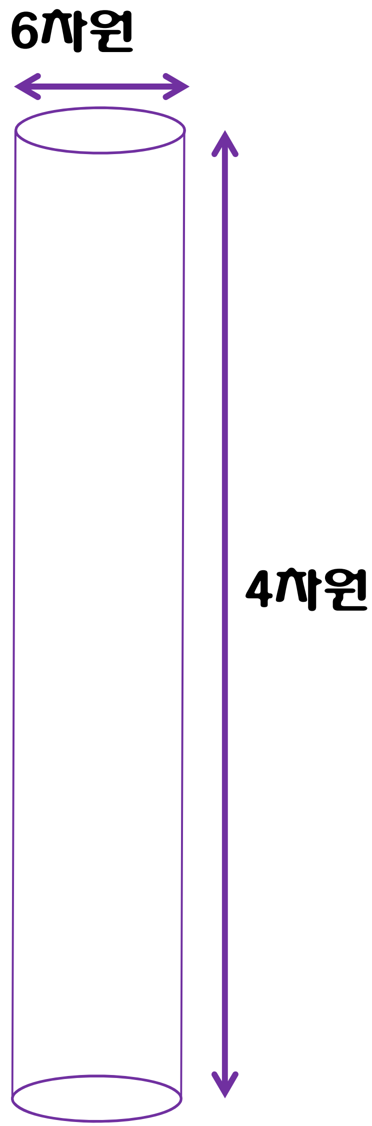
roughly,



roughly,

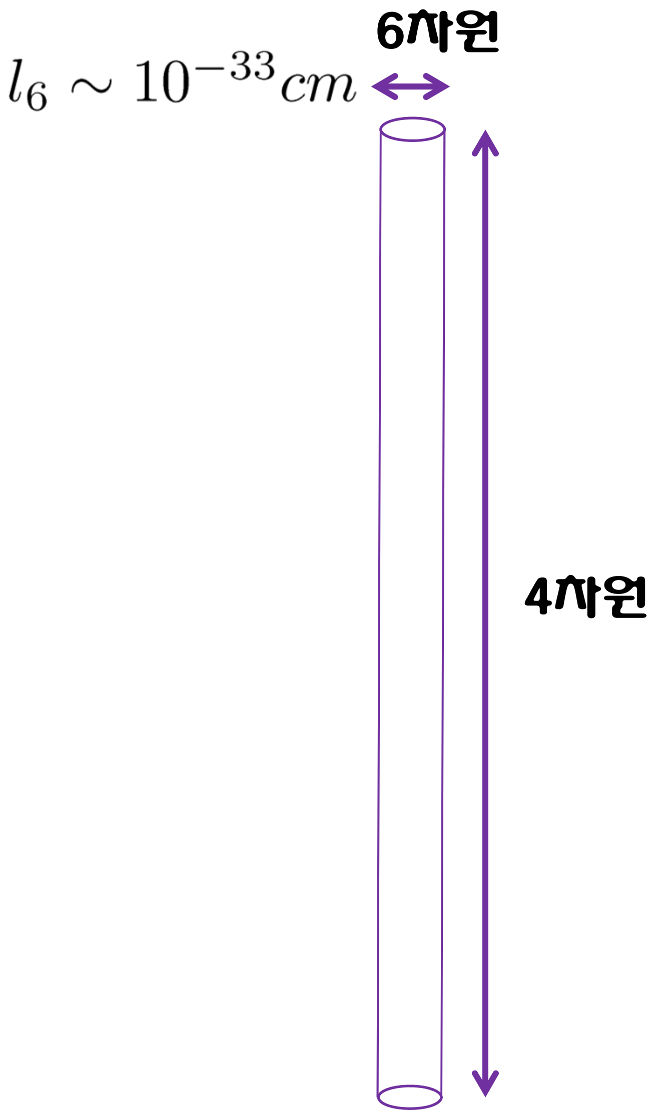


roughly,



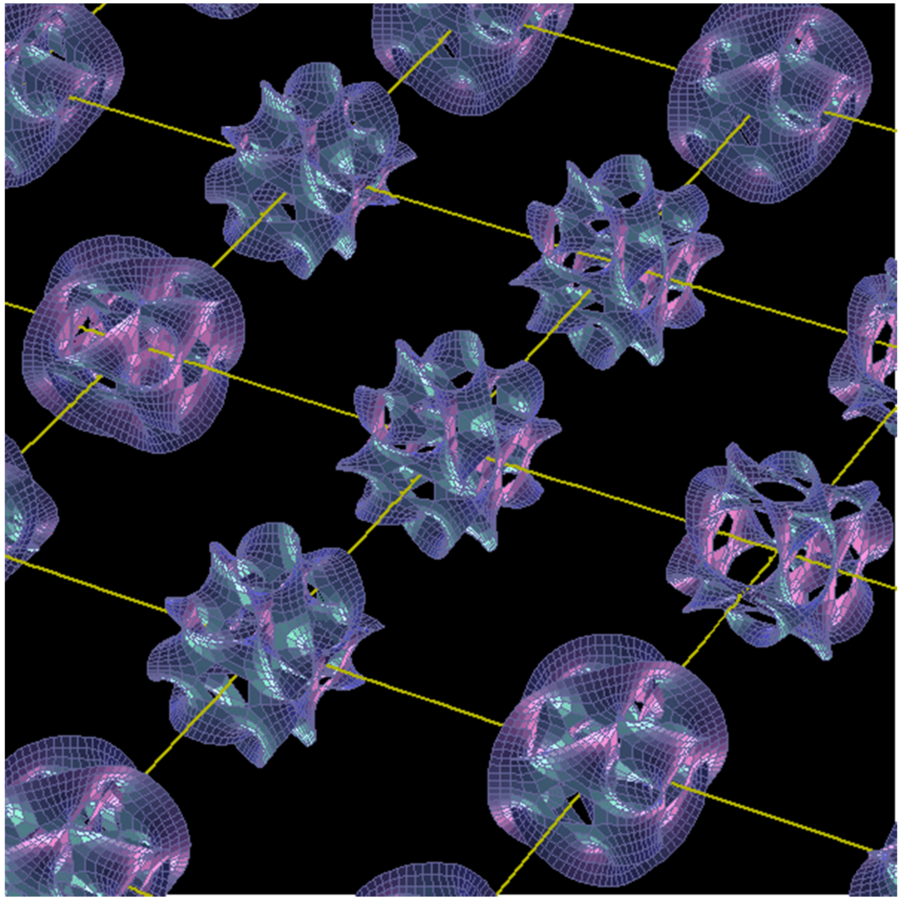
roughly,

roughly,



that is, more precisely,

$$10^{-13}cm \gg l_{size} > 10^{-33}cm$$



$$R_{AB} - \frac{1}{2}g_{AB}R = 8\pi\kappa_{9+1}^2 T_{AB}$$

superstring theory says :

spacetime is composed of 4+6 dimensions with
very small & tightly-curved (say, Calabi-Yau) 6D manifold
sitting at each and every point of usual 3D space,

which implies …………

a little bit of superstring theory

particles from geometry / geometry from particles

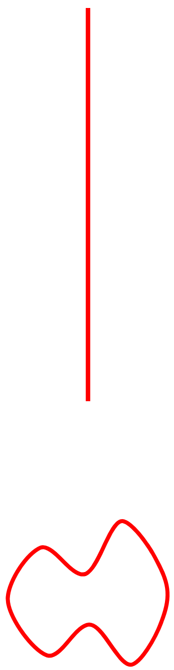
basic building blocks in superstring theory

fundamental
strings

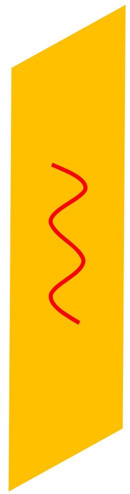
light particles
some of which mediate "forces"
such as gravity & electromagnetism

basic building blocks in superstring theory

fundamental
strings



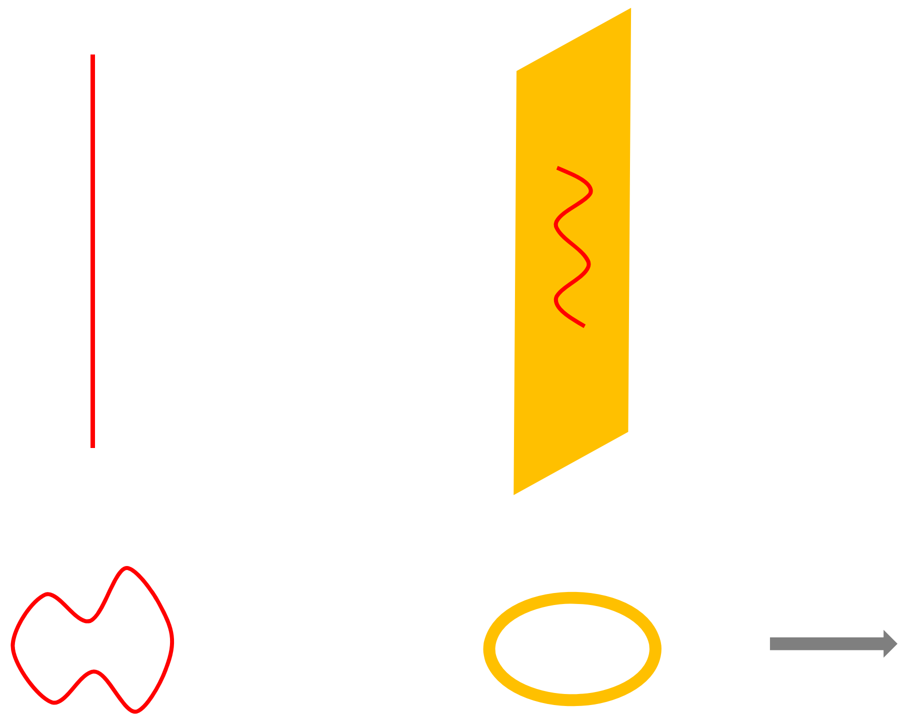
Dirichlet
branes
(as in membrane)



basic building blocks in superstring theory

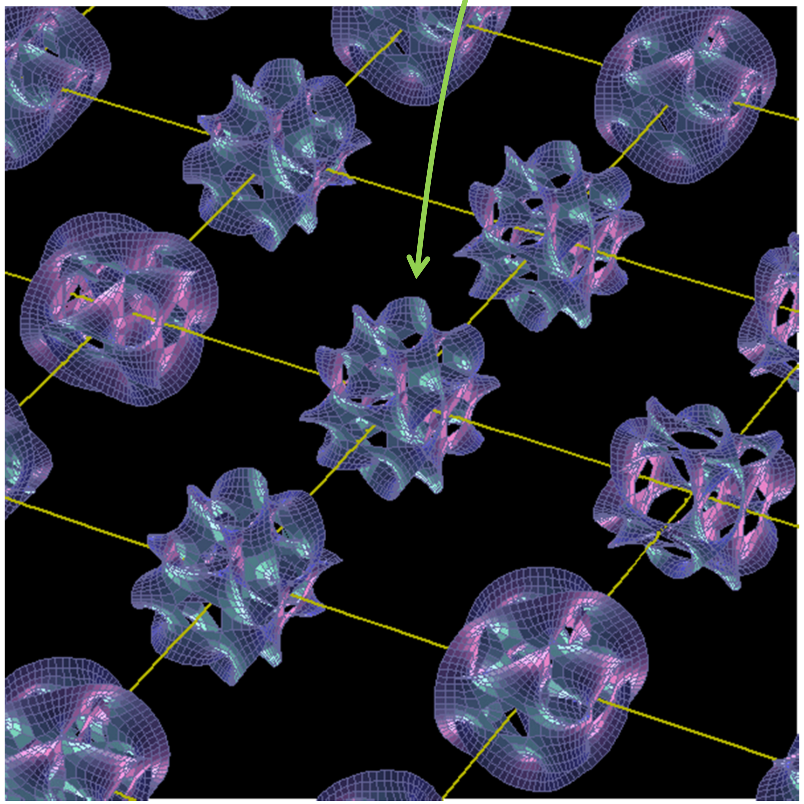
fundamental
strings

Dirichlet
branes

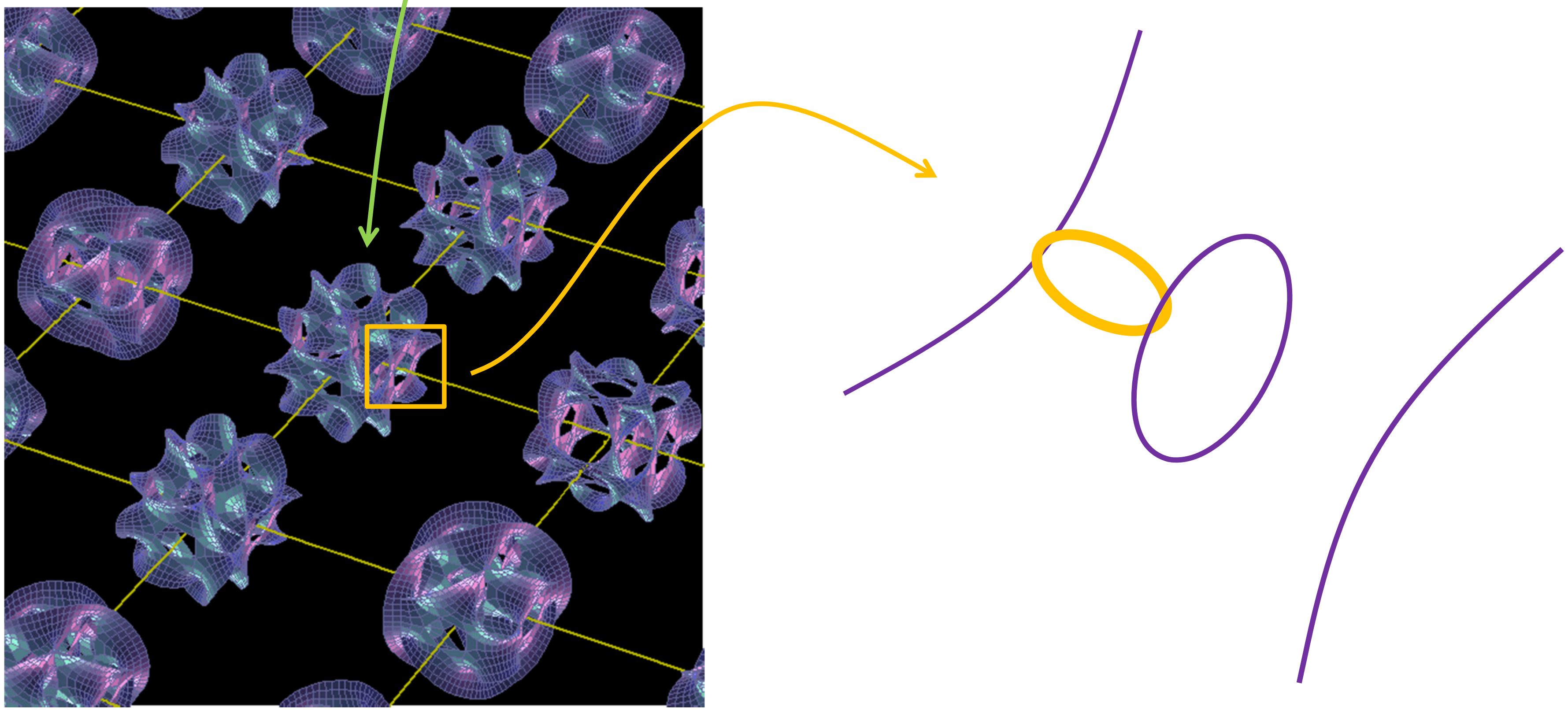


heavy particles,
typically with
electric/magnetic charges

a particle located somewhere in our visible space



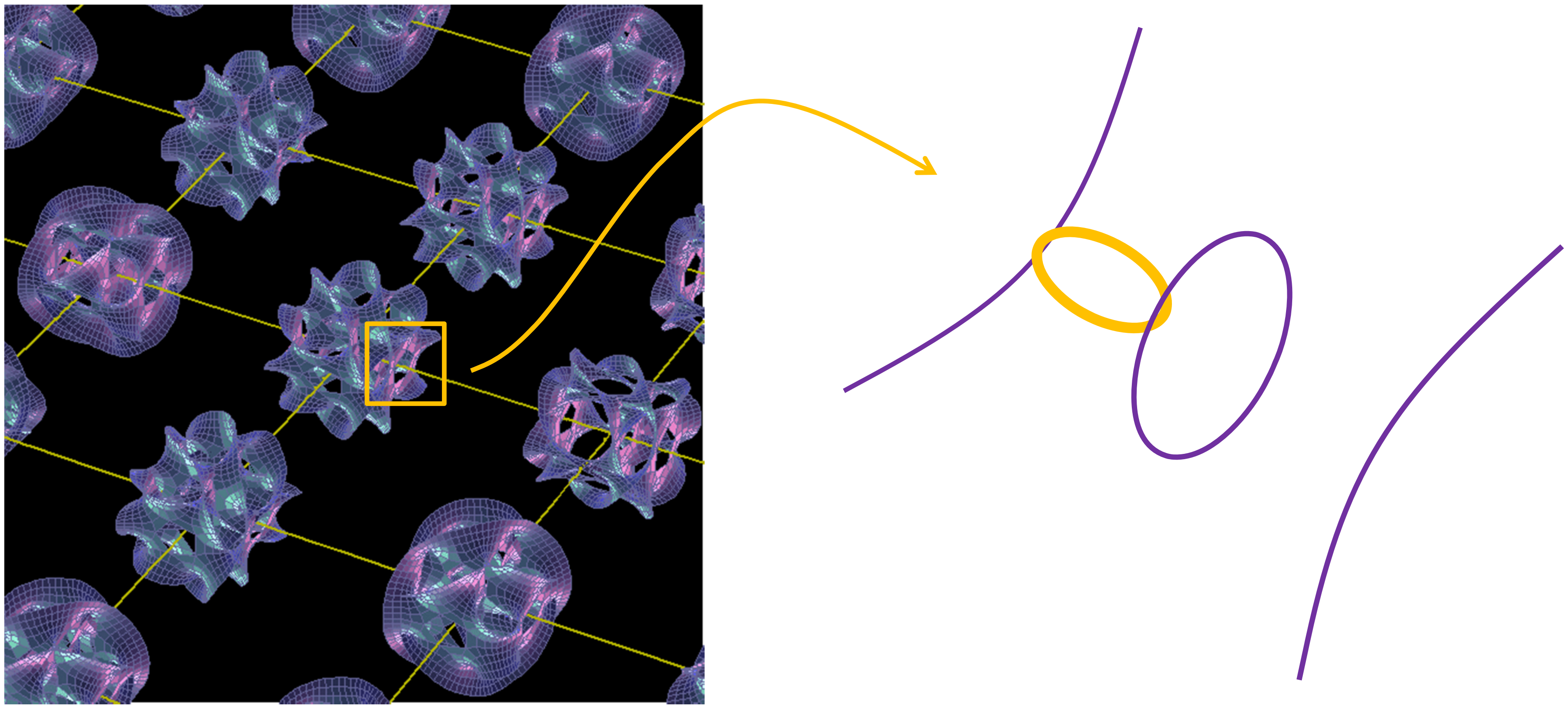
a particle located somewhere in our visible space
= a wrapped brane in the hidden Calabi-Yau at that point



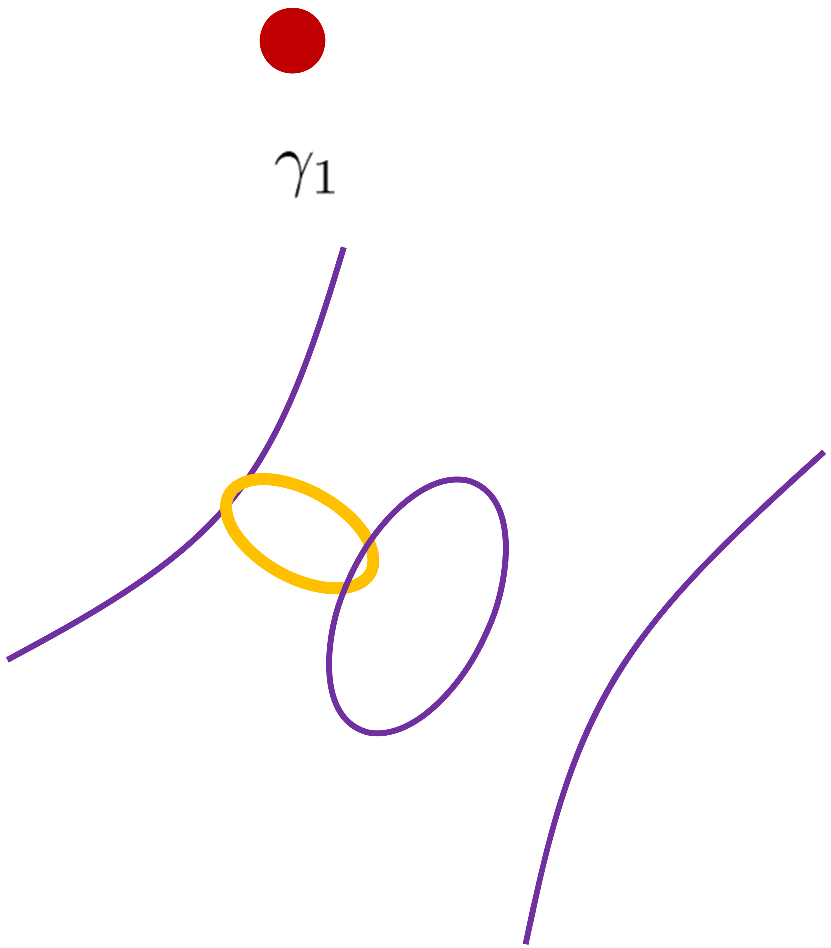
again, roughly,

$$l_6 \sim 10^{-33} cm$$

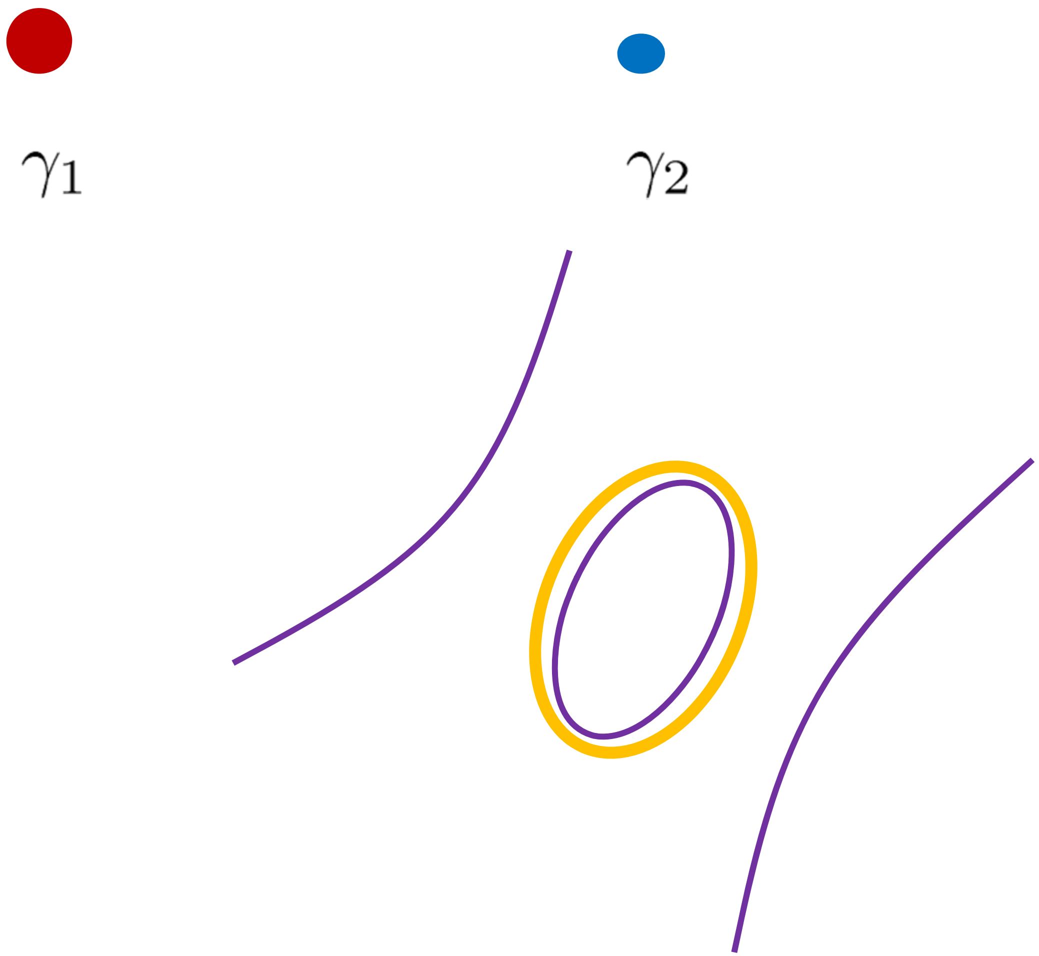
this means that we can actually detect
geometry (loops, holes, cavities, ……) of the hidden 6D space
by detecting what kind particles exist in visible 3D world



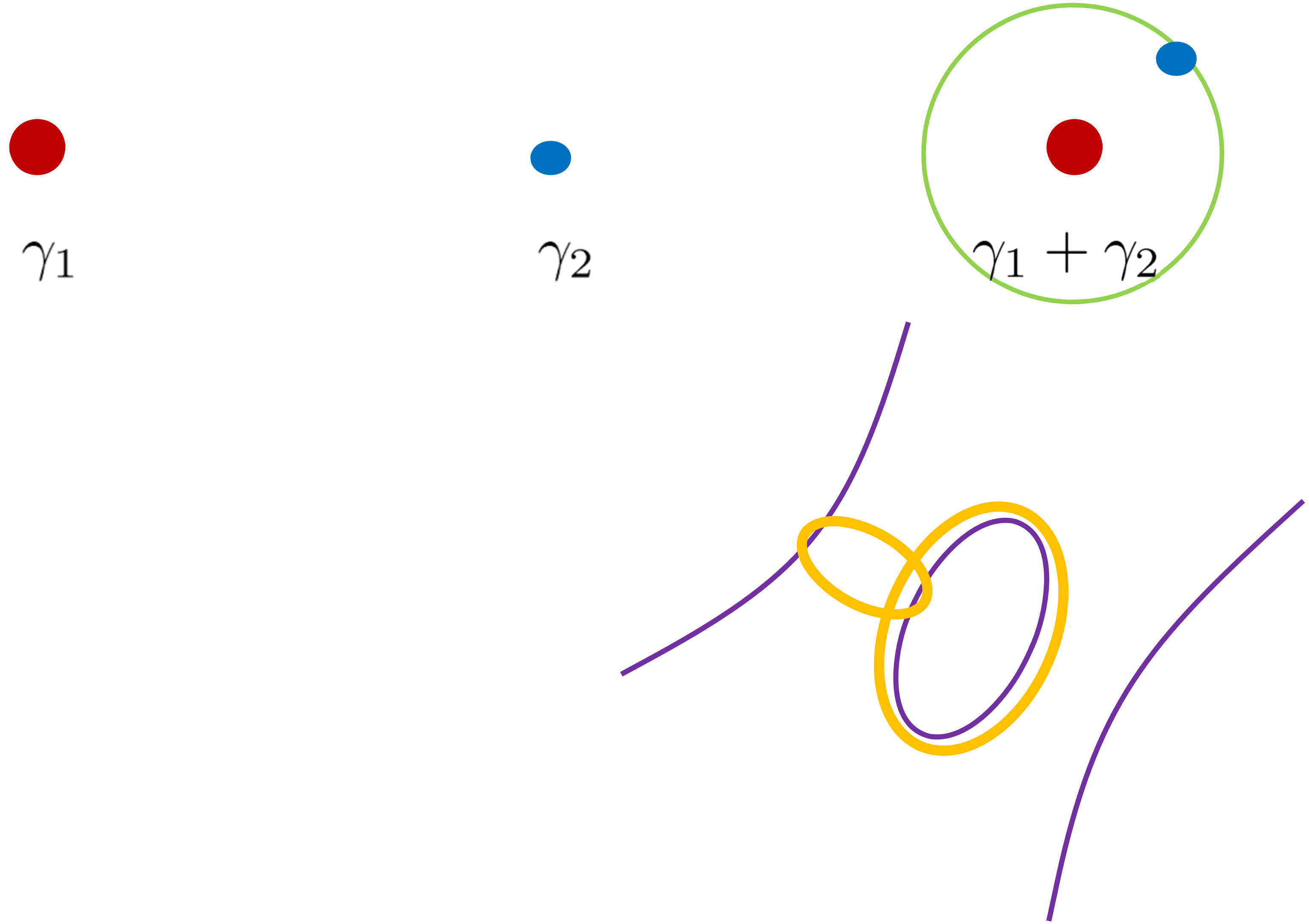
particle vs. geometry



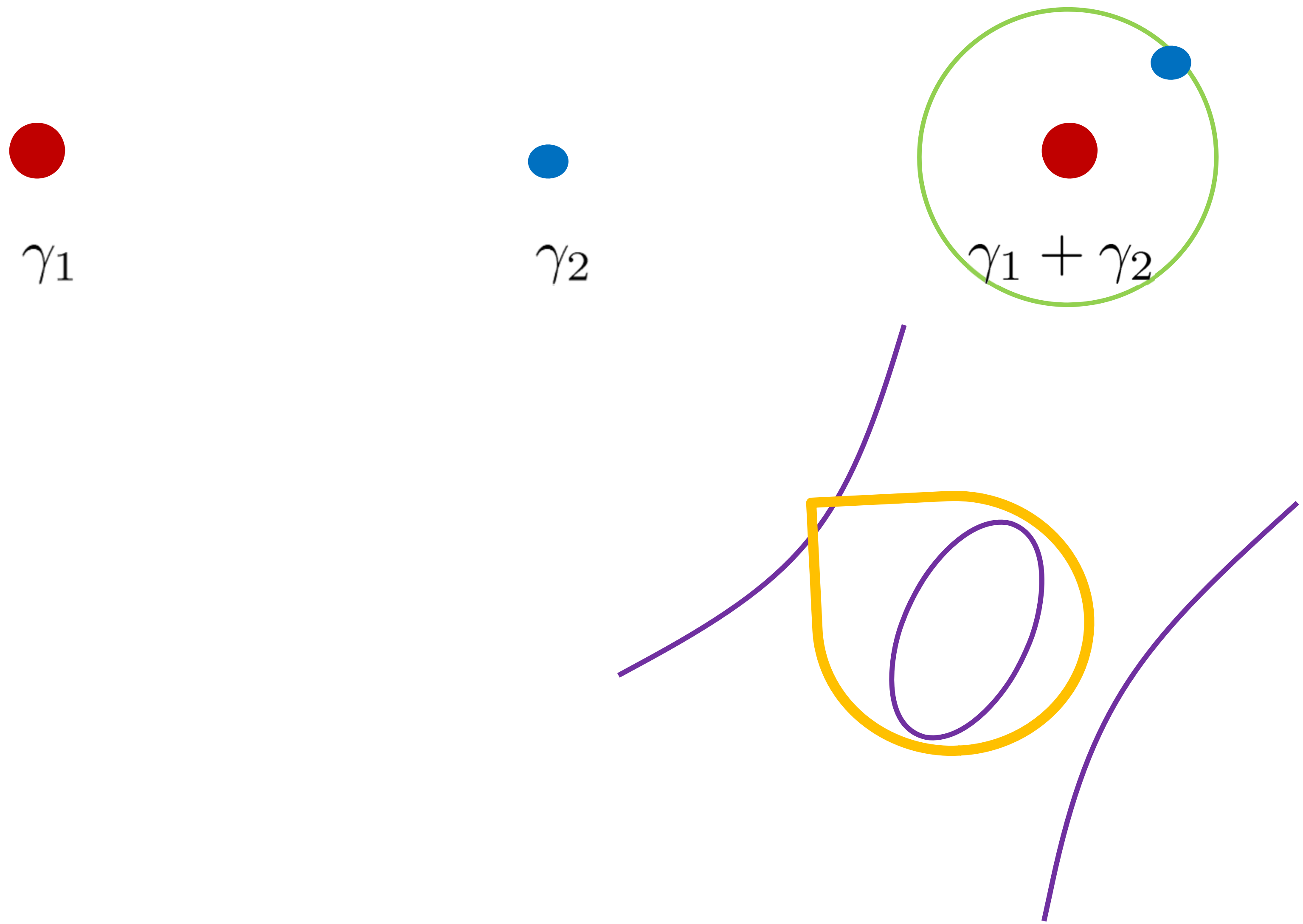
particle vs. geometry



particle vs. geometry



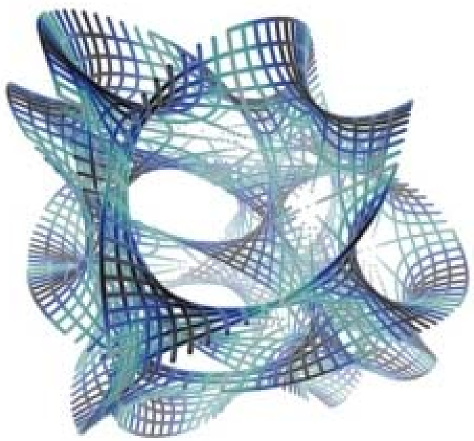
particle vs. geometry

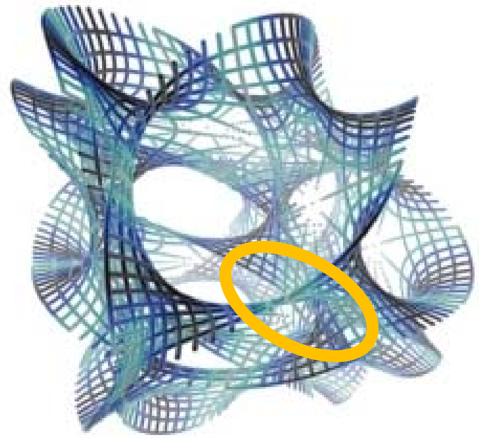


excursion : Calabi-Yau manifold & calibrated 3-cycles

$J^{(1,1)}$

$\Omega^{(3,0)}$

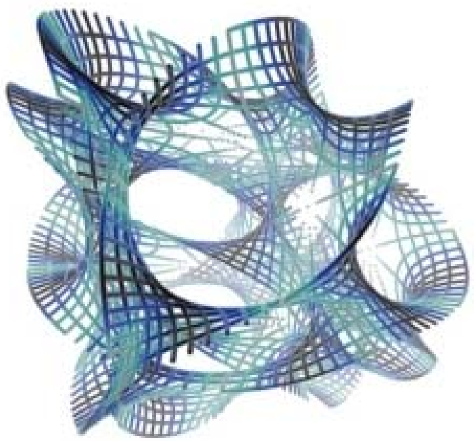


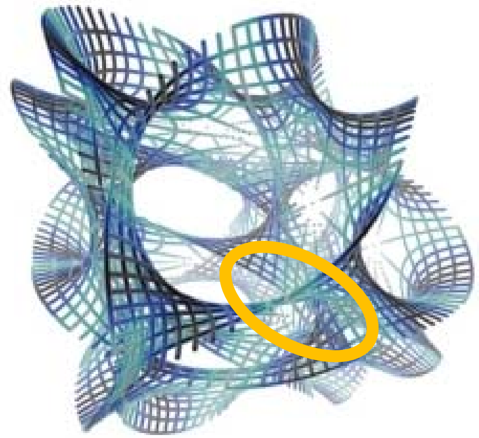


given a family of Calabi-Yau manifold with a fixed topology,
which topological 3-cycles can be calibrated ?

$$J^{(1,1)} \qquad\qquad J^{(1,1)}\Big| = 0$$

$$\Omega^{(3,0)} \qquad\qquad e^{-i\alpha}\Omega^{(3,0)}\Big| = \text{volume density of}$$



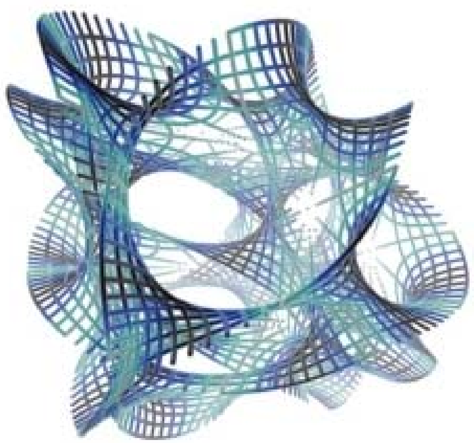


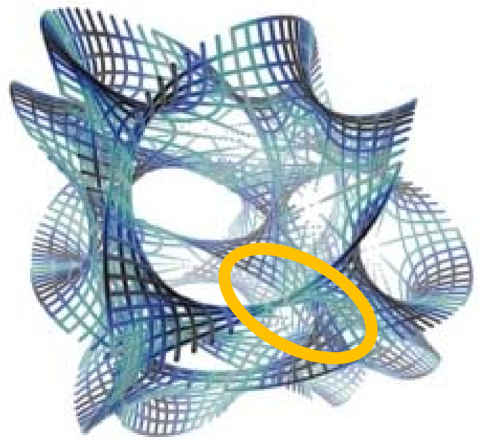
given a family of Calabi-Yau manifold with a fixed topology,
which topological 3-cycles can be calibrated ?

minimal volume condition
+
an extra "minimization" condition $\Longleftarrow$

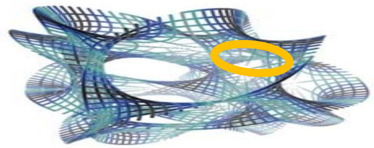
$$J^{(1,1)}\Big| = 0$$

$$e^{-i\alpha}\Omega^{(3,0)}\Big| = \text{volume density of}$$





which is the reason why topology alone
cannot guarantee existence of such cycles

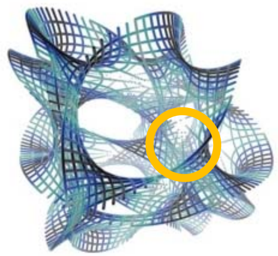


γ_1

γ_2

γ_3

$\vdots$



γ_1

γ_2

γ_3

$\vdots$

$\gamma_1 + \gamma_2$

$\gamma_2 + \gamma_3$

$\gamma_1 + \gamma_3$

$\vdots$

$\gamma_1 + \gamma_2 + \gamma_3$

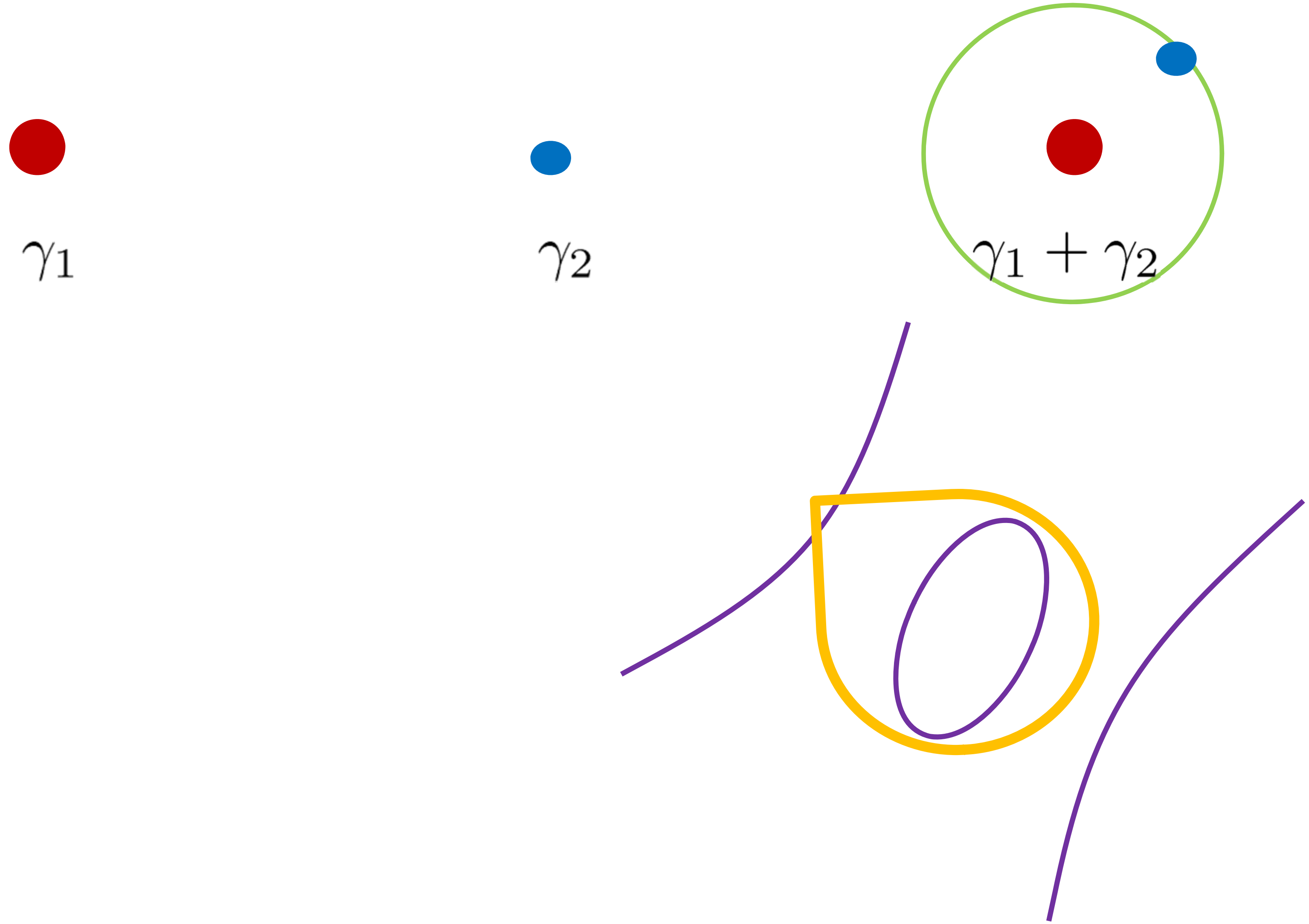
$\vdots$

$\sum k_i \gamma_i$

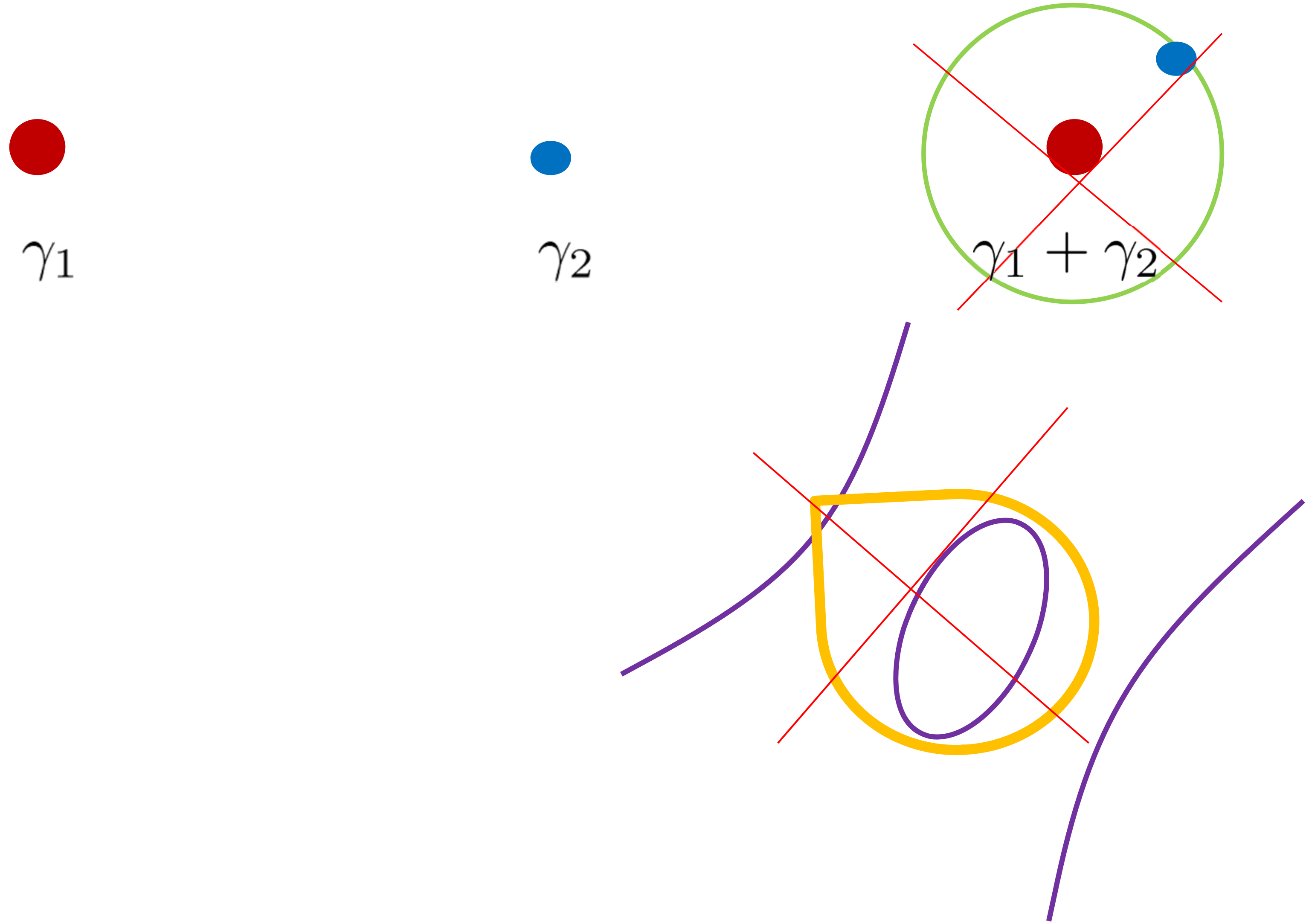
$\vdots$

existence of "quantum BPS (bound) states"

= existence of "calibrated 3-cycles"

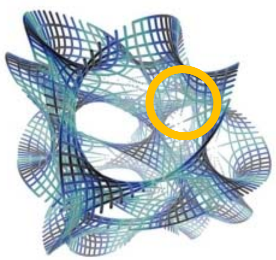


non-existence of "quantum BPS (bound) states"
= non-existence of "calibrated 3-cycles"

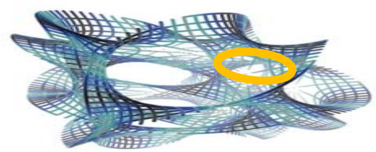


the answer depends on the precise shape/size of the Calabi-Yau

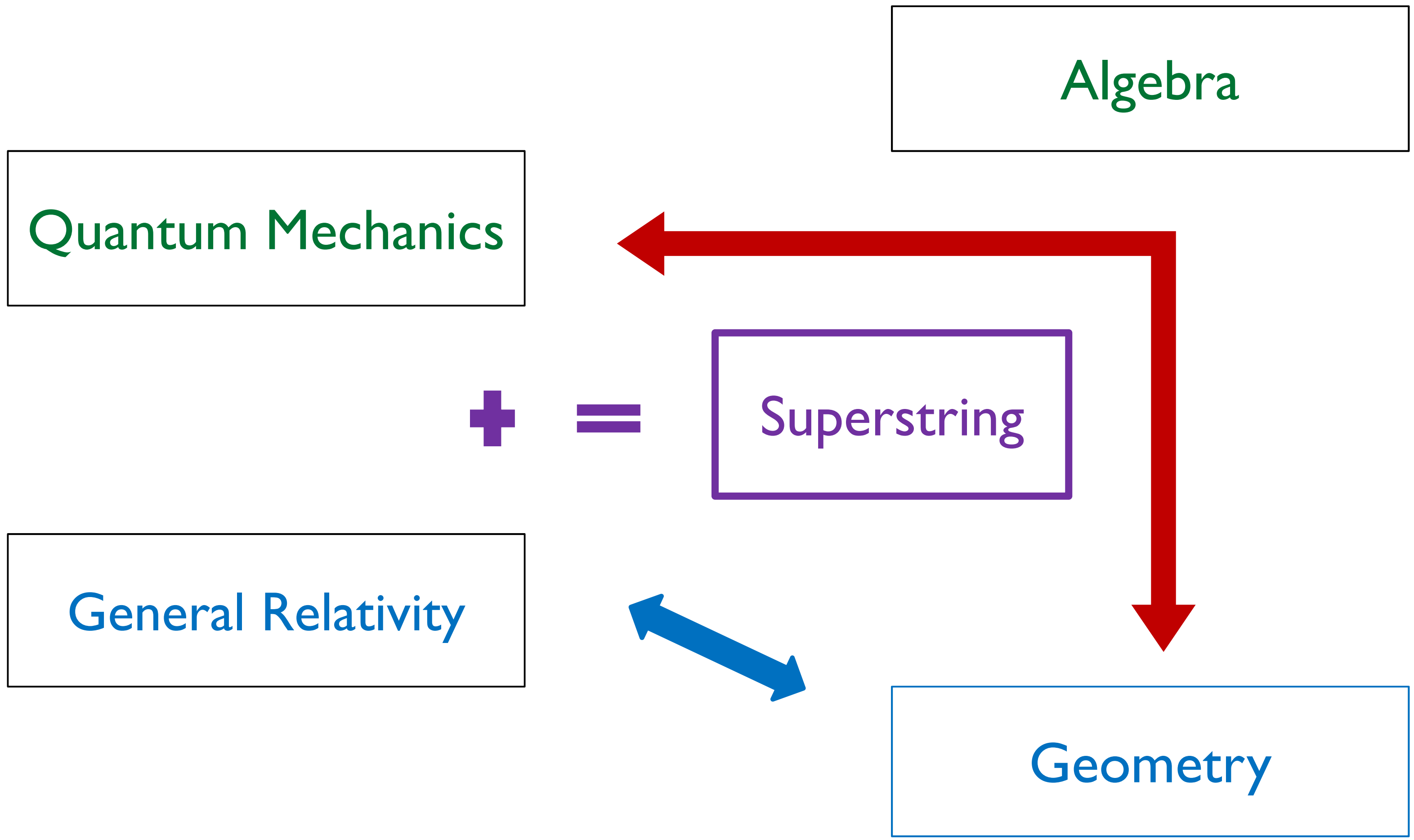
$$\gamma_1$$
$$\gamma_2$$
$$\gamma_3$$
$$\vdots$$



$$\gamma_1 \qquad \gamma_1+\gamma_2$$
$$\gamma_2 \qquad \gamma_2+\gamma_3$$
$$\gamma_3 \qquad \gamma_1+\gamma_3$$
$$\vdots \qquad \vdots$$
$$\gamma_1+\gamma_2+\gamma_3$$
$$\vdots$$
$$\sum k_i\gamma_i$$
$$\vdots$$



how quantum mechanics solved modern geometry


Algebra
Quantum Mechanics
+ =
Superstring
General Relativity
Geometry

Atiyah-Singer Index Theorem ~ 1963
Calabi-Yau ~ 1978
Calibrated Geometry ~ 1982
(Harvey & Lawson)
.
.
.
Homological Mirror Symmetry ~ 1994
(Kontsevich)
.
.
.
.
Stability & Derived Category ~ 2000
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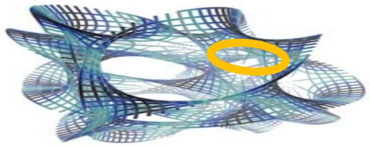
.
.
.
.
Wall-Crossing Conjecture~ 2008
(Conjecture by Kontsevich & Soibelman)
.
.
.
.

1975 ~ Bogomolnyi-Prasad-Sommerfeld (BPS)
1977 ~ Supersymmetry
1983 ~ Superstring Theory
.
1985 ~ Calabi-Yau Compactification
1988 ~ Mirror Symmetry
.
1994 ~ Wall-Crossing Discovered
(Seiberg & Witten)
1995 ~ Dirichlet Branes
1998 ~ Wall-Crossing is Bound State
Dissociation (Lee & P.Y.)
.
2001 ~ Wall-Crossing for Black Holes
(Denef)
.
.
.
.
2008 ~ Konstevich-Soibelman Explained
(Gaiotto & Moore & Neitzke)
.
2011 ~ KS Wall-Crossing proved
via Quatum Mechanics
Manschot , Pioline & Sen /
Kim , Park, Wang & P.Y. / Sen

wall-crossing is disappearance of BPS particles
as we deform a supersymmetric 4d theory continuously



γ_1

γ_2

γ_3

$\vdots$

$\gamma_1 + \gamma_2$

$\gamma_2 + \gamma_3$

$\gamma_1 + \gamma_3$

$\vdots$

$\gamma_1 + \gamma_2 + \gamma_3$

$\vdots$

$\sum k_i \gamma_i$

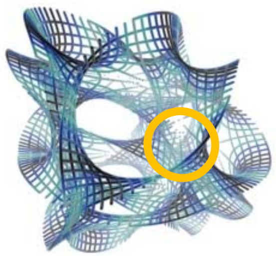
$\vdots$

γ_1

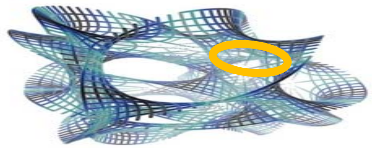
γ_2

γ_3

$\vdots$



wall-crossing is disappearance of calibrated 3-cycles as we deform a Calabi-Yau space continuously



γ_1

γ_2

γ_3

$\vdots$

$\gamma_1 + \gamma_2$

$\gamma_2 + \gamma_3$

$\gamma_1 + \gamma_3$

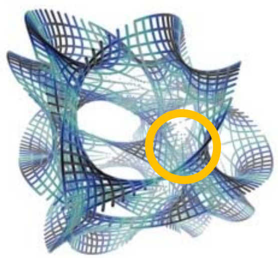
$\vdots$

$\gamma_1 + \gamma_2 + \gamma_3$

$\vdots$

$\sum k_i \gamma_i$

$\vdots$



γ_1

γ_2

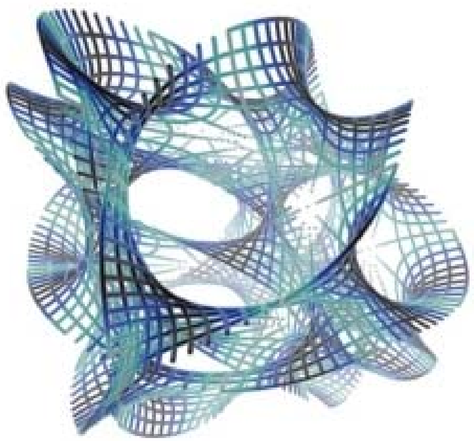
γ_3

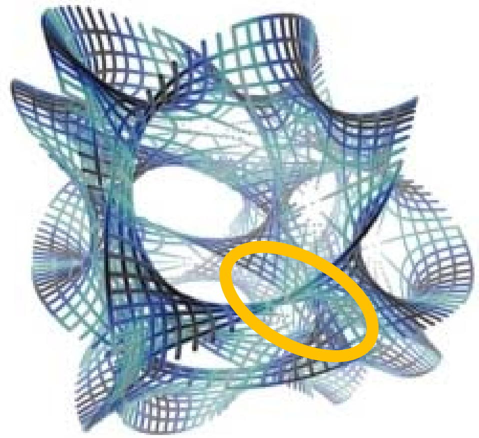
$\vdots$

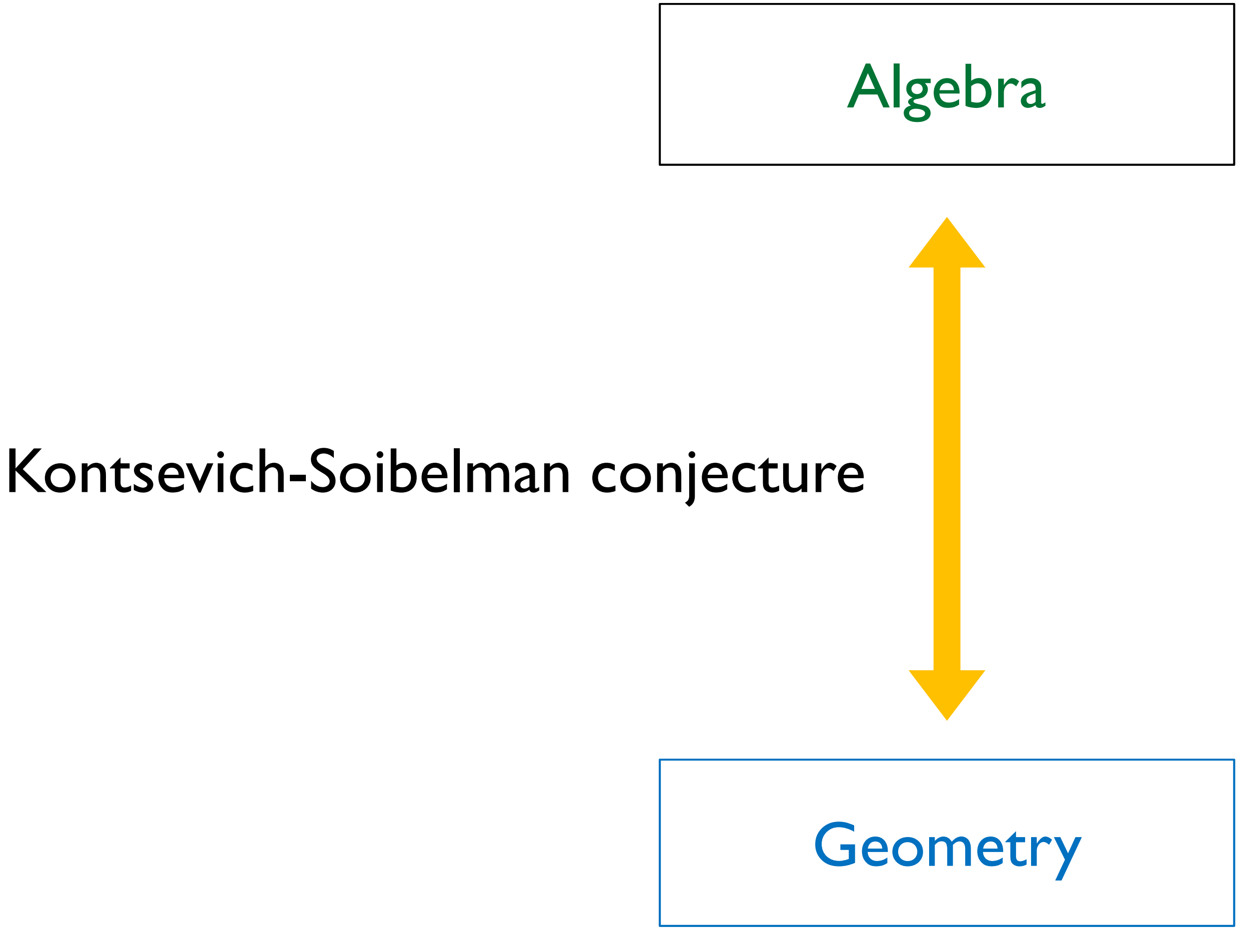
Calabi-Yau manifold & calibrated 3-cycles

$$J^{(1,1)} \qquad\qquad J^{(1,1)}\Big| = 0$$

$$\Omega^{(3,0)} \qquad\qquad e^{-i\alpha}\Omega^{(3,0)}\Big| = \text{volume density of}$$






Algebra
Kontsevich-Soibelman conjecture
Geometry

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$\langle \gamma, \gamma' \rangle = \langle (g, e), (g', e') \rangle = g \cdot e' - e \cdot g' \ \in \ \mathbf{Z}$$

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \ \in \ \mathbf{Z}^r \times \mathbf{Z}^r$$

$$\downarrow$$

$$V_\gamma$$

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

$$\gamma = (g, e) \in \mathbf{Z}^r \times \mathbf{Z}^r$$

$$\boxed{V_\gamma V_{\gamma'} - V_{\gamma'} V_\gamma = (-1)^{\langle\gamma,\gamma'\rangle} \langle\gamma, \gamma'\rangle V_{\gamma+\gamma'} \qquad \langle\gamma, \gamma'\rangle = 0, \pm 1, \pm 2, \ldots}$$

Kontsevich-Soibelman conjecture :
a universal algebra for wall-crossing ?

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$$\Omega(\gamma) = 0, \pm 1, \pm 2, \ldots$$

the "quantum degeneracy" of a given species of cycle / particle;
non-zero if and only if such a cycle exists in the geometric sense

Kontsevich-Soibelman conjecture : a universal algebra for wall-crossing ?

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$$\Omega(\gamma) = 0, \pm 1, \pm 2, \ldots$$

the "quantum degeneracy" of a given species of cycle / particle;
non-zero if and only if such a cycle exists in the geometric sense

mathematicians say, from 6d viewpoint

intersection number between the two cycles

$$\langle \gamma, \gamma' \rangle$$

Euler number of the moduli space of the cycle

$$\Omega(\gamma)$$

physicists say, from 4d viewpoint

Schwinger product

$$\langle \gamma, \gamma' \rangle = \langle (g,e), (g',e') \rangle = g \cdot e' - e \cdot g'$$

2nd helicity trace = the "number" of species of such particles

$$\Omega(\gamma) = -\frac{1}{2} \mathrm{tr}_\gamma \, (-1)^{2J_3} (2J_3)^2$$

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

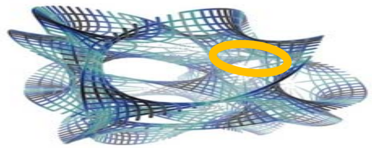
an infinite dimensional representation of Kontsevich-Soibelman wall-crossing algebra

$$V_\gamma V_{\gamma'} - V_{\gamma'} V_\gamma = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle V_{\gamma+\gamma'} \qquad \langle \gamma, \gamma' \rangle = 0, \pm 1, \pm 2, \ldots$$

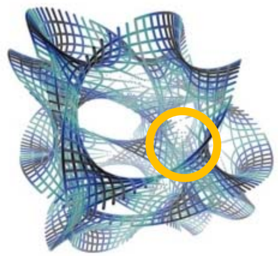
$$K_\gamma \equiv \exp\left(\sum_{n=1}^{\infty} \frac{V_{n\gamma}}{n^2} \right)$$

$$K_\gamma \;:\; X_\alpha \;\rightarrow\; X_\alpha (1 - \sigma(\gamma) X_\gamma)^{\langle \gamma, \alpha \rangle}$$

the conjecture : given the left-hand-side, the right-hand-side is entirely determined via the algebraic identity as follows



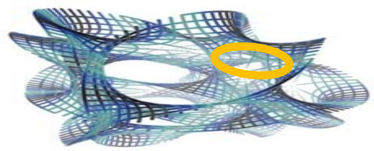
+ side $$\prod_{\gamma} K_{\gamma}^{\Omega^{+}(\gamma)} = \prod_{\gamma'}{}' K_{\gamma'}^{\Omega^{-}(\gamma')}$$ – side



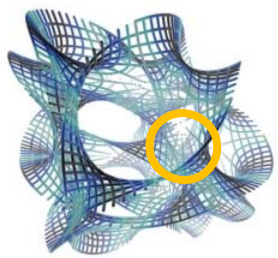
$$K_{\gamma} \equiv \exp\left(\sum_{n=1}^{\infty} \frac{V_{n\gamma}}{n^2}\right)$$

$$\exp(A) \equiv \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

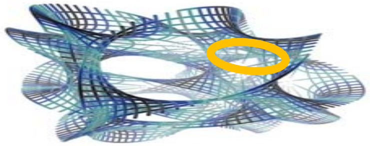
for example, $V_{\gamma_1} V_{\gamma_2} - V_{\gamma_2} V_{\gamma_1} = -V_{\gamma_1+\gamma_2}$



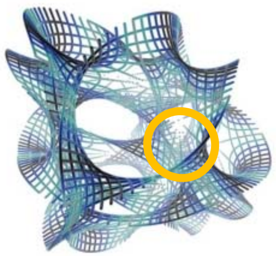
$$K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1+\gamma_2} K_{\gamma_1}$$



for example, $V_{\gamma_1} V_{\gamma_2} - V_{\gamma_2} V_{\gamma_1} = 2V_{\gamma_1+\gamma_2}$



$$K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1+2\gamma_2} K_{2\gamma_1+3\gamma_2} \cdots$$
$$\cdots K^{-2}_{\gamma_1+\gamma_2} \cdots K_{3\gamma_1+2\gamma_2} K_{2\gamma_1+\gamma_2} K_{\gamma_1}$$



which fits perfectly the wall-crossing prototype :
D=4 N=2 SU(2) Seiberg-Witten

$$W = (0,2)$$

$$\bar{D}_n = (-1, 2n)$$

$$M \equiv D_0 = (1,0)$$

$$\bar{D} \equiv \bar{D}_1 = (-1,2)$$

$$\Omega(\bar{W}) = \Omega(W) = -2$$

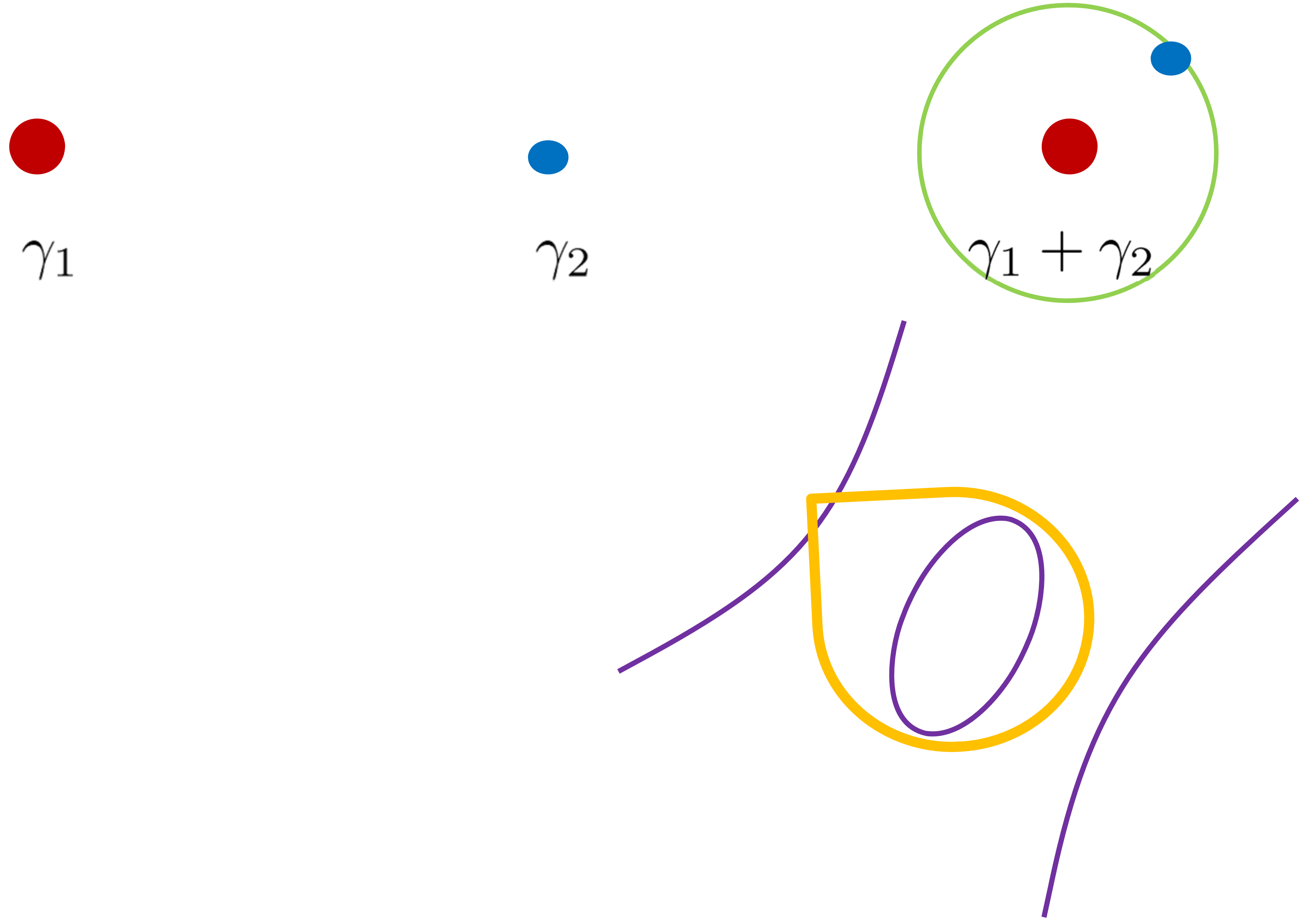
$$\Omega(\bar{D}_n) = \Omega(D_n) = 1$$

true ?

how to see from BPS state building/counting ?

existence of "quantum BPS (bound) states"

= existence of "calibrated 3-cycles"

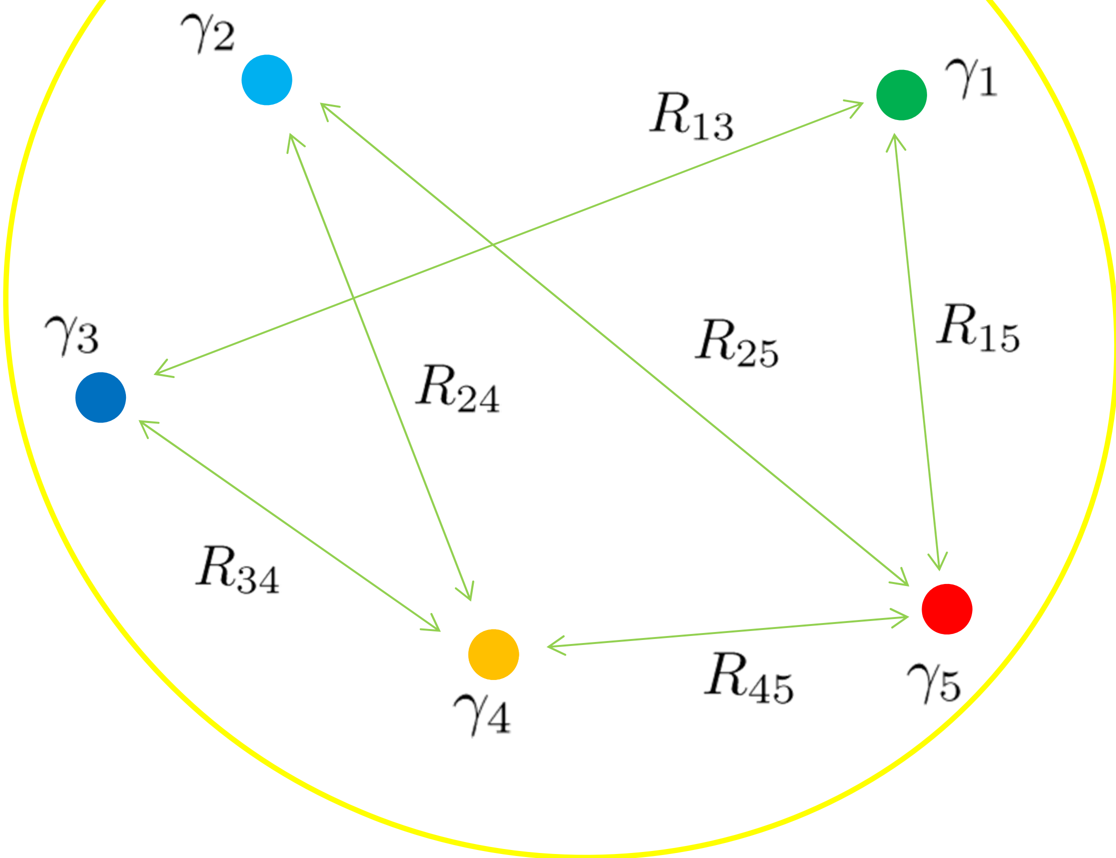


generic BPS particle is a multi-center bound state

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}}$$

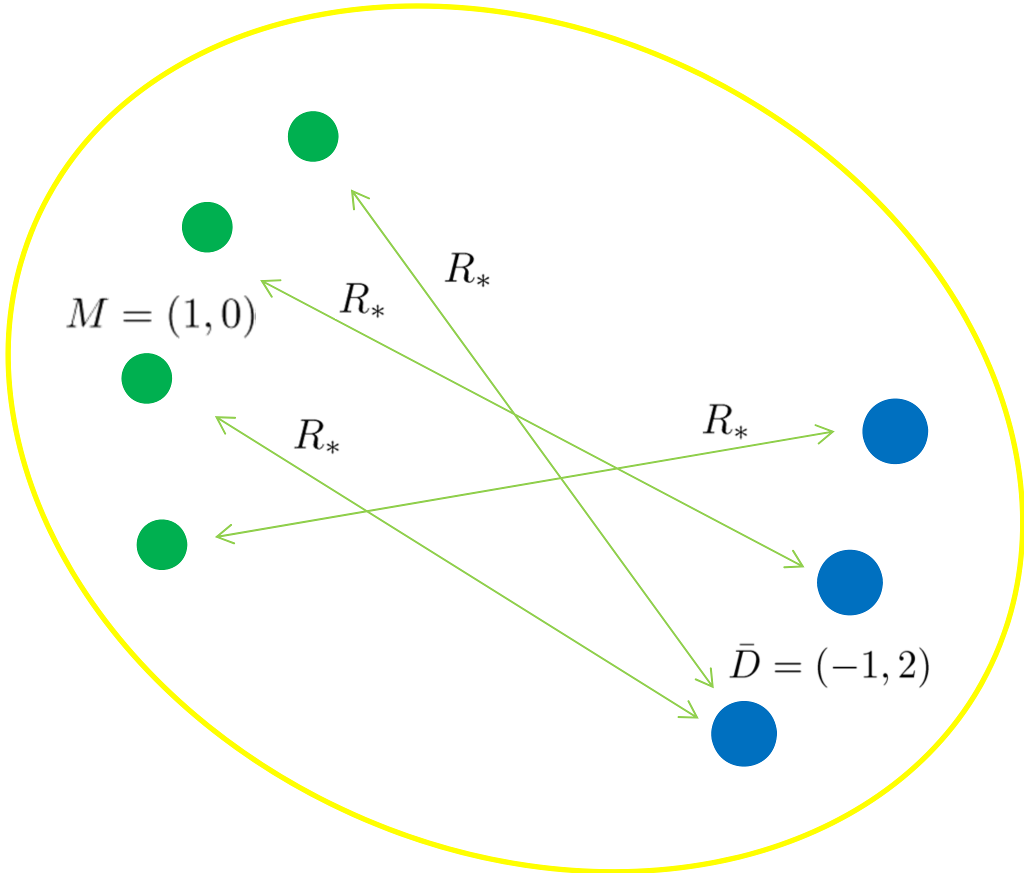
$$\gamma_{\mathrm{total}} = \sum_A \gamma_A$$

$$\zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$



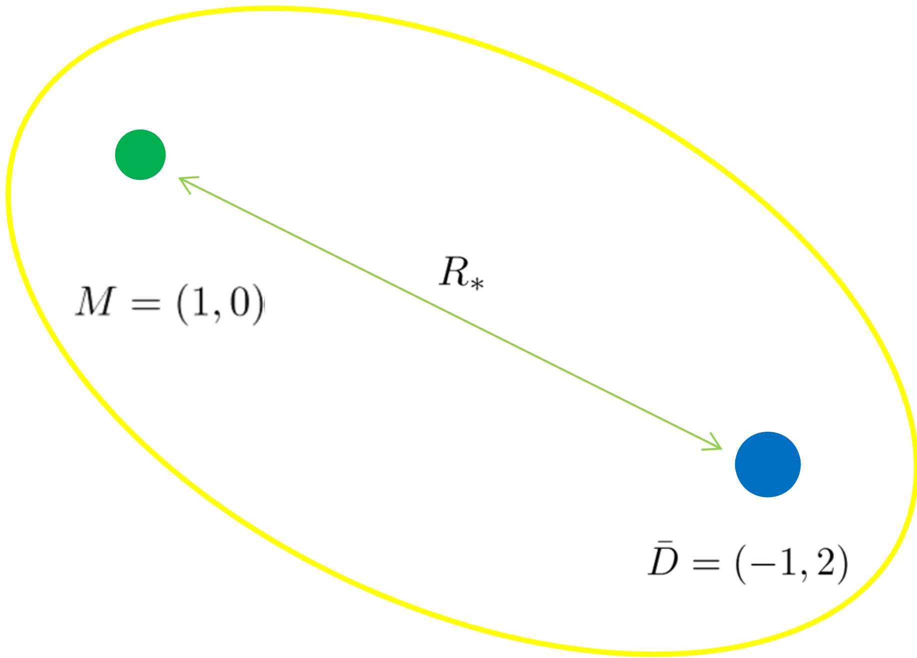
in particular, for SU(2) Seiberg-Witten

$$D_n = (n+1)M + n\bar{D}$$



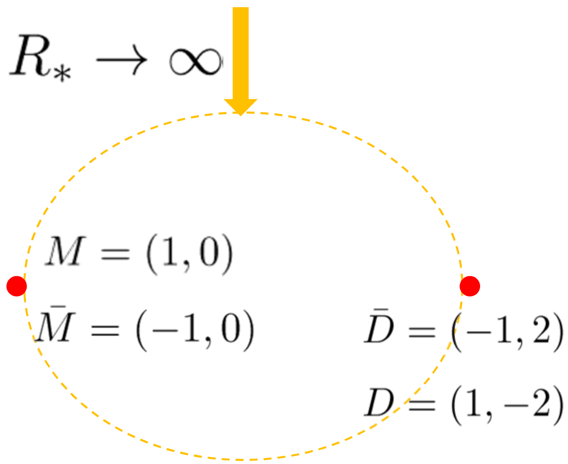
in particular, for SU(2) Seiberg-Witten

$$W^{+} = M + \bar{D}$$



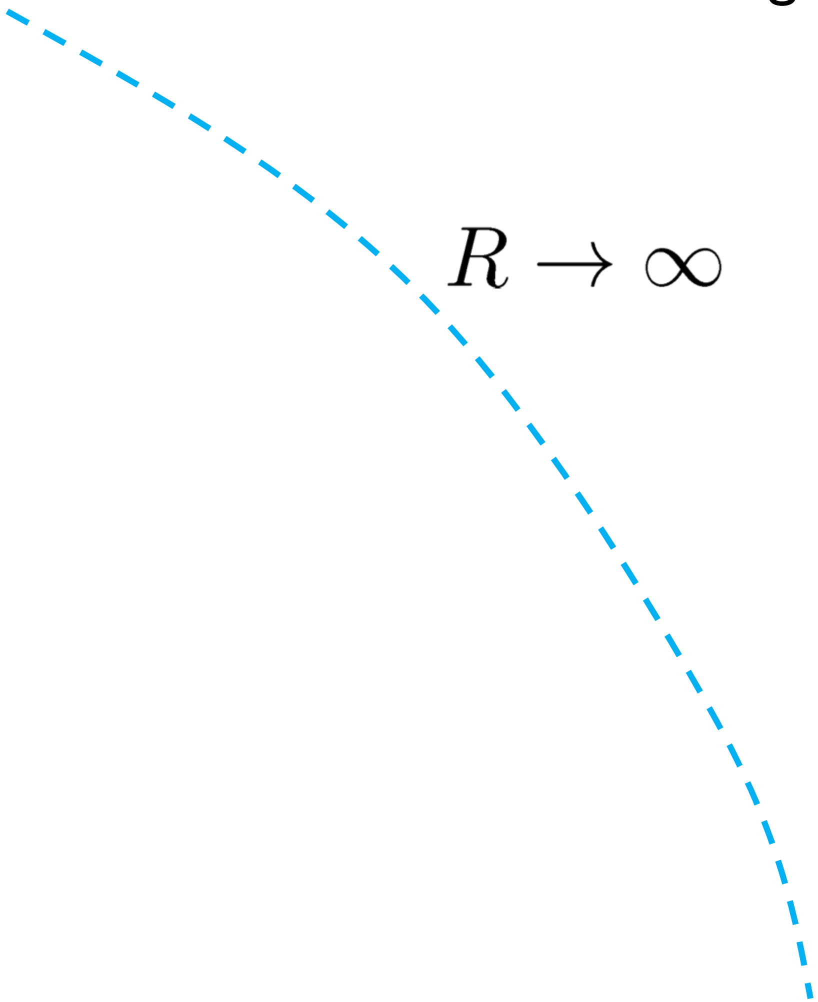
wall-crossing ← dissociation of supersymmetric bound states

$$W^+ = M + \bar{D}$$
$$W^- = \bar{M} + D$$
$$D_n = (n+1)M + n\bar{D}$$
$$\bar{D}_n = (n-1)M + n\bar{D}$$



$R_* \to 0$

thus, wall-crossing has a very simple and interpretation
in the particle / quantum mechanics viewpoint
as bound states becoming unbound



$$R \to \infty$$

2000 Stern + P.Y.

wall-crossing formula for simple magnetic charges; weak coupling regime

2002 Denef

quiver dynamics representation of N=2 supergravity BH's

•
•
•

2008 Kontsevich + Soibelman

•
•
•

2010/2011 Manschot + Pioline + Sen

general n-particle conjecture for Quantum Mechanics Counting

2011 Lee+P.Y. / Kim+Park+Wang+P.Y.

general n-particle solution to Quantum Mechanics Counting quantum mechanics for many such charged particles

Kim+Park+P.Y.+Wang 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

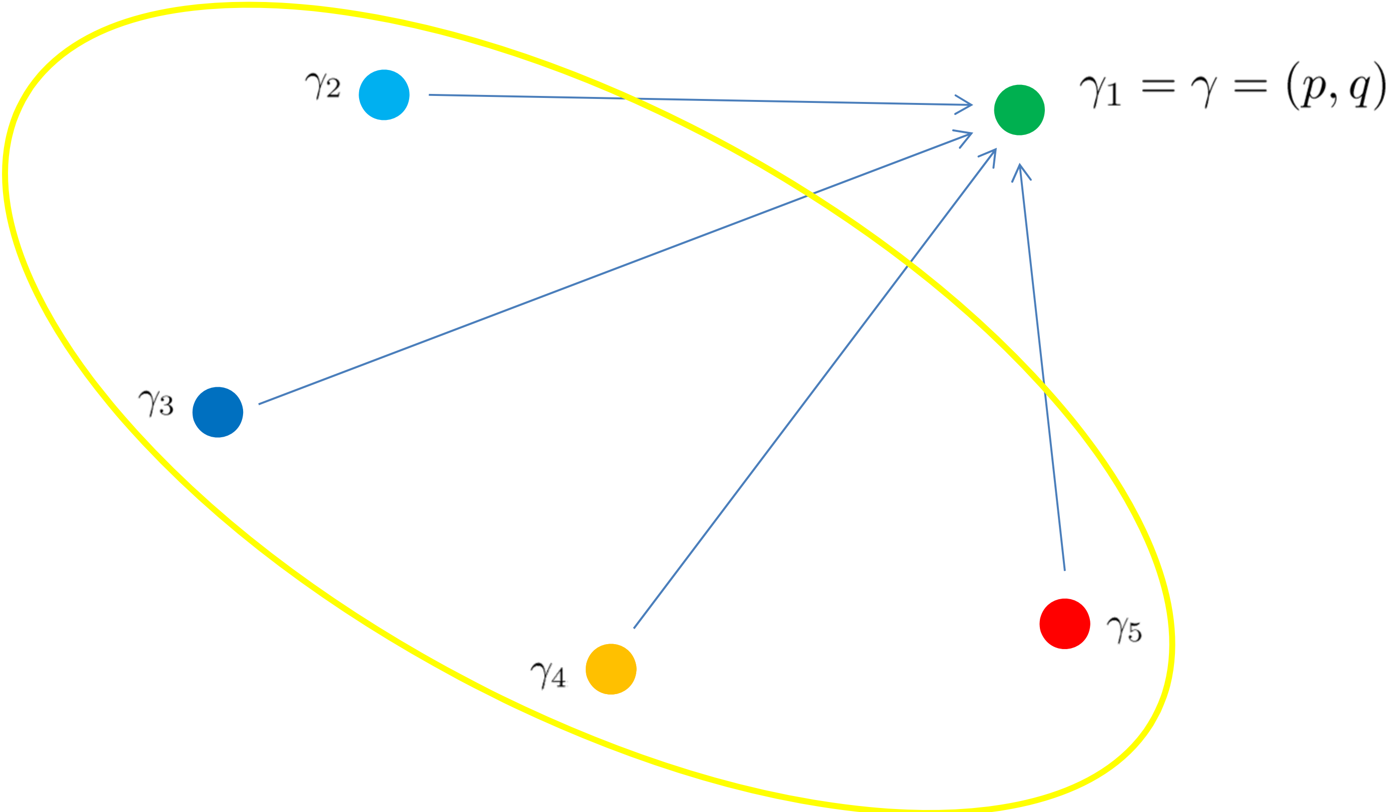
$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{|\vec{x}_A - \vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\ \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

$$F(\vec{x}_A) \simeq \sum_A |Z_A||\vec{x}_A|^2 = \sum_A m_A |\vec{x}_A|^2 \qquad \text{asymptotically}$$

each dyon feels the rest via long-range tails

$$\mathcal{Z}_{\gamma=(p,q)} \equiv [p^i \phi_D^i + q^i \phi^i]\Big|_{\gamma_{A'=2,3,4,\ldots}}$$



each dyon feels the rest via long-range tails

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_\gamma|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_\gamma] - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}\,|\mathcal{Z}_\gamma|\dot{\vec{x}}^2 - \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_\gamma])^2}{2|\mathcal{Z}_\gamma|} - \dot{\vec{x}}\cdot\vec{W}$$

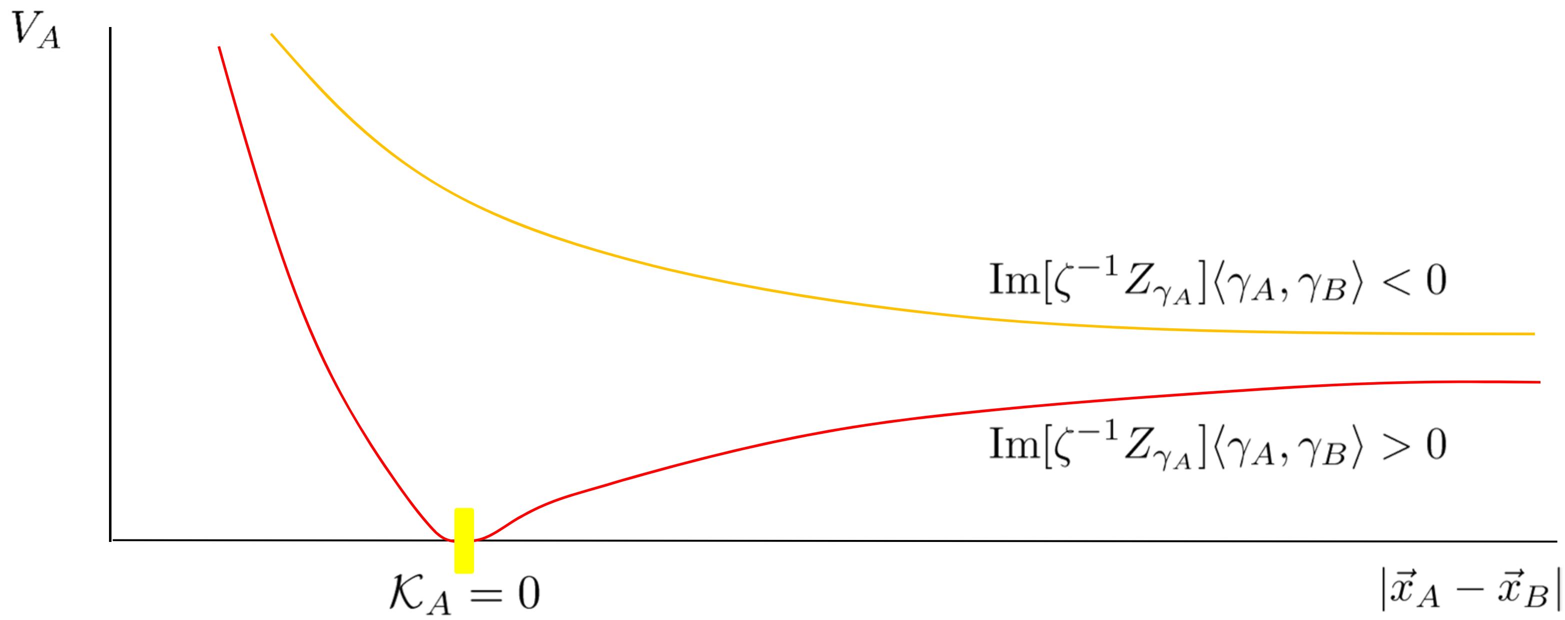
$$\mathrm{Im}[\zeta^{-1}\mathcal{Z}_{\gamma_A}] = \mathrm{Im}[\zeta^{-1}Z_{\gamma_A}] - \sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle/2}{|\vec{x}_A-\vec{x}_B|}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_\gamma]$$

$$\zeta^{-1}\mathcal{Z}_\gamma = |\mathcal{Z}_\gamma|e^{i\alpha}, \quad |\alpha| \ll 1$$

repulsive or attractive depending on a sign

$$V_A = \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_A])^2}{2|\mathcal{Z}_A|} \sim \left(\mathrm{Im}[\zeta^{-1}Z_{\gamma_A}] - \sum_B \frac{\langle\gamma_A,\gamma_B\rangle/2}{|\vec{x}_A - \vec{x}_B|}\right)^2$$



thus, wall-crossing has a very simple and interpretation
in the particle / quantum mechanics viewpoint
as bound states becoming unbound

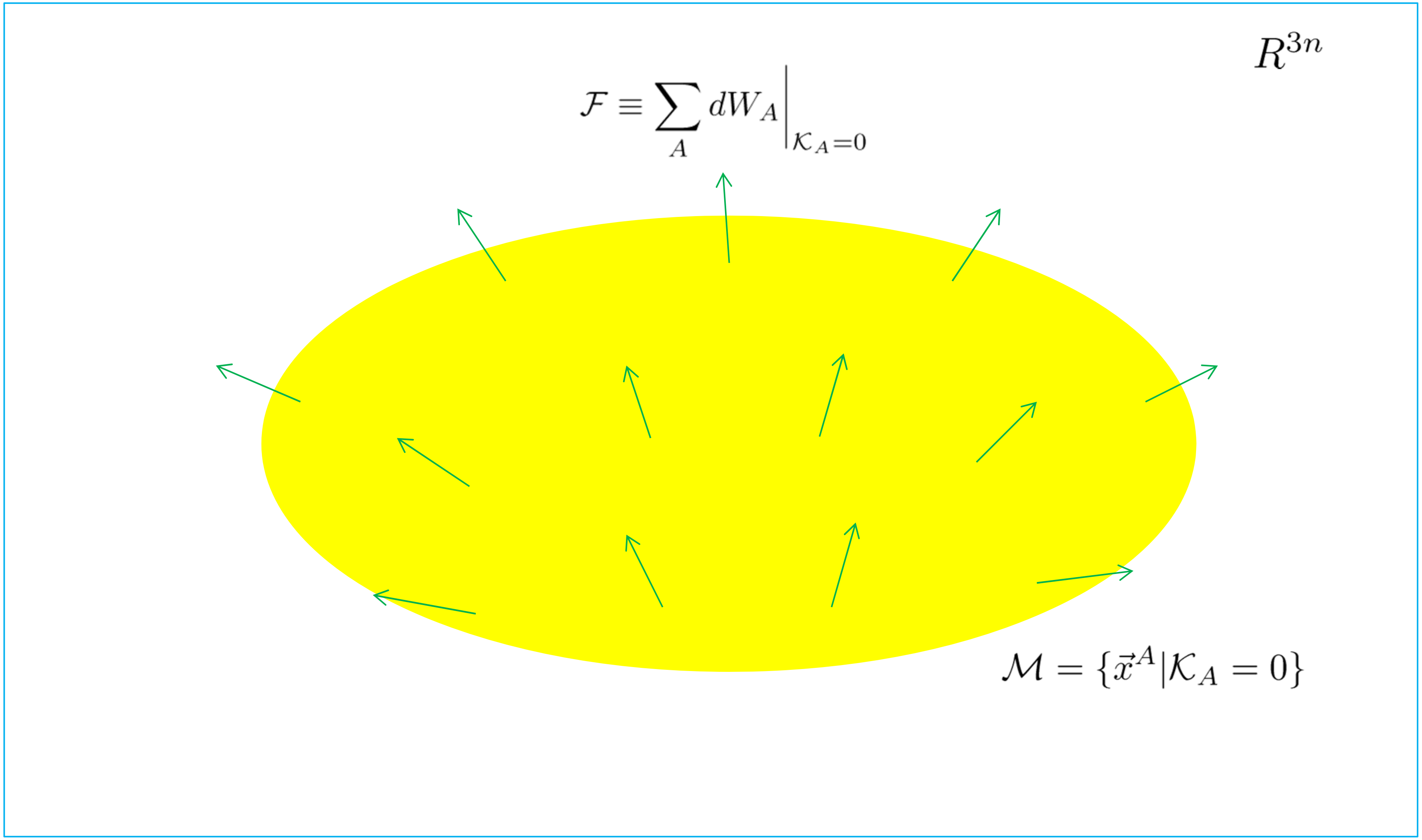
$$\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \end{array}$$

$$\begin{array}{cc} \gamma_1 & \gamma_1 + \gamma_2 \\ \gamma_2 & \gamma_2 + \gamma_3 \\ \gamma_3 & \gamma_1 + \gamma_3 \\ \vdots & \vdots \\ & \gamma_1 + \gamma_2 + \gamma_3 \\ & \vdots \end{array}$$

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] \langle \gamma_A, \gamma_B \rangle < 0$$

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] \langle \gamma_A, \gamma_B \rangle > 0$$

3n-dim dynamics → 3 + 2(n-1) dim nonlinear sigma model via deformation & localization that preserve the index



which reduces the problem to a N=1 Dirac index
on the manifold $\mathcal{K}_A = 0$ with Abelian magnetic fields

Kim+Park+Wang+P.Y. 2011

3(n-1) bosons + 4(n-1) fermions → 2(n-1) bosons + 2(n-1) fermions

$$\mathcal{L}^{for\ index\ only}_{deformed}\Big|_{L\rightarrow\infty} \quad \rightarrow \quad \mathcal{L}_{index}$$

$$\mathcal{L}_{index} \simeq \frac{1}{2}g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu - \dot{x}^\mu\cdot\mathcal{A}_\mu + \frac{i}{2}g_{\mu\nu}\psi^\mu\left(\dot{\psi}^\nu + \dot{z}^\alpha\Gamma^\nu_{\alpha\beta}\psi^\beta\right) + i\mathcal{F}_{\mu\nu}\psi^\mu\psi^\mu$$

$$\mathcal{F} = d\mathcal{A} \equiv \sum_A dW_A\Big|_{\mathcal{K}_A=0}$$

index theorem

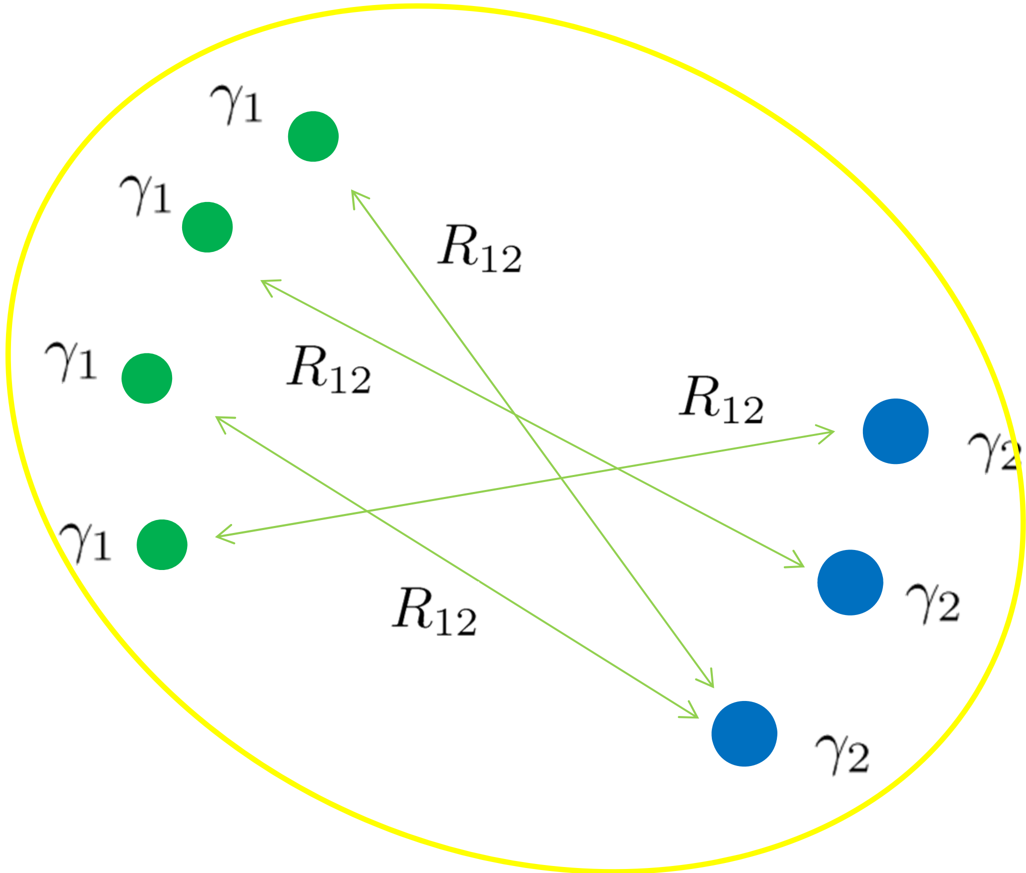
$$I_n(\{\gamma_A\}) = \mathrm{tr}\ \left[(-1)^F e^{-\beta H}\right] = \mathrm{tr}\ \left[(-1)^F e^{-\beta Q^2}\right]$$

$$= \int_{\mathcal{M}=\{\vec{x}_A\ \mid\ \mathcal{K}_A=0\}} ch(\mathcal{F}) \wedge \mathbf{A}(\mathcal{M})$$

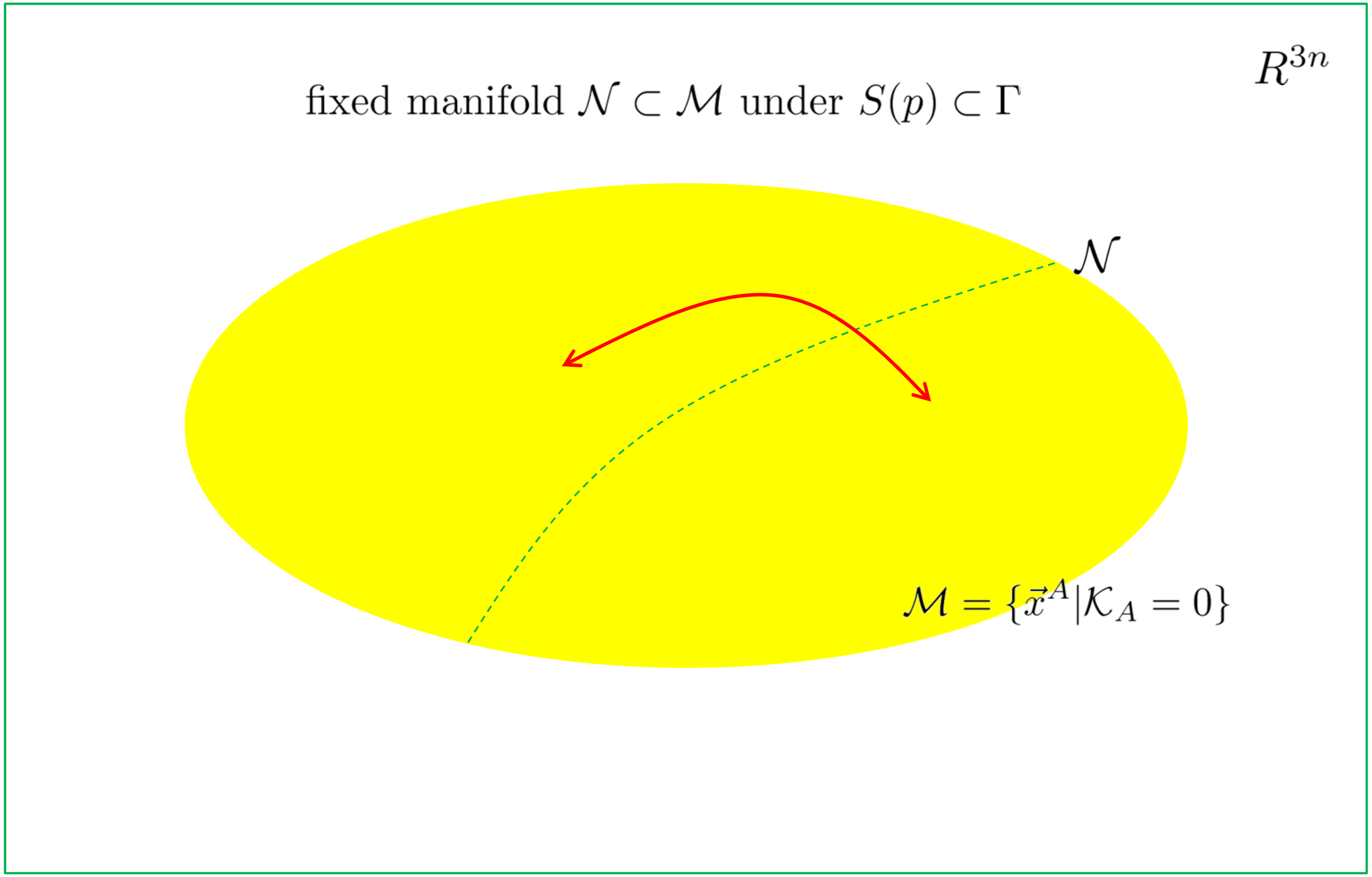
$$= \int_{\mathcal{M}=\{\vec{x}_A\ \mid\ \mathcal{K}_A=0\}} ch(\mathcal{F}) \wedge \det\left(\frac{R/2}{\sinh(R/2)}\right)$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\ \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

Bose/Fermi statistics is essential



incorporating Bose/Fermi statistics



universal wall-crossing formulae from quantum mechanics of BPS particles

Kim+Park+P.Y.+Wang 2011

$$\bar{\Omega}^-\left(\sum\gamma_A\right)-\bar{\Omega}^+\left(\sum\gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\}+n-1}\frac{\prod_A\bar{\Omega}^+(\gamma_A)}{|\Gamma|}\int_{\mathcal{M}}ch(\mathcal{F})$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\}+n'-1}\frac{\prod_{A'}\bar{\Omega}^+(\gamma'_{A'})}{|\Gamma'|}\int_{\mathcal{M}'}ch(\mathcal{F}')$$

$$\vdots$$

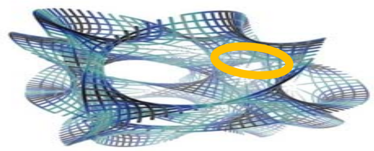
$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\}+n''-1}\frac{\prod_{A''}\bar{\Omega}^+(\gamma''_{A''})}{|\Gamma''|}\int_{\mathcal{M}''}ch(\mathcal{F}'')$$

$$\vdots$$

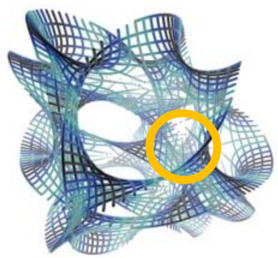
$$\bar{\Omega}(\gamma) = \sum_{p|\gamma}\Omega(\gamma/p)/p^2$$

$$\sum_{A=1}^{n}\gamma_A=\cdots=\sum_{A'=1}^{n'}\gamma'_{A'}=\cdots=\sum_{A''=1}^{n''}\gamma''_{A''}=\cdots$$

which fits an alternate form KS algebra with $\bar{\Omega}(\Gamma) \equiv \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$

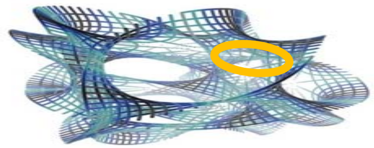


$+$ side $\quad \prod_{\gamma} e^{\bar{\Omega}^{+}(\gamma) V_\gamma} = {\prod_{\gamma'}}' e^{\bar{\Omega}^{-}(\gamma') V_{\gamma'}} \quad$ – side

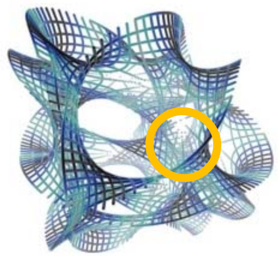


$$\cancel{K_\gamma \equiv \exp\left(\sum_{n=1}^{\infty} \frac{V_{n\gamma}}{n^2}\right)}$$

and precisely reproduces
answers from the conjectured Kontsevich-Soibelman Algebra



$$+\ \text{side} \qquad \prod_{\gamma} e^{\bar{\Omega}^{+}(\gamma) V_{\gamma}} = \prod_{\gamma'}{}' e^{\bar{\Omega}^{-}(\gamma') V_{\gamma'}} \qquad -\ \text{side}$$



2000 Stern + P.Y.

wall-crossing formula for simple magnetic charges; weak coupling regime

2002 Denef

quiver dynamics representation of N=2 supergravity BH's

⋮

2008 Kontsevich + Soibelman

⋮

2010/2011 Manschot + Pioline + Sen

general n-particle conjecture for Quantum Mechanics Counting

2011 Lee+P.Y. / Kim+Park+Wang+P.Y.

general n-particle solution to Quantum Mechanics Counting

2011 Sen

Quantum Mechanics Counting = Kontsevich-Soibelman Conjecture

quantum physics count states, gravity makes geometry,
and superstring theory combines quantum & gravity

quantum mechanics "count" geometry via superstring theory

→ quantum mechanical proof of the Kontsevich-Soibelman conjecture
which solves, partially, a 30-year-old geometry problem

geometry as mathematical tools for physics

string theory as physical tools for geometry